A NEW FRAMEWORK OF PARTIALLY ADDITIVE ALGEBRAIC GEOMETRY

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Abstract. In this article, we develop an elementary theory of partially additive rings as a foundation of $\mathbb{F}_1$-geometry. Our approach is so concrete that an analog of classical algebraic geometry is established very straightforwardly.

As applications, we construct (1) a kind of affine group scheme $\mathbb{GL}_n$ whose value at a commutative ring R is the group of $n \times n$ invertible matrices over R and at $\mathbb{F}_1$ is the n-th symmetric group, and (2) a projective space $\mathbb{P}^n$ as a kind of scheme and count the number of points of $\mathbb{P}^n(\mathbb{F}_q)$ for $q = 1$ or $q = p^n$ a power of a rational prime.

Key Words: Partially additive ring, Field with one element, Affine group scheme.

2020 *Mathematics Subject Classification*: Primary 14A23; Secondary 14L17.

1. Introduction

In his 1957 paper [5], J.Tits observed that the correspondence of geometries over a field k and the Chevalley groups over k developed in that paper specializes, when k is the hypothetical "field of characteristic one", to the correspondence of finite complexes and the Weyl groups of those Chevalley groups. In the early 1990s, Manin, based on the 'beautiful ideas of Deninger and Kurokawa', proposed to use this hypothetical field of characteristic one to solve the Riemann hypothesis. Since then, there have been many attempts to establish a foundation for these ideas, but it seems that the project has not been settled down yet.

In this paper, we develop an elementary theory of partially additive rings and based on it, we also develop a kind of scheme theory. As such, Deitmar's $\mathbb{F}_1$-schemes [2] and Lorscheid's blue schemes [3] are relevant to our theory. Deitmar's theory of $\mathbb{F}_1$-schemes is a direct analog of the classical scheme theory, where a commutative (multiplicative) monoid is used in place of a commutative ring. Lorscheid's theory of blue schemes is based on the theory of blueprints. A blueprint is a generalization of a commutative ring with identity in which addition is replaced by a congruence on a semiring $\mathbb{N}[A]$, the monoid-semiring of a commutative monoid A. A Partially additive ring, defined in this paper (and in [4]), is a special case of a blueprint developed in [3]. It is a part of blueprint which is the direct partially-additive analog of the non-additive setting of Deitmar and it can be thought of as an interpolation between commutative rings and commutative monoids. It is so concrete that an analog of classical algebraic geometry is established very straightforwardly. As applications, we construct a kind of affine group scheme $\mathbb{GL}_n$ whose value at a commutative ring R is the group of $n \times n$ invertible matrices over R and at $\mathbb{F}_1$ is the n-th symmetric group, and we construct a projective space $\mathbb{P}^n$ as a kind

The detailed version of this paper has been submitted to mathematics arXiv [4].

of scheme and count the number of points of $\mathbb{P}^n(\mathbb{F}_q)$ for $q = 1$ or $q = p^n$ a power of a rational prime.

2. Partially additive algebra

In this section we summarize definitions of the notions in partially additive algebra. The definition of a partial monoid given below is due to G.Segal.

Definition 1. A **partial monoid** is a set A, a distinguished element $0 \in A$, a subset A_2 of $A \times A$ and a map $+\colon A_2 \to A$ such that

(1) $A \times \{0\} \cup \{0\} \subseteq A_2$ and $a + 0 = a = 0 + a$, for any $a \in A$.
(2) $(a, b) \in A_2$ if $(b, a) \in A_2$ and $a + b = b + a$, for any $a, b \in A$.
(3) $(a, b) \in A_2$ and $(a + b, c) \in A_2$ if and only if $(b, c) \in A_2$ and $(a, b + c) \in A_2$, for any $a, b, c \in A$.

For example, any based set is considered as a partial monoid by giving it a trivial structure — only the base point 0 can be added to other elements. For another example, a commutative monoid (thus a commutative group) is a partial monoid where A_2 is taken to be the whole set $A \times A$.

Definition 2. Let A, B be partial monoids. A map $f\colon A \to B$ is a **homomorphism** of partial monoids if

(1) $f(0) = 0$ and
(2) for all $a, b \in A$, $(a, b) \in A_2$ implies $(f(a), f(b)) \in B_2$ and $f(a + b) = f(a) + f(b)$.

Definition 3. A **partial ring** is a partial monoid with a bilinear, associative and commutative product $\cdot\colon A \times A \to A$ and a multiplicative identity $1 \in A$. More explicitly, a partial ring is a partial monoid with a product $\cdot\colon A \times A \to A$ such that, for all $a, b, c \in A$,

(1) $0 \cdot a = 0$,
(2) if $(a, b) \in A_2$ then $(a \cdot c, b \cdot c) \in A_2$ and $(a + b) \cdot c = a \cdot b + a \cdot c$,
(3) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
(4) $a \cdot b = b \cdot a$, and
(5) $1 \cdot a = a$.

If every element except for 0 of a partial ring is invertible, we call it a **partial field**.

For example, any commutative monoid with an absorbing element 0 is a partial ring in which only 0 can be added to other elements, and any commutative semiring (thus a commutative ring) with identity is a partial ring. A commutative group with an absorbing element 0 adjoined and a field are examples of a partial field.

Definition 4. Let A and B be partial rings. A **homomorphism** of $A \to B$ is a homomorphism of partial monoids $f\colon A \to B$ which is compatible with the multiplications in A and B.

For example, if A and B are commutative monoids with an absorbing element, considered as partial rings, then a homomorphism $A \to B$ as partial rings is nothing but a homomorphism $A \to B$ as commutative monoids. For another example, if A and B are commutative rings with identity considered as partial rings, then a homomorphism

$A \to B$ as partial rings is nothing but a homomorphism $A \to B$ as commutative rings. Thus we have full embeddings of categories $\mathcal{CM}on_0 \to \mathcal{PR}ing$ and $\mathcal{CR}ing \to \mathcal{PR}ing$, where $\mathcal{CM}on_0, \mathcal{PR}ing$ and $\mathcal{CR}ing$ denote the category of commutative monoids with absorbing element, of partial rings and of commutative rings with identity. In the rest of this paper, a commutative monoid with absorbing element and a commutative ring with identity are considered as a partial ring via this embedding.

In the rest of this paper, A is a partial ring.

Definition 5. An **A-module** is a partial monoid M with an action of A which is bilinear, associative and unital. More explicitly, an A-module is a partial monoid M and a map $\cdot\colon A \times M \to M$ such that

(1) for any $a \in A$, the map $M \to M$ given by $m \mapsto a \cdot m$ is a homomorphism of partial monoids,
(2) for any $m \in M$, the map $A \to M$ given by $a \mapsto a \cdot m$ is a homomorphism of partial monoids,
(3) $(a \cdot b) \cdot m = a \cdot (b \cdot m)$ for any $a, b \in A$ and $m \in M$,
(4) $1 \cdot m = m$ for any $m \in M$.

For example, a based set M with an action of a commutative monoid A on it is an A-module. For another example, an A-module in the usual sense, where A is a commutative ring, is an A-module in our sense.

Definition 6. Let M and N be A-modules. A **homomorphism** of A-modules is a homomorphism of partial monoids $f\colon M \to N$ which is compatible with the actions of A on M and N.

Let M be an A-module and $S \subseteq A$ be a multiplicative subset. As usual, let $\frac{m}{s}$ denote the equivalence class of $(s, m) \in S \times M$ under the equivalence relation $(s, m) \sim (t, n) \iff \exists u \in S$ s.t. $usn = utm$. If we put

$$S^{-1}M = \left\{ \frac{m}{s} \,\middle|\, m \in M, s \in S \right\}, \text{ and}$$
$$(S^{-1}M)_2 = \left\{ \left(\frac{m}{s}, \frac{n}{s} \right) \,\middle|\, (m, n) \in M_2 \right\},$$

then $S^{-1}M$ is an A-module in a natural manner. (The definition of $(S^{-1}M)_2$ may seem too easy at first look, but it can do, since we can make a common denominator.)

Definition 7. An A-submodule of A is called an **ideal** of A. An ideal I is a **prime ideal** if $A \setminus I$ is multiplicatively closed.

3. Examples

In this section, partial rings and A-modules of major interests are listed.

Partial rings.

- Partial rings of order 2
  If $A = \{0, 1\}$ is a partial ring, operations other than $1 + 1$ are determined by the axioms. There are three possibilities of $1+1$, namely, $1+1 = 0, 1$ and 'undefined'. If $1 + 1 = 0, A = \mathbb{F}_2$, the field of two elements. If $1 + 1 = 1, A = \mathbb{B}$, a "Boolean"

semiring. If $1+1$ is undefined, we denote this partial ring by $\mathbb{F}_1$. All these three are partial fields.

- Let $\langle x_1, \ldots, x_n \rangle$ be the commutative monoid generated freely by n indeterminates $x_1, \ldots, x_n$ with an absorbing element 0 adjoined.
- Let S be a set of elements of $\mathbb{N}[x_1, \ldots, x_n]$. We denote by $\mathbb{F}_1\langle x_1, \ldots, x_n | S \rangle$ the smallest partial subring of $\mathbb{N}[x_1, \ldots, x_n]$ which contains $\langle x_1, \ldots, x_n \rangle$ and any subsum of an element of S can be calculated in it.
- As a special case of the previous example, we consider the case where S consists of a single element $x_1 + \cdots + x_n$. In this case, we can show that

$$\mathbb{F}_1\langle x_1, \ldots, x_n | x_1 + \cdots + x_n \rangle = \{ \text{ subsum of } (x_1 + \cdots + x_n)^r \mid r \in \mathbb{N} \}$$

and any two elements $\alpha, \beta \in \mathbb{F}_1\langle x_1, \ldots, x_n | x_1 + \cdots + x_n \rangle$ are summable in this partial ring if the sum $\alpha + \beta$ taken in $\mathbb{N}[x_1, \ldots, x_n]$ is contained in $\mathbb{F}_1\langle x_1, \ldots, x_n | x_1 + \cdots + x_n \rangle$. This partial ring represents the summable n-tupples, as there exists an isomorphism of A-modules

$$\mathrm{Hom}_{\mathcal{PR}ing}(F_1\langle x_1, \ldots, x_n | x_1 + \cdots + x_n \rangle, A) \cong A_n$$

for any partial ring A.of partial rings.

<u>A-modules</u>.

- Direct product

$$A^n = \{(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in A\},$$
$$(A^n)_2 = \{(a_1, \ldots, a_n; b_1, \ldots, b_n) \mid (a_i, b_i) \in A_2, \forall i\}.$$

- Summable n-tuples

$$A_n = \{(a_1, \ldots, a_n) \mid a_1 + \cdots + a_n \text{ can be calculated in } A\},$$
$$(A_n)_2 = \left\{ (a_1, \ldots, a_n; b_1, \ldots, b_n) \;\middle|\; \sum_{i=1}^{n} (a_i + b_i) \text{ can be calculated in } A \right\}.$$

- "Hyper summable" n-tuples

$$A_{(m)} = \{(a_1, \ldots, a_m) \in A^m \mid (c_1 a_1, \ldots, c_m a_m) \in A_m \, \forall c_i \in A\},$$
$$(A_{(m)})_2 = \{(a_1, \ldots, a_m; b_1, \ldots, b_n) \mid (a_1 + b_1, \ldots, a_n + b_n) \in A_{(m)}\}.$$

4. Congruences

In this section, we summarize some facts about congruences on a partial ring. Congruence is one of the main points where the analogy between partial algebras and classical commutative algebras does not go smoothly. This inconsistency is explained simply by stating the fact that congruences on a partial ring does not correspond bijectively with ideals in that partial ring. But since the congruence is one of the main tools to construct a new partial algebra from another, we need to establish the theory of congruences on a partial ring.

Recall from [1] that an **equivalence relation** on an object X of a category $\mathcal{C}$ with small limits is a subobject R of $X \times X$ such that the injection $\mathrm{Hom}_{\mathcal{C}}(C, R) \to \mathrm{Hom}_{\mathcal{C}}(C, X) \times \mathrm{Hom}_{\mathcal{C}}(C, X)$ induced by the monomorphism $R \to X \times X$ gives rise to an equivalence relation on the set $\mathrm{Hom}_{\mathcal{C}}(C, X)$ for all object C of $\mathcal{C}$. An equivalence relation R is called **effective** if it is a kernel pair of a morphism in $\mathcal{C}$. In this paper the word **congruence** will be used as a synonym of effective equivalence relation.

If $\mathcal{C} = \mathcal{CR}ing$, the following are true:

(1) Every equivalence relation is a congruence.
(2) A congruence R on a ring A gives rise to an ideal $I(R) = \{a - b \mid (a, b) \in C\}$ of A and, conversely, an ideal J of A gives rise to a congruence $C(J) = \{(a, b) \mid a - b \in J\}$ on A. This establishes a bijective correspondence between congruences on A and ideals of A.

On the other hand, if $\mathcal{C} = \mathcal{PR}ing$, the following are true:

(1) An equivalence relation R is a congruence if and only if $R_2 = (R \times R) \cap (A \times A)_2$.
(2) For a congruence R on a ring A, let $I(R)$ be the ideal of A defined by

$$I(R) = \{a \in A \mid (a, 0) \in R\}.$$

Conversely, for an ideal J of A, let $C(J)$ be the smallest congruence on A which contains $J \times J$. This establishes a two way correspondence

$$C\colon (\text{ideals of } A) \rightleftarrows (\text{congruences on } A) : I$$

We have $CI(R) \subseteq R$ and $J \subseteq IC(J)$ for any congruence R on A and for any ideal J of A. Thus we have a bijective correspondence between the congruences of the form $C(J)$ and the ideals of the form $I(R)$.

5. Partial schemes

Let X be the set of the prime ideals of A. For any $a \in A$, let $D(a)$ denote the set of prime ideals of A which does not contain a. Since $D(a) \cap D(b) = D(ab)$ for any $a, b \in A$, $D(a)$'s for all $a \in A$ constitute a base for a topology on X_A, with which we make X_A a topological space.

For any open set $U \subseteq X_A$, we put $S_U = \{s \in A | s \notin P, \forall P \in U\}$ and $\mathcal{O}'_A(U) = S_U^{-1}A$. Then $\mathcal{O}'_A(U)$ is a presheaf of partial rings on X_A. Let $\mathcal{O}_A$ denote the sheafification of $\mathcal{O}'_A$. We put $\mathrm{Spec}A = (X_A, \mathcal{O}_A)$.

Definition 8. An **affine partial scheme** is a partial-ringed space $(X, \mathcal{O}_X)$ which is isomorphic to $\mathrm{Spec}A$ for some partial ring A. A **partial scheme** is a partial-ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine partial scheme.

Proposition 9. *Let* $\mathit{Spec}\, A = (X, \mathcal{O}_X)$.

(1) X *is quasi-compact.*
(2) *We have a natural monomorphism* $A \to \mathcal{O}_X(X)$.
(3) *If* A *is a partial field,* $A \cong \mathcal{O}_X(X)$.

For a proof, see [4].

6. Projective space $\mathbb{P}^n$ as a partial scheme

Let B be a partial ring given by $B = \mathbb{F}_1\langle y_0, \dots, y_n \mid y_0 + \cdots + y_n\rangle$. Let A_i be the 0-th part of the localization of B by the multiplicative set $\{x_i^r \mid r \in \mathbb{N}\}$. It is readily seen that A_i consists of subsums of $\left(\frac{y_1}{y_i} + \cdots + \frac{y_n}{y_i}\right)^r_.$ If $A_{i,j}$ denotes the localization of A_i by the multiplicative set $\left\{\left(\frac{x_j}{x_i}\right)^r \mid r \in \mathbb{N}\right\}_,$ we have that $A_{i,j} = A_{j,i}$. This allows us to patch affine pieces $X_i = \mathrm{Spec} A_i$ together to get a partial scheme, which we denote by $\mathbb{P}^n_B$. If $\{i_1, \dots, i_k\} \subseteq \{0, 1, \dots, n\}$ is a k-element set, let $A_{i_1,\dots,i_k}$ denote the 0-th part of the localization of B by the multiplicative set generated by $x_{i_1}, \dots, x_{i_k}$. Then we have an isomorphism $X_{i_1} \cap \cdots \cap X_{i_k} \cong \mathrm{Spec} A_{i_1,\dots,i_k}$. Let F be a partial field. Assume that there exists a finite subset E of F such that for any $a_1, \dots, a_r \in E, (1, a_1, \dots, a_r) \in F_{r+1}$ and elements in $F \setminus E$ is not summable with 1. If the cardinal of E is $\kappa(F)$, we have that the number of F-valued points of $X_{i_1} \cap \cdots \cap X_{i_k}$, *i.e.* that of homomorphisms from $A_{i_1,\dots,i_k}$ to F is $(\kappa(F) - 1)^k \kappa(F)^{n-k}$. So we can calculate the number of F-valued points of $\mathbb{P}^n_B$ as

$$\begin{aligned}
\#\mathbb{P}^n_B(F) &= \sum_{k=0}^{n} (-1)^k \binom{n+1}{k} (\kappa(F) - 1)^k \kappa(F)^{n-k} \\
&= \begin{cases} \kappa(F)^n + \cdots + \kappa(F) + 1 & \text{if } \kappa(F) \neq 1 \\ (n+1)\kappa(F)^n = n + 1 & \text{if } \kappa(F) = 1. \end{cases}
\end{aligned}$$

Since $\kappa(\mathbb{F}_q) = q$ including the case $q = 1$, we have that

$$\#\mathbb{P}^n_B(\mathbb{F}_q) = \begin{cases} q^n + \cdots + q + 1 & \text{if } q = p^d \text{ is a power of a prime } p \\ n + 1 & \text{if } q = 1. \end{cases}$$

Since $n + 1$ is the number of vertices of an n-simplex, this result can be thought of as a supportive evidence for a part of the conjecture of Tits [5].

7. Linear algebra of A-modules

Let $\varphi \colon A^m \to A^n$ be a homomorphism of A-modules. If e_j $(1 \leq j \leq m)$ and f_i $(1 \leq i \leq n)$ denote the elementary vectors of M and N respectively, then φ determines an $n \times m$ matrix $\alpha = (a_{ij})$, where $\varphi(e_j) = \sum_{i=1}^n a_{ij} f_i$. For any $(c_1, \dots, c_m) \in A^m$, we have $(c_1 e_1, \dots, c_m e_m) \in (A^m)_m$. Since φ is a homomorphism, this implies that $(c_1\varphi(e_1), \dots, c_m\varphi(e_m)) \in (A^n)_m$. This implies that $(c_1 a_{i1}, \dots c_m a_{im}) \in A_2$ for all $i = 1, \dots, n$, so that for a matrix $\alpha = (a_{ij})$ determined by a homomorphism $A^m \to A^n$, we have $(a_{i1}, \dots, a_{im}) \in A_{(m)}$ for all $i = 1, \dots, n$. Conversely, a matrix α with this property determines a homomorphism $A^m \to A^n$ in an obvious way.

Proposition 10. *There exists a bijection between the set of homomorphisms $A^m \to A^n$ and the set of $n \times m$ matrices whose rows are in $A_{(m)}$. $A_{(m)}$ is naturally isomorphic to the A-module dual of A^m.*

Let $M_{n,m}(A)$ denote the set of $n \times m$ matrices whose rows are in $A_{(m)}$. Unfortunately, $M_{n,m}(A)$ is not a functor of A, since the correspondence $A \mapsto A_{(n)}$ is not functorial. We remedy this defect by considering A_n as a functorial approximation to $A_{(n)}$ and use A_n in place of $A_{(n)}$. A supporting evidence that we can use A_n as an approximation to

$A_{(n)}$ is that these two A-modules coincide for an important class of partial rings such as commutative monoids with absorbing element and commutative rings with identity. This observation leads to the following definition:

In this paper we say that a partial ring A **is good** if $A_n = A_{(n)}$.

Now, let $M'_{n,m}(A)$ denote the set of $n \times m$ matrices whose rows are in A_m. Since an element of $M'_{n,m}(A)$ is not "genuine" matrices, *i.e.* does not correspond to an A-module homomorphism, $M'_n(A) = M'_{n,n}(A)$ is only a non-commutative partial magma, while $M_n(A) = M_{n,n}(A)$ is a (genuine) monoid by the usual matrix product. Similarly, if we put $GL'_n(A) = GL_n(A) \cap M'_n(A)$, where $GL_n(A)$ is the group of invertible matrices in $M_n(A)$, then $GL'_n(A)$ is only a non-commutative partial group. A definition of partial group is given in [4].

8. Main result

Let $\mathcal{PG}rp$ and $\mathcal{G}rp$ denote the category of partial groups and groups, respectively. We will give a definition of a good partial ring in the talk. Commutative monoids with absorbing element 0 and commutative rings with identity are examples of good partial rings.

Theorem 11. *There exists a representable functor* $\mathbb{GL}'_n \colon \mathcal{PR}ing \to \mathcal{PG}rp$ *which enjoys the following properties:*

1. *its restriction to the category of good partial rings factors as* $\iota \circ \mathbb{GL}_n$*, where* ι *is the canonical inclusion* $\mathcal{G}rp \to \mathcal{PG}rp$*.*
2. *If* A *is a commutative rings with identity, then* $\mathbb{GL}'_n(A) = \mathbb{GL}_n(A)$ *is the group of* n*-th general linear group with entries in* A*, and*
3. $\mathbb{GL}'_n(\mathbb{F}_1) = \mathbb{GL}_n(\mathbb{F}_1) = \mathfrak{S}_n$ *is the* n*-th symmetric group.*

Proof. Let $\mathbb{N}[x_{ij}, y_{ij}\,(1 \le i,j \le n)]$ be the semiring of polynomials of $2n^2$ indeterminates $x_{ij}, y_{ij}\,(1 \le i,j \le n)$. Consider $n \times n$ matrices $X = (x_{ij}), Y = (y_{ij}), Z = XY = (z_{ij})$ and $W = YX = (w_{ij})$. Let K be the subset of $\mathbb{N}[x_{ij}, y_{ij}\,(1 \le i,j \le n)]$ consisting of $4n$ elements

$$\begin{aligned}
&x_i = x_{i1} + \cdots + x_{in}\,(1 \le i \le n),\\
&y_i = y_{i1} + \cdots + y_{in}\,(1 \le i \le n),\\
&z_i = z_{i1} + \cdots + z_{in}\,(1 \le i \le n) \text{ and}\\
&w_i = w_{i1} + \cdots + w_{in}\,(1 \le i \le n).
\end{aligned}$$

We put $G' = \mathbb{F}_1\langle x_{ij}, y_{ij}\,(1 \le i,j \le n) | K\rangle$. Let Q be the smallest congruence on G' which contains $2n^2$ pairs (z_{ij}, δ_{ij}) and $(w_{ij}, \delta_{ij})\,(1 \le i,j \le n)$. Then we put $G = G'/Q$.

Next, let $N = \mathbb{N}[x_{ij}, y_{ij}, x'_{ij}, y'_{ij}\,(1 \le i,j \le n)]$ be the semiring of polynomials of $4n^2$ indeterminates $x_{ij}, y_{ij}, x'_{ij}, y'_{ij}\,(1 \le i,j \le n)$. Consider $n \times n$ matrices

$$\begin{aligned}
&X = (x_{ij}), Y = (y_{ij}), Z = XY = (z_{ij}), W = YX = (w_{ij}),\\
&X' = (x'_{ij}), Y' = (y'_{ij}), Z' = X'Y' = (z'_{ij}), W' = Y'X' = (w'_{ij}),\\
&S = XX' = (s_{ij}), T = Y'Y = (t_{ij}), U = ST = (u_{ij}), V = TS = (v_{ij}).
\end{aligned}$$

We put

$$L = \{x_i, y_i, z_i, w_i, x'_i, y'_i, z'_i, w'_i, s_i, t_i, u_i, v_i | 1 \le i \le n\},$$

where $*_i$ denotes the sum of i-th row of a matrix indicated by the capital of the same letter $*$. We put $H' = \mathbb{F}_1\langle x_{ij}, y_{ij}, x'_{ij}, y'_{ij}\,(1 \leq i, j \leq n)|L\rangle$. Let R be the smallest congruence on H' which contains $6n^2$ pairs (z_{ij}, δ_{ij}), (w_{ij}, δ_{ij}), (z'_{ij}, δ_{ij}), (w'_{ij}, δ_{ij}), (u_{ij}, δ_{ij}) and $(v_{ij}, \delta_{ij})\,(1 \leq i, j \leq n)$. Then we put $H = H'/R$. A partial cogroup structure on G is given by a series of partial ring homomorphisms, for the details of which we refer the reader to [4]. $\square$

Remark 12. Above theorem suggests that analogies between the symmetric group and the general linear group can be unified to a single statement about a single object $\mathbb{GL}_n$. It also shows that if we have an intermediate good partial ring A between $\mathbb{F}_1$ and $\mathbb{F}_q$, where $q = p^d$ is a power of a prime p, there exists an intermidiate group $\mathbb{GL}_n(A)$. It is proved that commutative submonoids with absorbing element and subfields of $\mathbb{F}_q$ exhausts the intermediate partial rings between $\mathbb{F}_1$ and $\mathbb{F}_q$. Anyway, a more detailed investigation is needed in this direction.

References

[1] F. Borceux, *Handbook of categorical algebra. 2*, Encyclopedia of Mathematics and its Applications, 51, Cambridge University Press, Cambridge, (1994).

[2] A. Deitmar, Anton, *Schemes over* $\mathbb{F}_1$, Number fields and function fields—two parallel worlds, Progr. Math., 239 (2005), 87–100.

[3] O. Lorscheid, *The geometry of blueprints: Part I: Algebraic background and scheme theory*, Advances in Mathematics (2012), 1804–1846.

[4] S. Okuyama, *Partially additive rings and group schemes over* $\mathbb{F}_1$, arXiv e-prints (2022), arXiv:2206.06084.

[5] J. Tits, *Sur les analogues algébriques des groupes semi-simples complexes*, Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956 (1957), 261–289.

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