THE REDUCTION NUMBER OF STRETCHED IDEALS

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Abstract. The homological property of the associated graded ring of an ideal is an important problem in commutative algebra. In this report we explore the structure of the associated graded ring of stretched $\mathfrak{m}$-primary ideals in the case where the reduction number attains almost minimal value in a Cohen-Macaulay local ring $(A, \mathfrak{m})$. As an application, we present complete descriptions of the associated graded ring of stretched $\mathfrak{m}$-primary ideals with small reduction number.

Key Words: commutative ring, stretched local ring, stretched ideal, Cohen-Macaulay local ring, associated graded ring, Hilbert function, Hilbert coefficient.

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1. Introduction

Throughout this report, let A be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. For simplicity, we may assume the residue class field $A/\mathfrak{m}$ is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and let

$$R = \mathrm{R}(I) := A[It] \quad \subseteq A[t] \text{ and } \quad R' = \mathrm{R}'(I) := A[It, t^{-1}] \quad \subseteq A[t, t^{-1}]$$

denote, respectively, the Rees algebra and the extended Rees algebra of I. Let

$$G = \mathrm{G}(I) := R'/t^{-1}R' \cong \bigoplus_{n \geq 0} I^n/I^{n+1}$$

denotes the associated graded ring of I. Let $\ell_A(N)$ denotes, for an A-module N, the length of N.

Let $Q = (a_1, a_2, \cdots, a_d) \subseteq I$ be a parameter ideal in A which forms a reduction of I. We set

$$n_I = n_Q(I) := \min\{n \geq 0 \mid I^{n+1} \subseteq Q\} \quad \text{and} \quad r_I = r_Q(I) := \min\{n \geq 0 \mid I^{n+1} = QI^n\},$$

respectively, denote the index of nilpotency and the reduction number of I with respect to Q. Then it is easy to see that the inequality $r_I \geq n_I$ always holds true.

The notion of *stretched* Cohen-Macaulay local rings was introduced by J. Sally. We say that the ring A is *stretched* if $\ell_A(Q + \mathfrak{m}^2/Q + \mathfrak{m}^3) = 1$ holds true, i.e. the ideal $(\mathfrak{m}/Q)^2$ is principal, for some parameter ideal Q in A which forms a reduction of $\mathfrak{m}$ ([9]). We note here that this condition depends on the choice of a reduction Q (see [8, Example 2.3]). She showed that the equality $r_Q(\mathfrak{m}) = n_Q(\mathfrak{m})$ holds true if and only if the associated graded ring $\mathrm{G}(\mathfrak{m})$ of $\mathfrak{m}$ is Cohen-Macaulay in the case where the base local ring A is stretched.

The detailed version of this paper has been submitted for publication elsewhere.

In 2001, Rossi and Valla [8] gave the notion of stretched $\mathfrak{m}$-primary ideals. We say that the $\mathfrak{m}$-primary ideal I is stretched if the following two conditions

(1) $Q \cap I^2 = QI$ and
(2) $\ell_A(Q + I^2/Q + I^3) = 1$

hold true for some parameter ideal Q in A which forms a reduction of I. We notice that the first condition is naturally satisfied if $I = \mathfrak{m}$ so that this extends the classical definition of stretched local rings given in [9].

Throughout this report, a stretched $\mathfrak{m}$-primary ideal I will come always equipped with a parameter ideal Q in A which forms a reduction of I such that $Q \cap I^2 = QI$ and $\ell_A(I^2 + Q/I^3 + Q) = 1$. Rossi and Valla [8] showed that the equality $r_I = n_I$ holds true if and only if the associated graded ring G is Cohen-Macaulay in the case where I is stretched. Thus stretched $\mathfrak{m}$-primary ideals whose reduction number attends to minimal value enjoy nice properties.

In this report we will also study the Hilbert coefficients of stretched $\mathfrak{m}$-primary ideals. As is well known, for a given $\mathfrak{m}$-primary ideal I, there exist integers $\{\mathrm{e}_k(I)\}_{0 \le k \le d}$ such that the equality

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_d(I)$$

holds true for all integers $n \gg 0$. For each $0 \le k \le d$, $\mathrm{e}_k(I)$ is called the k-th *Hilbert coefficient* of I. We set the power series

$$HS_I(z) = \sum_{n=0}^{\infty} \ell_A(I^n/I^{n+1}) z^n$$

and call it the Hilbert series of I. It is also well known that this series is rational and that there exists a polynomial $h_I(z)$ with integer coefficients such that $h_I(1) \neq 0$ and

$$HS_I(z) = \frac{h_I(z)}{(1-z)^d}.$$

The purpose of this report is to explore the structure of associated graded ring of stretched $\mathfrak{m}$-primary ideal I in the case where the reduction number attains almost minimal value.

The main result of this report is the following.

Theorem 1. *Suppose that I is stretched and assume that the equality $r_I = n_I + 1$ is satisfied. Then the following assertions hold true where $s = \min\{n \ge 1 \mid Q \cap I^{n+1} \neq QI^n\}$.*

(1) $\operatorname{depth} G = d - 1$,
(2) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \binom{n_I+1}{2} - s + 1$,
(3) $\mathrm{e}_k(I) = \binom{n_I+2}{k+1} - \binom{s}{k}$ *for all* $2 \le k \le d$,
(4) $\ell_A(A/I^{n+1}) = \sum_{k=0}^{d} (-1)^k \mathrm{e}_k(I) \binom{n+d-k}{d-k}$ *for all* $n \ge \max\{0, n_I - d + 1\}$, *and*
(5) *the Hilbert series* $HS_I(z)$ *of* I *is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \{\mathrm{e}_0(I) - \ell_A(A/I) - n_I + 1\}z + \sum_{2 \le i \le n_I+1, i \neq s} z^i}{(1-z)^d}.$$

Corollary 2. *Suppose that I is stretched and assume that $I^{n_I+2} = QI^{n_I+1}$ (i.e. $r_I \leq n_I + 1$), then* $\operatorname{depth} G \geq d - 1$.

As an application, we give the depth of the associated graded ring of stretched $\mathfrak{m}$-primary ideals with reduction number at most four as follows.

Corollary 3 (Corollary 19)**.** *Suppose that I is stretched and assume that $I^5 = QI^4$ (i.e. $r_I \leq 4$), then* $\operatorname{depth} G \geq d - 1$.

2. Preliminary Steps

The purpose of this section is to summarize some results on the structure of the stretched $\mathfrak{m}$-primary ideals, which we need throughout this report.

We set $\alpha_n = \ell_A(I^{n+1}/QI^n)$ for $n \geq 1$. We then have the following lemma which was given by Rossi and Valla.

Lemma 4. ([7, Lemma 2.4]) *Suppose that I is stretched. Then we have the following.*

(1) *There exists $x, y \in I \backslash Q$ such that $I^{n+1} = QI^n + (x^n y)$ holds true for all $n \geq 1$.*
(2) *The map*
$$I^{n+1}/QI^n \xrightarrow{\widehat{x}} I^{n+2}/QI^{n+1}$$
is surjective for all $n \geq 1$. Therefore $\alpha_n \geq \alpha_{n+1}$ for all $n \geq 1$.
(3) $\ell_A(I^{n+1}/QI^n + I^{n+2}) \leq 1$ *for all $n \geq 1$.*

We set
$$\Lambda := \Lambda_I = \Lambda_Q(I) = \{n \geq 1 \mid QI^{n-1} \cap I^{n+1}/QI^n \neq (0)\}$$
and $|\Lambda|$ denotes the cardinality of the set Λ. Then the following proposition is satisfied.

Proposition 5. *Suppose that I is stretched. Then we have the following.*

(1) $\alpha_1 = \ell_A(I^2 + Q/Q) = n_I - 1$.
(2) $\alpha_n = \alpha_{n-1} - 1$ *if* $n \notin \Lambda$, *and* $\alpha_n = \alpha_{n-1}$ *if* $n \in \Lambda$ *for all* $2 \leq n \leq r_I - 1$.
(3) $|\Lambda| = r_I - n_I$.

We notice here that $\alpha_1 = \ell_A(I^2/QI) = \mathrm{e}_0(I) + (d-1)\ell_A(A/I) - \ell_A(I/I^2)$ holds true, so that $n_I = \alpha_1 + 1$ doesn't depend on a minimal reduction Q of I for stretched $\mathfrak{m}$-primary ideals I.

3. The structure of Sally modules

In this report we need the notion of Sally modules for computing to the Hilbert coefficients of ideals. The purpose of this section is to summarize some results and techniques on the Sally modules which we need throughout this report. Remark that in this section $\mathfrak{m}$-primary ideals I are not necessarily stretched.

Let $T = \mathrm{R}(Q) = A[Qt] \subseteq A[t]$ denotes the Rees algebra of Q. Following Vasconcelos [10], we consider
$$S = \mathrm{S}_Q(I) = IR/IT \cong \bigoplus_{n \geq 1} I^{n+1}/Q^n I$$
the Sally module of I with respect to Q.

We give one remark about Sally modules. See [3, 10] for further information.

Remark 6 ([3, 10]). We notice that S is a finitely generated graded T-module and $\mathfrak{m}^n S = (0)$ for all $n \gg 0$. We have $\mathrm{Ass}_T S \subseteq \{\mathfrak{m}T\}$ so that $\dim_T S = d$ if $S \neq (0)$.

From now on, let us introduce some techniques, being inspired by [1, 2], which plays a crucial role throughout this report. See [6, Section 3] (also [5, Section 2] for the case where $I = \mathfrak{m}$) for the detailed proofs.

We denote by $E(m)$, for a graded module E and each $m \in \mathbb{Z}$, the graded module whose grading is given by $[E(m)]_n = E_{m+n}$ for all $n \in \mathbb{Z}$.

We have an exact sequence

$$0 \to K^{(-1)} \to F \overset{\varphi_{-1}}{\to} G \to R/IR + T \to 0 \qquad (\dagger_{-1})$$

of graded T-modules induced by tensoring the canonical exact sequence

$$0 \to T \overset{i}{\hookrightarrow} R \to R/T \to 0$$

of graded T-modules with A/I where $\varphi_{-1} = A/I \otimes i$, $K^{(-1)} = \mathrm{Ker}\, \varphi_{-1}$, and $F = T/IT \cong (A/I)[X_1, X_2, \cdots, X_d]$ is a polynomial ring with d indeterminates over the residue class ring A/I.

Lemma 7. ([5]) *There exists an exact sequence*

$$0 \to K^{(0)}(-1) \to ([R/IR + T]_1 \otimes F)(-1) \overset{\varphi_0}{\to} R/IR + T \to S/IS(-1) \to 0 \qquad (\dagger_0)$$

of graded T-modules where $K^{(0)} = \mathrm{Ker}\, \varphi_0$.

Notice that $\mathrm{Ass}_T K^{(m)} \subseteq \{\mathfrak{m}T\}$ for all $m = -1, 0$, because $F \cong (A/I)[X_1, X_2, \cdots, X_d]$ is a polynomial ring over the residue ring A/I and $[R/IR + T]_1 \otimes F$ is a maximal Cohen-Macaulay module over F.

We then have the following proposition by the exact sequences $(\dagger_{-1})$ and $(\dagger_0)$.

Proposition 8. ([6, Lemma 3.3]) *We have*

$$\begin{aligned}\ell_A(I^n/I^{n+1}) &= \ell_A(A/[I^2 + Q])\binom{n+d-1}{d-1} - \ell_A(I/[I^2+Q])\binom{n+d-2}{d-2} \\ &+ \ell_A([S/IS]_{n-1}) - \ell_A(K_n^{(-1)}) - \ell_A(K_{n-1}^{(0)})\end{aligned}$$

for all $n \geq 0$.

We also need the notion of *filtration of the Sally module* which was introduced by M. Vaz Pinto [11] as follows.

Definition 9. ([11]) We set, for each $m \geq 1$,

$$S^{(m)} = I^m t^{m-1} R / I^m t^{m-1} T (\cong I^m R / I^m T(-m+1)).$$

We notice that $S^{(1)} = S$, and $S^{(m)}$ are finitely generated graded T-modules for all $m \geq 1$, since R is a module-finite extension of the graded ring T.

The following lemma follows by the definition of the graded module $S^{(m)}$.

Lemma 10. *Let $m \geq 1$ be an integer. Then the following assertions hold true.*

(1) $\mathfrak{m}^n S^{(m)} = (0)$ *for integers $n \gg 0$; hence* $\dim_T S^{(m)} \leq d$.

(2) *The homogeneous components* $\{S_n^{(m)}\}_{n\in\mathbb{Z}}$ *of the graded* T*-module* $S^{(m)}$ *are given by*

$$S_n^{(m)} \cong \begin{cases} (0) & \text{if } n \le m-1, \\ I^{n+1}/Q^{n-m+1}I^m & \text{if } n \ge m. \end{cases}$$

Let $L^{(m)} = TS_m^{(m)}$ be a graded T-submodule of $S^{(m)}$ generated by $S_m^{(m)}$ and

$$\begin{aligned} D^{(m)} &= (I^{m+1}/QI^m) \otimes (A/\operatorname{Ann}_A(I^{m+1}/QI^m))[X_1, X_2, \cdots, X_d] \\ &\cong (I^{m+1}/QI^m)[X_1, X_2, \cdots, X_d] \end{aligned}$$

for $m \ge 1$ (c.f. [11, Section 2]).

We then have the following lemma.

Lemma 11. ([11, Section 2]) *The following assertions hold true for* $m \ge 1$.

(1) $S^{(m)}/L^{(m)} \cong S^{(m+1)}$ *so that the sequence*

$$0 \to L^{(m)} \to S^{(m)} \to S^{(m+1)} \to 0$$

is exact as graded T*-modules.*

(2) *There is a surjective homomorphism* $\theta_m : D^{(m)}(-m) \to L^{(m)}$ *graded* T*-modules.*

For each $m \ge 1$, tensoring the exact sequence

$$0 \to L^{(m)} \to S^{(m)} \to S^{(m+1)} \to 0$$

and the surjective homomorphism $\theta_m : D^{(m)}(-m) \to L^{(m)}$ of graded T-modules with A/I, we get the exact sequence

$$0 \to K^{(m)}(-m) \to D^{(m)}/ID^{(m)}(-m) \stackrel{\varphi_m}{\to} S^{(m)}/IS^{(m)} \to S^{(m+1)}/IS^{(m+1)} \to 0 \qquad (\dagger_m)$$

of graded F-modules where $K^{(m)} = \operatorname{Ker}\varphi_m$.

Notice here that, for all $m \ge 1$, we have $\operatorname{Ass}_T K^{(m)} \subseteq \{\mathfrak{m}T\}$ because $D^{(m)}/ID^{(m)} \cong (I^{m+1}/QI^m + I^{m+2})[X_1, X_2, \cdots, X_d]$ is a maximal Cohen-Macaulay module over F.

We then have the following result by Proposition 8 and exact sequences $(\dagger_m)$ for $m \ge 1$.

Proposition 12. *The following assertions hold true:*

(1) *We have*

$$\begin{aligned} \ell_A(I^n/I^{n+1}) &= \{\ell_A(A/I^2+Q) + \sum_{m=1}^{r_I-1} \ell_A(I^{m+1}/QI^m + I^{m+2})\}\binom{n+d-1}{d-1} \\ &+ \sum_{k=1}^{r_I}(-1)^k \left\{\sum_{m=k-1}^{r_I-1}\binom{m+1}{k}\ell_A(I^{m+1}/QI^m + I^{m+2})\right\}\binom{n+d-k-1}{d-k-1} \\ &- \sum_{m=-1}^{r_I-1}\ell_A(K_{n-m-1}^{(m)}) \end{aligned}$$

for all $n \ge \max\{0, r_I - d + 1\}$.

(2) $\mathrm{e}_0(I) = \ell_A(A/I^2+Q) + \sum_{m=1}^{r_I-1}\ell_A(I^{m+1}/QI^m + I^{m+2}) - \sum_{m=-1}^{r_I-1}\ell_{T_\mathcal{P}}(K_\mathcal{P}^{(m)})$ *where* $\mathcal{P} = \mathfrak{m}T$.

4. Proof of Main Theorem

Let $B = T/\mathfrak{m}T \cong (A/\mathfrak{m})[X_1, X_2, \cdots, X_d]$ which is a polynomial ring with d indeterminates over the field $A/\mathfrak{m}$. We notice that $D^{(m)}/ID^{(m)} \cong B$ for $m \geq 1$ if I is stretched. Then, thanks to Lemma 4 and Proposition 12, we can get the following proposition.

Proposition 13. *Suppose that I is stretched. Then the following assertions hold true:*

(1) *We have*

$$\begin{aligned}\ell_A(A/I^{n+1}) &= \{\mathrm{e}_0(I) + r_I - n_I\}\binom{n+d}{d} \\ &- \{\mathrm{e}_0(I) - \ell_A(A/I) + \binom{r_I}{2} + r_I - n_I\}\binom{n+d-1}{d-1} \\ &+ \sum_{k=2}^{r_I}(-1)^k\binom{r_I+1}{k+1}\binom{n+d-k}{d-k} - \sum_{m=-1}^{r_I-1}\sum_{i=0}^{n}\ell_A(K_{i-m-1}^{(m)})\end{aligned}$$

for all $n \geq \max\{0, r_I - d\}$.

(2) $\sum_{m=-1}^{r_I-1} \ell_{T_\mathcal{P}}(K_\mathcal{P}^{(m)}) = r_I - n_I = |\Lambda|$ *where $\mathcal{P} = \mathfrak{m}T$.*

Now we get the following result of Sally and Rossi-Valla as a corollary.

Corollary 14. ([9, Corollary 2.4], [8, Theorem 2.6]) *Suppose that I is stretched. Then the equality $r_I = n_I$ holds true if and only if the associated graded ring G is Cohen-Macaulay. When this is the case the following assertions also follow.*

(1) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \binom{n_I}{2}$.

(2) $\mathrm{e}_k(I) = \binom{n_I+1}{k+1}$ *for $2 \leq k \leq d$.*

(3) *The Hilbert series $HS_I(z)$ of I is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \{\mathrm{e}_0(I) - \ell_A(A/I) - n_I + 1\}z + \sum_{2 \leq k \leq n_I} z^k}{(1-z)^d}.$$

The following proposition plays an important role for our proof of Theorem 1.

Proposition 15. *Suppose that I is stretched and assume that $r_I = n_I + 1$. We set $s = \min\{n \geq 1 \mid Q \cap I^{n+1} \neq QI^n\}$. Then the following conditions hold true:*

(1) *$K^{(m)} \cong B(-s+m+1)$ as graded T-modules and $K^{(n)} = (0)$ for all $n \neq m$ for either of $m = -1$ or $m = 0$.*

(2) $\operatorname{depth} G = d - 1$.

5. Applications

In this section let us introduce some applications of Theorem 1. We study the structure of stretched $\mathfrak{m}$-primary ideals with small reduction number.

Remark 16. Suppose that I is stretched. We notice that we have $n_I, r_I \geq 2$, and G is Cohen-Macaulay if $r_I = 2$.

We have the following proposition for the case where the reduction number is three.

Proposition 17. *Suppose that I is stretched and assume that $r_I = 3$. Then we have $\Lambda \subseteq \{2\}$ and the following condition holds true.*

(1) *Suppose $\Lambda = \emptyset$. Then*
 (*i*) $n_I = 3$, $\alpha_1 = 2$, $\alpha_2 = 1$,
 (*ii*) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 3$, $\mathrm{e}_2(I) = 4$ *if* $d \geq 2$, $\mathrm{e}_3(I) = 1$, *if* $d \geq 3$, *and* $\mathrm{e}_i(I) = 0$ *for* $4 \leq i \leq d$, *and*
 (*iii*) *G is Cohen-Macaulay.*

(2) *Suppose $\Lambda = \{2\}$. Then*
 (*i*) $n_I = 2$, $\alpha_1 = \alpha_2 = 1$,
 (*ii*) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 2$, $\mathrm{e}_2(I) = 3$ *if* $d \geq 2$, $\mathrm{e}_3(I) = 1$, *if* $d \geq 3$, *and* $\mathrm{e}_i(I) = 0$ *for* $4 \leq i \leq d$, *and*
 (*iii*) $\operatorname{depth} G = d - 1$.

Proof. Since $n_I = r_I - |\Lambda|$, the assertions (1) and (2) follow by Corollary 14 and Theorem 1 respectively. □

The following theorem determine the structure of stretched $\mathfrak{m}$-primary ideals with reduction number four.

Theorem 18. *Suppose that I is stretched and assume that $r_I = 4$. Then we have $\Lambda \subseteq \{2, 3\}$ and the following conditions hold true.*

(1) *Suppose $\Lambda = \emptyset$. Then*
 (*i*) $n_I = 4$, $\alpha_1 = 3$, $\alpha_2 = 2$, $\alpha_3 = 1$,
 (*ii*) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 6$, $\mathrm{e}_2(I) = 10$ *if* $d \geq 2$, $\mathrm{e}_3(I) = 5$, *if* $d \geq 3$, $\mathrm{e}_4(I) = 1$ *if* $d \geq 4$, *and* $\mathrm{e}_i(I) = 0$ *for* $5 \leq i \leq d$, *and*
 (*iii*) *G is Cohen-Macaulay.*

(2) *Suppose $\Lambda = \{2\}$. Then*
 (*i*) $n_I = 3$, $\alpha_1 = \alpha_2 = 2$, $\alpha_3 = 1$,
 (*ii*) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 5$, $\mathrm{e}_2(I) = 9$ *if* $d \geq 2$, $\mathrm{e}_3(I) = 5$, *if* $d \geq 3$, $\mathrm{e}_4(I) = 1$ *if* $d \geq 4$, *and* $\mathrm{e}_i(I) = 0$ *for* $5 \leq i \leq d$, *and*
 (*iii*) $\operatorname{depth} G = d - 1$

(3) *Suppose $\Lambda = \{3\}$. Then*
 (*i*) $n_I = 3$, $\alpha_1 = 2$, $\alpha_2 = \alpha_3 = 1$,
 (*ii*) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 4$, $\mathrm{e}_2(I) = 7$ *if* $d \geq 2$, $\mathrm{e}_3(I) = 4$, *if* $d \geq 3$, $\mathrm{e}_4(I) = 1$ *if* $d \geq 4$, *and* $\mathrm{e}_i(I) = 0$ *for* $5 \leq i \leq d$, *and*
 (*iii*) $\operatorname{depth} G = d - 1$.

(4) *Suppose $\Lambda = \{2, 3\}$. Then*
 (*i*) $n_I = 2$, $\alpha_1 = \alpha_2 = \alpha_3 = 1$,
 (*ii*) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 3$, $\mathrm{e}_2(I) = 6$ *if* $d \geq 2$, $\mathrm{e}_3(I) = 4$, *if* $d \geq 3$, $\mathrm{e}_4(I) = 1$ *if* $d \geq 4$, *and* $\mathrm{e}_i(I) = 0$ *for* $5 \leq i \leq d$, *and*
 (*iii*) $\operatorname{depth} G = d - 1$.

Proof. The assertions (1), (2), and (3) follow by Corollary 14 and Theorem 1. Suppose that $\Lambda = \{2, 3\}$ then we have $\alpha_1 = n_I - 1 = r_I - |\Lambda| - 1 = 1$. Thanks to [7, Theorem 2.1], [12, Theorem 3.1], and [4, Corollary 2.11], we obtain the assertion (4). □

We can get the following corollary by Proposition 17 and Theorem 18.

Corollary 19. *Suppose that I is stretched and assume that $I^5 = QI^4$ (i.e. $r_I \leq 4$), then* $\operatorname{depth} G \geq d-1$.

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