HIGHER VERSIONS OF MORPHISMS REPRESENTED BY MONOMORPHISMS

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ABSTRACT. In this article, we introduce and study a new class of morphisms which includes morphisms represented by monomorphisms in the sense of Auslander and Bridger. As an application, we give a common generalization of several results due to Auslander and Bridger that describe relationships between torsionfreeness and the grades of Ext modules.

Key Words: morphism represented by monomorphisms, n-torsionfree module, syzygy, (Auslander) transpose, grade.

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1. INTRODUCTION

Throughout this article, let R be a two-sided noetherian ring. We assume that all modules are finitely generated right ones. It is a natural and classical question to ask when a given homomorphism of R-modules is stably equivalent to another homomorphism satisfying certain good properties. A well-studied one is about stable equivalence to a monomorphism: A homomorphism $f : X \to Y$ of R-modules is said to be *represented by monomorphisms* if there is an R-homomorphism $t : X \to P$ with P projective such that $\binom{f}{t} : X \to Y \oplus P$ is a monomorphism. This notion has been introduced by Auslander and Bridger [1], and later studied in detail by Kato [3]. Among other things, Kato gave the following characterization; we denote by $\operatorname{Tr}(-)$ the (Auslander) transpose.

Theorem 1 (Kato)**.** *Let $f : X \to Y$ be an R-homomorphism. Then the following are equivalent.*

(1) *The morphism f is represented by monomorphisms.*
(2) *The map $\operatorname{Ext}^1_{R^{\mathrm{op}}}(\operatorname{Tr} f, R) : \operatorname{Ext}^1_{R^{\mathrm{op}}}(\operatorname{Tr} X, R) \to \operatorname{Ext}^1_{R^{\mathrm{op}}}(\operatorname{Tr} Y, R)$ is injective.*

Motivated by the above theorem, we define a condition which we call (T_n) for each integer $n \geq 0$ so that (T_1) is equivalent to being represented by monomorphisms, and find out several properties.

The notion of n-torsionfree modules was also introduced by Auslander and Bridger [1]: An R-module M is called *n-torsionfree* if $\operatorname{Ext}^i_{R^{\mathrm{op}}}(\operatorname{Tr} M, R) = 0$ for all $1 \leq i \leq n$. Auslander and Bridger found various important properties related to n-torsionfree modules. For example, for an R-module M, Auslander and Bridger figured out the relationship between the grade of the Ext module $\operatorname{Ext}^i_R(M, R)$ and the torsionfreeness of the syzygy $\Omega^i M$. This result has been playing an important role in studies on n-torsionfree modules. As an application of a result stated in Section 2, we give a higher version of Auslander and

The detailed version [5] of this article has been submitted for publication elsewhere.

Bridger's theorem in Section 3. It gives a common generalization of [1, Propositions 2.26, 2.28 and Corollary 2.32].

2. The condition (T_n)

We begin with introducing the new condition (T_n) for R-homomorphisms. It is a natural extension of the condition (2) in Theorem 1.

Definition 2. Let $n \geq 1$ be an integer. We say that a homomorphism $f : X \to Y$ of R-modules satisfies (T_n) if the map $\operatorname{Ext}^i_{R^{\mathrm{op}}}(\operatorname{Tr} f, R)$ is bijective for all $1 \leq i \leq n-1$ and $\operatorname{Ext}^n_{R^{\mathrm{op}}}(\operatorname{Tr} f, R)$ is injective. In addition, we provide that every R-homomorphism satisfies (T_0).

In order to describe several properties related to the condition (T_n), we use the following terminology.

Definition 3. [3, Definition and Lemma 2.11] Let $f : X \to Y$ be a homomorphism of R-modules. Let $s : P \to Y$ be an epimorphism with P projective. The module $\underline{\operatorname{Ker}} f$ is defined as $\operatorname{Ker}((f, s) : X \oplus P \to Y)$. The module $\underline{\operatorname{Ker}} f$ is uniquely determined by f up to projective summands.

For an R-homomorphism f, we denote by $\operatorname{Cok} f$ the cokernel of f. The kernel and the cokernel of the homomorphim f are related to the module $\underline{\operatorname{Ker}} f$ as follows.

Lemma 4. [3, Lemma 2.17, Theorem 4.12] *Let $f : X \to Y$ be a homomorphism of R-modules. Let $t : Q \to Y$ be an epimorphism with Q projective. Then there exists an exact sequence*

$$0 \to \operatorname{Ker} f \to \underline{\operatorname{Ker}} f \to Q \to \operatorname{Cok} f \to 0.$$

Let M be an R-module. The *grade* of M, which is denoted by $\operatorname{grade}_R M$, is defined to be the infimum of integers i such that $\operatorname{Ext}^i_R(M, R) = 0$. The relationship between the grades of Ext modules and the torsionfreeness of modules has been actively studied; the works of Auslander and Bridger [1] and Auslander and Reiten [2] are among the most celebrated studies. The following theorem is the first main theorem of this article, which interprets the condition (T_n) in terms of grades and torsionfreeness.

Theorem 5. *Let $n \geq 1$ be an integer. Consider the following conditions for an R-homomorphism $f : X \to Y$.*

- (a_n) *The homomorphism f satisfies the condition (T_n).*
- (b_n) *The R-module $\underline{\operatorname{Ker}} f$ is n-torsionfree.*
- (c_n) *There is an inequality $\operatorname{grade}_{R^{\mathrm{op}}} \operatorname{Ker} \operatorname{Ext}^1_R(f, R) \geq n$.*

Then the following implications hold.

$$(a_n) \wedge (b_{n+1}) \Longrightarrow (c_n), \qquad (b_n) \wedge (c_n) \Longrightarrow (a_n), \qquad (a_n) \wedge (c_{n-1}) \Longrightarrow (b_n).$$

Let us consider an application of the above theorem. The following corollary is none other than [3, Theorem 4.2], which gives a simple characterization of the morphisms represented by monomorphisms when R is commutative and generically Gorenstein (e.g., when R is an integral domain). We can deduce it from Theorem 5.

Corollary 6 (Kato)**.** *Suppose that R is commutative and the total ring $\mathrm{Q}(R)$ of fractions of R is Gorenstein. Let $f : X \to Y$ be a homomorphism of R-modules. Then f is represented by monomorphisms if and only if $\operatorname{Ker} f$ is torsionless.*

Proof. Since $\mathrm{Q}(R)$ is Gorenstein, the torsionless property is closed under extensions; see [4, Theorem 2.3] for instance. Hence, by Lemma 4, $\operatorname{Ker} f$ is torsionless if and only if so is $\underline{\operatorname{Ker}} f$. Suppose that $\underline{\operatorname{Ker}} f$ is torsionless. By [1, Proposition 4.21], we have $\operatorname{grade} \operatorname{Ker} \operatorname{Ext}^1(f, R) \geq 1$. It follows from Theorem 5 that f is represented by monomorphisms. □

3. Grade inequalities of Ext modules

In this section, as an application of Theorem 5, we consider the grades of Ext modules. Let M be an R-module and $n \geq 1$ an integer. Auslander and Bridger [1] state and prove a criterion for $\Omega^i M$ to be i-torsionfree for $1 \leq i \leq n$. By using Theorem 5, we can recover [1, Proposition 2.26], which is the most fundamental theorem in studies on n-torsionfree modules.

Corollary 7 (Auslander–Bridger)**.** *Let $n \geq 1$ be an integer and M an R-module. The following are equivalent.*

(1) *The inequality $\operatorname{grade}_{R^{\mathrm{op}}} \operatorname{Ext}_R^i(M, R) \geq i - 1$ holds for all $1 \leq i \leq n$.*
(2) *The syzygy $\Omega^i M$ is i-torsionfree for all $1 \leq i \leq n$.*

Proof. We use induction on n. The assertion is trivial for $n = 1$. Let $n > 1$. Assume that (1) or (2) holds. Then $\Omega^i M$ is i-torsionfree for all $1 \leq i \leq n-1$ by the induction hypothesis. Let $f : P \to \Omega^{n-1} M$ be an epimorphism with P projective. Then f satisfies (T_n). It follows from Theorem 5 that $\operatorname{grade} \operatorname{Ker} \operatorname{Ext}^1(f, R) \geq n - 1$ if and only if $\underline{\operatorname{Ker}} f$ is n-torsionfree. As $\operatorname{Ker} \operatorname{Ext}^1(f, R) \cong \operatorname{Ext}^1(\Omega^{n-1} M, R) \cong \operatorname{Ext}^n(M, R)$ and $\underline{\operatorname{Ker}} f \cong \Omega^n M$, we have the desired result. □

The results [1, Proposition 2.26 and Corollary 2.32] describe the relationship between the grades of Ext modules and the torsionfreeness of syzygy modules, and the relationship of them with the natural map $\psi_M^i : \operatorname{Tr} \Omega^i \operatorname{Tr} \Omega^i M \to M$ being represented by monomorphisms. The main result of this section is the following theorem, which gives a common generalization of [1, Propositions 2.26, 2.28 and Corollary 2.32].

Theorem 8. *Let $n \geq 1$ and $j \geq 0$ be integers and M an R-module. The following are equivalent.*

(1) *The inequality $\operatorname{grade}_{R^{\mathrm{op}}} \operatorname{Ext}_R^i(M, R) \geq i + j - 1$ holds for all $1 \leq i \leq n$.*
(2) *The syzygy $\Omega^i M$ is i-torsionfree and the natural map $\psi_M^i : \operatorname{Tr} \Omega^i \operatorname{Tr} \Omega^i M \to M$ satisfies the condition (T_j) for all $1 \leq i \leq n$.*

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