LOCALIZATION OF TRIANGULATED CATEGORIES WITH RESPECT TO EXTENSION-CLOSED SUBCATEGORIES

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Abstract. The aim of this article is to develop a framework of the localization theory of triangulated categories via extriangulated categories. Actually, given the pair of a triangulated category $\mathcal{C}$ and an extension-closed subcategory $\mathcal{N}$, we establish an exact sequence $\mathcal{N} \to \mathcal{C} \to \mathcal{C}/\mathcal{N}$ of extriangulated categories. Such a construction unifies the Verdier quotient and the heart of a t-structure.

1. Introduction

The abelian categories and triangulated categories, introduced by A. Grothendieck and J.-L. Verdier [5, 7], serve a foundation of the homological algebra. In the many blanches of mathematics, we often encounter interplays between abelian categories and triangulated categories. To name just a few important instances in the representation theory of algebra:

- for a given t-structure of a triangulated category $\mathcal{C}$, there exists a cohomological functor from $\mathcal{C}$ to the abelian heart [2];
- for a 2-cluster tilting subcategory $\mathcal{U}$ of a triagnulated category $\mathcal{C}$, the ideal quotient $\mathcal{C}/[\mathcal{U}]$ is abelian [9];
- for a Frobenius exact category $\mathcal{C}$ and the subcategory $\mathcal{N}$ of projective-injective objects, the ideal quotient $\mathcal{C}/[\mathcal{N}]$ is triangulated [6];
- the derived category $\mathsf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is defined to be the localization of the abelian category $\mathsf{C}(\mathcal{A})$ of complexes in $\mathcal{A}$ with respect to the subcategory of acyclic complexes.

To capture such phenomena in a more conceptual framework, the notion of *extriangulated category* was introduced by Nakaoka and Palu [10] as a simultaneous generalization of exact and triangulated categories. As a benefit of revealing an extriangulated structure, it is closed under basic categorical operations; taking extension-closed subcategories, ideal quotients by projective-injective objects and the relative theory [8]. Recently, it was shown that the extriangulated structure is still closed by certain localizations which were introduced in [11] as a unification of the Serre/Verdier quotients. Our aim is to formulate a new framework of localization of triangulated categories as an application of [11], namely, we establish an exact sequence

$$\mathcal{N} \to \mathcal{C} \to \mathcal{C}/\mathcal{N} \tag{1.1}$$

of extriangulated categories arising from the pair of a triangulated category $\mathcal{C}$ and an extension-closed subcategory $\mathcal{N}$. Precisely, the main theorem summarized as follows.

The detailed version of this paper will be submitted for publication elsewhere.

Theorem 1. [13, Thm. 2.20, 3.2, 4.2] *Let $\mathcal{C}$ be a triangulated category and regard it as a natural extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Assume that a full subcategory $\mathcal{N}$ of $\mathcal{C}$ is closed under direct summands and isomorphisms.*

(0) *If $\mathcal{N}$ is extension-closed in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, then $\mathcal{N}$ naturally defines a relative extriangulated structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$.*

(1) *If $\mathcal{N}$ is extension-closed in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, then $\mathcal{N}$ is thick with respect to the relative structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$. Moreover, we have an extriangulated localization $(Q, \mu) : (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}) \to (\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$.*

(2) *Suppose that $\mathcal{N}$ is extension-closed. Then, $\mathcal{N}$ is thick in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ if and only if $\mathcal{N}$ is biresolving with respect to the relative structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$ if and only if the resulting category $(\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ is triangulated. In this case, the localization (Q, μ) is nothing but the Verdier quotient.*

(3) *Suppose that $\mathcal{N}$ is extension-closed and functorially finite. Then, $\mathcal{N}$ satisfies* $\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$ *in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ if and only if $\mathcal{N}$ is Serre with respect to the relative structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$ if and only if the resulting category $\mathcal{C}/\mathcal{N}$ is abelian. Furthermore, the functor $Q : (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to \mathcal{C}/\mathcal{N}$ from the original triangulated category is cohomological.*

$\mathcal{N}$	*extension-closed*	*thick*	$\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$	*in* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$
	thick	*biresolving*	*Serre*	*in* $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$
$\mathcal{C}/\mathcal{N}$	*extriangulated*	*triangulated*	*abelian*	

The assertion (1) in the above theorem shows that the Verdier quotient is a typical example of the exact sequence (1.1). The assertion (2) contains some types of cohomological functors such as the heart of a t-structures, see Examples 10, 11 and 12.

Notation and convention. All categories and functors in this article are always assumed to be additive. All subcategory $\mathcal{U} \subseteq \mathcal{C}$ is always assumed to be full, additive and closed under isomorphisms. For $X \in \mathcal{C}$, if $\mathcal{C}(U, X) = 0$ for any $U \in \mathcal{U}$, we write abbreviately $\mathcal{C}(\mathcal{U}, X) = 0$. Similar notations will be used in obvious meanings.

2. Localization with respect to extension-closed subcategories

In the reset, we fix a triangulated category $\mathcal{C}$ with a suspension [1] and an extension-closed subcategory $\mathcal{N}$ of $\mathcal{C}$ and regard $\mathcal{C}$ as a natural extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Note that $\mathbb{E}(C, A) = \mathcal{C}(C, A[1])$ for any objects $A, C \in \mathcal{C}$. First, we show that $\mathcal{N}$ naturally determines an extriangulated structure on $\mathcal{C}$ relative to the triangulated structure.

Proposition 2. *For any objects $A, C \in \mathcal{C}$, we define subsets of $\mathbb{E}(C, A)$ as follows.*

(1) *A subset $\mathbb{E}^L_{\mathcal{N}}(C, A)$ is defined to be the set of morphisms $h : C \to A[1]$ satisfying the condition that, for any morphism $N \xrightarrow{x} C$ with $N \in \mathcal{N}$, $h \circ x$ factors through an object in $\mathcal{N}[1]$.*

(2) *A subset $\mathbb{E}^R_{\mathcal{N}}(C, A)$ is defined to be the set of morphisms $h : C \to A[1]$ satisfying the condition that, for any morphism $A \xrightarrow{y} N$ with $N \in \mathcal{N}$, $y \circ h[-1]$ factors through an object in $\mathcal{N}[-1]$.*

Then, both $\mathbb{E}^L_{\mathcal{N}}$ *and* $\mathbb{E}^R_{\mathcal{N}}$ *give rise to closed subfunctors of* $\mathbb{E}$ *in the sense of* [8, Prop. 3.16]. *In particular, putting* $\mathbb{E}_{\mathcal{N}} := \mathbb{E}^L_{\mathcal{N}} \cap \mathbb{E}^R_{\mathcal{N}}$, *we have three extriangulated structures*

$$(\mathcal{C}, \mathbb{E}^L_{\mathcal{N}}, \mathfrak{s}^L_{\mathcal{N}}), \quad (\mathcal{C}, \mathbb{E}^R_{\mathcal{N}}, \mathfrak{s}^R_{\mathcal{N}}), \quad (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$$

which are relative to $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. *Here* $\mathfrak{s}_{\mathcal{N}}$ *is a restriction of* $\mathfrak{s}$ *to* $\mathbb{E}_{\mathcal{N}}$ *and other symbols are used in similar meanings.*

To understand the above relative extriangulated structures, we observe the following two extremal cases.

Example 3. (1) Suppose that the subcategory $\mathcal{N}$ is *thick* in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, namely it is closed under taking cones and cocones. Then, since $\mathcal{N} = \mathcal{N}[1] = \mathcal{N}[-1]$, we have equalities $\mathbb{E}^L_{\mathcal{N}} = \mathbb{E}^R_{\mathcal{N}} = \mathbb{E}$. In particular, the relative structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$ coincides with the original triangulated structure.

(2) Suppose that the subcategory $\mathcal{N}$ is *rigid*, namely $\mathbb{E}(\mathcal{N}, \mathcal{N}) = 0$. Then, $\mathcal{N}$ forms a subcategory of projective-injective objects in $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$. Moreover, the structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$ is maximal with respect to the above property. In this case, due to [10, Prop. 3.30], the ideal quotient $\overline{\mathcal{C}} := \mathcal{C}/[\mathcal{N}]$ admits a natural extriangulated structure $(\overline{\mathcal{C}}, \overline{\mathbb{E}}_{\mathcal{N}}, \overline{\mathfrak{s}}_{\mathcal{N}})$.

Recall that a subcategory $\mathcal{N}$ of an arbitrary extriangulated category is said to be *thick* if it satisfies the 2-out-of-3 property for $\mathfrak{s}$-conflations, namely, for any $\mathfrak{s}$-conflation $A \longrightarrow B \longrightarrow C$, if two of $\{A, B, C\}$ belong to $\mathcal{N}$, so does the third[1]. It is easily checked that any extension-closed subcategory $\mathcal{N}$ of $\mathcal{C}$ becomes a thick subcategory of $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$. Keeping in mind the case of the Verdier localization, we define the class $\mathcal{S}_{\mathcal{N}}$ of morphisms which one would like to consider to be isomorphisms in the quotient category $\mathcal{C}/\mathcal{N}$.

Definition 4. For a thick subcategory $\mathcal{N}$ of an arbitrary extriangulated category, we associate the following classes of morphisms.

(1) $\mathcal{L} = \{f \in \operatorname{Mor}\mathcal{C} \mid f \text{ is an } \mathfrak{s}\text{-inflation with } \mathsf{Cone}(f) \in \mathcal{N}\}$.
(2) $\mathcal{R} = \{g \in \operatorname{Mor}\mathcal{C} \mid g \text{ is an } \mathfrak{s}\text{-deflation with } \mathsf{CoCone}(g) \in \mathcal{N}\}$.

Define $\mathcal{S}_{\mathcal{N}}$ to be the smallest subclass closed by compositions containing both $\mathcal{L}$ and $\mathcal{R}$.

For the pair of triangulated category $\mathcal{C}$ and an extension-closed subcategory $\mathcal{N}$, the above class $\mathcal{S}_{\mathcal{N}}$ possesses nice properties.

Lemma 5. *We consider the class* $\overline{\mathcal{S}_{\mathcal{N}}}$ *of morphisms* $\overline{s}$ *with* $s \in \mathcal{S}_{\mathcal{N}}$.

(1) *Let us denote by* $\overline{\mathcal{S}_{\mathcal{N}}}^{*}$ *the closure of* $\overline{\mathcal{S}_{\mathcal{N}}}$ *with respect to compositions with isomorphisms in* $\overline{\mathcal{C}}$. *Then, we have* $\overline{\mathcal{S}_{\mathcal{N}}} = \overline{\mathcal{S}_{\mathcal{N}}}^{*}$.
(2) *The class* $\mathcal{S}_{\mathcal{N}}$ *forms a* multiplicative system *in the ideal quotient* $\overline{\mathcal{C}}$. *In particular, we have the additive localization* $Q : \mathcal{C} \to \mathcal{C}/\mathcal{N} := \mathcal{C}[\mathcal{S}_{\mathcal{N}}^{-1}]$ *as follows:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad Q \quad} & \mathcal{C}/\mathcal{N} \\ & {\scriptstyle \text{ideal quot.}} \searrow \quad \nearrow {\scriptstyle \text{Localization}} & \\ & \overline{\mathcal{C}} & \end{array} \tag{2.1}$$

[1]Note that this definition is a generalization of thick subcategories of triangulated categories.

(3) *The class* $\overline{\mathcal{S}_{\mathcal{N}}}$ *is saturated in the sense that, for any morphism* $f \in \operatorname{Mor}\mathcal{C}$*, if* $Q(f)$ *is an isomorphism, then* $f \in \overline{\mathcal{S}_{\mathcal{N}}}$*.*

Theorem 1 (2) says that the multiplicative system $\overline{\mathcal{S}_{\mathcal{N}}}$ satisfies the needed *compatibility with extriangulation* (see the conditions (MR1), …, (MR4) in [11, Thm. 3.5]). In particular, the relative structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$ determines a natural extriangulated structures $(\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ on the localization $\mathcal{C}/\mathcal{N}$ which makes the natural quotient functor $Q : \mathcal{C} \to \mathcal{C}/\mathcal{N}$ exact. The construction so far is depicted below.

$$
\begin{array}{ccccc}
\underset{\text{extension-closed sub.}}{(\mathcal{N}, \mathbb{E}|_{\mathcal{N}}, \mathfrak{s}|_{\mathcal{N}})} & \xrightarrow{\mathsf{inc}} & \underset{\text{triangulated cat.}}{(\mathcal{C}, \mathbb{E}, \mathfrak{s})} & & \\
{\scriptstyle \mathsf{id}}\uparrow & & {\scriptstyle \mathsf{id}}\uparrow & & \\
\underset{\text{thick sub.}}{(\mathcal{N}, \mathbb{E}_{\mathcal{N}}|_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}|_{\mathcal{N}})} & \xrightarrow{\mathsf{inc}} & \underset{\text{extriangulated cat.}}{(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})} & \xrightarrow{Q} & \underset{\text{extriangulated cat.}}{(\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})}
\end{array}
$$

Note that the all appearing functors are *exact* in the sense in [11, Def. 2.11].

We push further an observation on what the above diagram means in Example 3.

Example 6. (1) Suppose that the subcategory $\mathcal{N}$ is thick in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Then, we get $\mathbb{E} = \mathbb{E}_{\mathcal{N}}$ and the quotient functor Q is the usual Verdier quotient.

(2) Suppose that the subcategory $\mathcal{N}$ is *rigid*, namely $\mathbb{E}(\mathcal{N}, \mathcal{N}) = 0$. Then, $\overline{\mathcal{S}_{\mathcal{N}}}$ becomes the set of isomorphisms and the quotient functor Q is nothing other than the ideal quotient $\mathcal{C} \to \mathcal{C}/\mathcal{N} = \overline{\mathcal{C}}$. The extriangulated structure $(\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ coincides with the natural one in $\overline{\mathcal{C}}$.

3. The triangulated case

As we have already seen in Example 6, if a given subcategory $\mathcal{N} \subseteq \mathcal{C}$ is thick, our extriangulated category $\mathcal{C}/\mathcal{N}$ corresponds to a triangulated category. Conversely, if the quotient $\mathcal{C}/\mathcal{N}$ is triangulated, then $\mathcal{N}$ must be thick. To sharpen this assertion, we recall that a thick subcategory $\mathcal{N}$ is said to be *biresolving* if, for any object $C \in \mathcal{C}$, there exist an $\mathfrak{s}$-inflation $C \to N$ and an $\mathfrak{s}$-deflation $N' \to C$ with $N, N' \in \mathcal{N}$

Corollary 7. *We consider the extriangulated category* $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$ *and the localization* $Q : (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}) \to (\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ *with respect to the subcategory* $\mathcal{N}$*. Then the following three conditions are equivalent.*

(i) *The extriangulated category* $(\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ *corresponds to a triangulated category.*

(ii) $\mathcal{N}$ *is a thick subcategory of the triangulated category* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$*.*

(iii) $\mathcal{N}$ *is a biresolving subcategory of the extriangulated category* $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$*.*

Under the above equivalent conditions, the localization $Q : (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}) \to (\widetilde{\mathcal{C}}_{\mathcal{N}}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ *coincides with the usual Verdier quotient.*

4. The exact case

It is natural to ask when the extriangulated category $\mathcal{C}/\mathcal{N}$ corresponds to an exact category. We denote by $\mathsf{Cone}(\mathcal{N}, \mathcal{N})$ the subcategory of $\mathcal{C}$ consisting of objects X appearing in a triangle $N' \longrightarrow N \longrightarrow X \longrightarrow N'[1]$ with $N, N' \in \mathcal{N}$. The following is an exact version of Corollary 7.

Corollary 8. *Let us consider the following conditions.*

(i) *The extriangulated category* $(\widetilde{\mathcal{C}}_{\mathcal{N}}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ *corresponds to an exact category.*
(ii) $\mathcal{N}$ *satisfies the condition* $\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$ *in the triangulated category* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$.
(iii) $\mathcal{N}$ *is a Serre subcategory of the extriangulated category* $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$.

The condition (ii) *always implies* (i) *and* (iii). *Suppose that* $\mathcal{N}$ *is functorially finite in* $\mathcal{C}$. *Then, the all conditions are equivalent.*

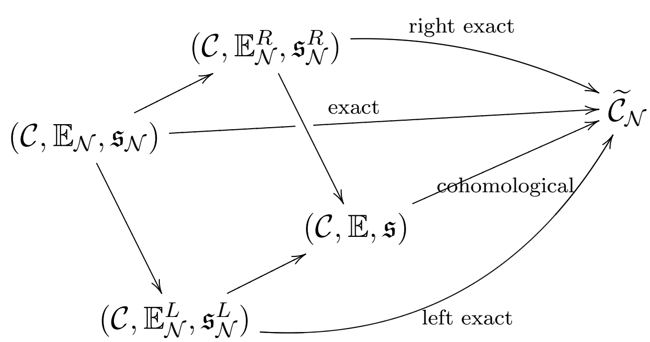
We do not know whether the functorial finiteness on $\mathcal{N}$ are really needed for the above corollary.

The following shows that $\mathcal{C}/\mathcal{N}$ is actually an abelian category under the assumption $\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$ and provide a new construction of cohomological functors.

Corollary 9. *Assume that* $\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$ *holds in the triangulated category* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. *Then, the following assertions hold.*

(1) *The resulting extriangulated category* $(\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ *corresponds to an abelian exact category.*
(2) *The exact functor* $Q : (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}) \to (\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ *induces a cohomological functor* $Q : (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to \mathcal{C}/\mathcal{N}$ *from the original triangulated category.*
(3) *The exact functor* Q *induces a right exact functor* $Q : (\mathcal{C}, \mathbb{E}^R_{\mathcal{N}}, \mathfrak{s}^R_{\mathcal{N}}) \to \mathcal{C}/\mathcal{N}$ *and a left exact functor* $Q : (\mathcal{C}, \mathbb{E}^L_{\mathcal{N}}, \mathfrak{s}^L_{\mathcal{N}}) \to \mathcal{C}/\mathcal{N}$ *in the sense of* [12, Def. 2.7].

As mentioned so far, we have half/left/right exact functors $Q : \mathcal{C} \to \mathcal{C}/\mathcal{N}$ from an extension closed subcategory $\mathcal{N}$ with $\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$ as depicted in the following commutative diagram.



Unlike the triangulated case, the above contains many important examples.

Example 10. [9] Let $\mathcal{C}$ be a triangulated category and assume that $\mathcal{U}$ is a 2-cluster tilting subcategory of $\mathcal{C}$, equivalently, $(\mathcal{U}, \mathcal{U})$ forms a cotorsion pair. Then, the ideal quotient $\mathcal{C}/[\mathcal{U}]$ is abelian and the natural functor $\pi : \mathcal{C} \to \mathcal{C}/[\mathcal{U}]$ is cohomological.

Sketch. Due to [1, Thm. 5.7], the pair $(\mathcal{U}, \mathcal{U})$ forms a cotorsion pair and we get its abelian heart $\mathcal{C}/[\mathcal{U}]$. We put $\mathcal{N} := \mathcal{U}$ and consider the relative structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$. Since $\mathcal{N}$ is rigid and $\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$, Example 6 guarantees that our quotient functor $Q : \mathcal{C} \to \mathcal{C}/\mathcal{N}$ is nothing but the ideal quotient $\mathcal{C} \to \mathcal{C}/[\mathcal{N}]$. Corollary 9(2) shows Q is cohomological. $\square$

Example 11. [2] Let $\mathcal{C}$ be a triangulated category and $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ be a t-structure of $\mathcal{C}$. Then, the subcategory $\mathcal{H} := \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$ is abelian and there exists a natural cohomological functor $H : \mathcal{C} \to \mathcal{H}$.

Sketch. Due to [1, Thm. 5.7], the pair $(\mathcal{U}, \mathcal{V}) := (\mathcal{C}^{\leq -1}, \mathcal{C}^{\geq 1})$ forms a cotorsion pair and we get its heart $\mathcal{H}$. We put $\mathcal{N} := \mathsf{add}(\mathcal{U} * \mathcal{V})$ and consider the relative structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$. Then, since $\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$ holds, by Corollary 9, we have the cohomolgical functor $Q : \mathcal{C} \to \mathcal{C}/\mathcal{N}$. By the universality, we can easily check an equivalence $\mathcal{C}/\mathcal{N} \simeq \mathcal{H}$. $\square$

Note that the general heart construction due to Abe-Nakaoka unifies the heart of a t-structure and Koenig-Zhu's abelian quotient $\mathcal{C}/[\mathcal{N}]$ as mentioned above. Abe-Nakaoka's construction can be still understood through Corollary 9. However, we skip the details. The following example can not be explained by Abe-Nakaoka's construction.

Example 12. [3, 4] Let $\mathcal{C}$ be a triangulated category and $\mathcal{U}$ a contravariantly finite rigid subcategory of $\mathcal{C}$. Then, we have a cohomological functor $H := \mathcal{C}(\mathcal{U}, -) : \mathcal{C} \to \mathsf{mod}\,\mathcal{U}$ which is factored as follows:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad H \quad} & \mathsf{mod}\,\mathcal{U} \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle \mathsf{Loc}} & \\ & \mathcal{C}/[\mathcal{U}^{\perp}] & \end{array} \tag{4.1}$$

where $\mathcal{U}^{\perp}$ denotes the subcategory of objects X in $\mathcal{C}$ with $\mathcal{C}(\mathcal{U}, X) = 0$ and Loc is a Gabriel-Zisman localization which admits left and right fractions.

Sketch. We clarify how the diagram (4.1) relates to our localization. Firstly, we put $\mathcal{N} := \mathcal{U}^{\perp}$ and note that $\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$ holds. Thus, the localization $Q : \mathcal{C} \to \mathcal{C}/\mathcal{N}$ is, by definition, factored as the ideal quotient $\pi : \mathcal{C} \to \overline{\mathcal{C}}$ followed by the localization of $\overline{\mathcal{C}}$ with respect to the multiplicative system $\overline{\mathcal{S}_{\mathcal{N}}}$ which is same as Loc in (4.1). As a bit more advantage of our results, Corollary 9 explains how the abelian exact structure on $\mathsf{mod}\,\mathcal{U} \simeq \mathcal{C}/\mathcal{N}$ inherits from the relative extriangulated structure on the triangulated category $\mathcal{C}$. Thus, their diagram (4.1) is nothing but our construction (2.1) of the quotient functor Q. $\square$

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