ON IE-CLOSED SUBCATEGORIES

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Abstract. We study IE-closed subcategories of module categories, subcategories closed under taking images and extensions. The class of IE-closed subcategories contains that of torsion classes, torsion-free classes and wide subcategories, which are important objects in representation theory of algebras. We give a characterization of τ-tilting finiteness in terms of IE-closed subcategories. When we consider a hereditary algebra, we introduce the concept of twin rigid modules and give a classification result of IE-closed subcategories.

1. Introduction

Let Λ be a finite dimensional algebra over a field k and $\mathsf{mod}\Lambda$ the category of finitely generated right Λ-modules. It is one of the main subjects of representation theory of finite dimensional algebras to study subcategories of $\mathsf{mod}\Lambda$. For example, torsion classes are studied actively in connection with tilting and τ-tilting theory [1]. We focus on subcategories of $\mathsf{mod}\Lambda$ closed under some operations. In this note, we always assume that all subcategories are full and closed under isomorphisms.

Definition 1. Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\Lambda$.

(1) $\mathcal{C}$ is *closed under extensions* if for every short exact sequence

$$0 \to L \to M \to N \to 0$$

in $\mathsf{mod}\Lambda$ with $L, N \in \mathcal{C}$, we have $M \in \mathcal{C}$.

(2) $\mathcal{C}$ is *closed under quotients (resp. submodules) in* $\mathsf{mod}\Lambda$ if, for every object $C \in \mathcal{C}$, every quotient (resp. submodule) of C in $\mathsf{mod}\Lambda$ belongs to $\mathcal{C}$.

(3) $\mathcal{C}$ is a *torsion class (resp. torsion-free class) in* $\mathsf{mod}\Lambda$ if $\mathcal{C}$ is closed under extensions and quotients in $\mathsf{mod}\Lambda$ (resp. extensions and submodules).

(4) $\mathcal{C}$ is closed under *images (resp. kernels, cokernels)* if, for every map $\varphi\colon C_1 \to C_2$ with $C_1, C_2 \in \mathcal{C}$, we have $\mathsf{Im}\varphi \in \mathcal{C}$ (resp. $\mathsf{Ker}\varphi \in \mathcal{C}$, $\mathsf{Coker}\varphi \in \mathcal{C}$).

(5) $\mathcal{C}$ is a *wide subcategory of* $\mathsf{mod}\Lambda$ if $\mathcal{C}$ is closed under kernels, cokernels, and extensions.

(6) $\mathcal{C}$ is an *IE-closed subcategory of* $\mathsf{mod}\Lambda$ if $\mathcal{C}$ is closed under images and extensions.

It is easy to check that torsion classes, torsion-free classes and wide subcategories are IE-closed subcategories. The notion of *ICE-closed subcategories*, subcategories closed under images, cokernels and extensions is considered in [4].

For a collection $\mathcal{C}$ of Λ-modules in $\mathsf{mod}\Lambda$, we denote by $\mathsf{T}(\mathcal{C})$ (resp. $\mathsf{F}(\mathcal{C})$) the smallest torsion class (resp. torsion-free class) containing $\mathcal{C}$. The following proposition implies that

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an IE-closed subcategory is same as an intersection of a torsion class and a torsion-free class.

Proposition 2. [5, Proposition 2.3] *The following conditions are equivalent for a subcategory $\mathcal{C}$ of $\mathsf{mod}\Lambda$.*

(1) *$\mathcal{C}$ is an IE-closed subcategory of $\mathsf{mod}\Lambda$.*

(2) *There exist a torsion class $\mathcal{T}$ and a torsion-free class $\mathcal{F}$ in $\mathsf{mod}\Lambda$ satisfying $\mathcal{C} = \mathcal{T} \cap \mathcal{F}$.*

In this case, $\mathcal{C} = \mathsf{T}(\mathcal{C}) \cap \mathsf{F}(\mathcal{C})$ holds.

2. Functorial finiteness

In this section, we consider some finiteness conditions of subcategories and give implications among them. Using these, we characterize tau-tilting finite alegebras by fuctorial finiteness of IE-closed subcategories. We start introducing the concept of functorial finiteness.

Definition 3. Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\Lambda$ and M an object in $\mathsf{mod}\Lambda$.

(1) A morphism $f\colon M \to C$ in $\mathsf{mod}\Lambda$ is a *left $\mathcal{C}$-approximation* of M if C belongs to $\mathcal{C}$ and every morphism $f'\colon M \to C'$ with $C' \in \mathcal{C}$ factors through f. Dually, a *right $\mathcal{C}$-approximation* is defined.

(2) A subcategory $\mathcal{C}$ is *covariantly finite (resp. contravariantly finite) in* $\mathsf{mod}\Lambda$ if for any object M in $\mathsf{mod}\Lambda$, there exists a left (resp. right) $\mathcal{C}$-approximation of M.

(3) A subcategory is *functorially finite in* $\mathsf{mod}\Lambda$ if it is covariantly finite and contravariantly finite in $\mathsf{mod}\Lambda$.

Every torsion class $\mathcal{T}$ in $\mathsf{mod}\Lambda$ is contravariantly finite since it gives a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathsf{mod}\Lambda$. The notion of functorial finiteness appears in [2], which considers the existance of Auslander-Reiten sequences in subcategories of $\mathsf{mod}\Lambda$. Next we introduce the notion of Ext-projective.

Definition 4. Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\Lambda$ closed under extensions.

(1) An object P of $\mathcal{C}$ is Ext-*projective* if it satisfies $\mathrm{Ext}^1_\Lambda(P, C) = 0$ for any $C \in \mathcal{C}$.

(2) $\mathcal{C}$ *has enough* Ext-*projectives* if for any $C \in \mathcal{C}$, there exists a short exact sequence

$$0 \to C' \to P \to C \to 0$$

such that P is an Ext-projective object in $\mathcal{C}$ and $C' \in \mathcal{C}$.

(3) P is an Ext-*progenerator* of $\mathcal{C}$ if $\mathcal{C}$ has enough Ext-projectives and Ext-projective objects are precisely objects in $\mathsf{add}P$.

(4) If $\mathcal{C}$ has an Ext-progenerator, then $\mathbf{P}(\mathcal{C})$ denotes a unique basic Ext-progenerator of $\mathcal{C}$, that is, a direct sum of all indecomposable Ext-projective objects in $\mathcal{C}$ up to isomorphism.

Dually, the notions for Ext-injectives are defined, and $\mathbf{I}(\mathcal{C})$ denotes a unique basic Ext-injective cogenerator of $\mathcal{C}$ (if it exists).

It is well-known that the above notions are related to each other for torsion classes.

Proposition 5. *The following conditions are equivalent for a torsion class $\mathcal{T}$ in $\mathsf{mod}\Lambda$.*

(1) $\mathcal{T}$ *has an* Ext*-progenerator.*
(2) $\mathcal{T}$ *has a finite cover, that is, there is* $M \in \mathcal{T}$ *such that* $\mathcal{T} \subseteq \mathsf{Fac}M$.
(3) $\mathcal{T}$ *is covariantly finite in* $\mathsf{mod}\Lambda$.
(4) $\mathcal{T}$ *has enough* Ext*-projective objects.*

The following proposition gives relations among the above finiteness conditions for IE-closed subcategories.

Proposition 6. [5, Lemma 2.6, 2.9] *Consider the following conditions for an IE-closed subcategory* $\mathcal{C}$ *in* $\mathsf{mod}\Lambda$.

(1) $\mathcal{C}$ *is left finite, that is,* $\mathsf{T}(\mathcal{C})$ *is functorially finite in* $\mathsf{mod}\Lambda$.
(2) *There exist a torsion class* $\mathcal{T}$ *and a torsion-free class* $\mathcal{F}$ *such that* $\mathcal{C} = \mathcal{T} \cap \mathcal{F}$ *and* $\mathcal{T}$ *is functorially finite in* $\mathsf{mod}\Lambda$.
(3) $\mathcal{C}$ *has an* Ext*-progenerator.*
(4) $\mathcal{C}$ *has a finite cover.*
(5) $\mathcal{C}$ *is covariantly finite in* $\mathsf{mod}\Lambda$.
(6) $\mathcal{C}$ *has enough* Ext*-projective objects.*

The assertions (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5) $\Rightarrow$ (6) *hold. If* Λ *is hereditary, all conditions are equivalent.*

The notion of τ-tilting finiteness introduced in [3] is an analogue of representation finiteness in the perspective of τ-tilting theory. A finite dimensional algebra Λ is *τ-tilting finite* if the set of functorially finite torsion classes in $\mathsf{mod}\Lambda$ is a finite set. This definition coincides with the condition that there are only finitely many support τ-tilting Λ-modules up to isomorphisms, see Theorem 9. In [3], τ-tilting finiteness is characterized as follows:

Theorem 7. [3, Theorem 3.8] *Let* Λ *be a finite dimensional algebra. The following conditions are equivalent.*

(1) Λ *is τ-tilting finite.*
(2) *The set of torsion classes in* $\mathsf{mod}\Lambda$ *is a finite set.*
(3) *The set of torsion-free classes in* $\mathsf{mod}\Lambda$ *is a finite set.*
(4) *Every torsion class in* $\mathsf{mod}\Lambda$ *is functorially finite.*
(5) *Every torsion-free class in* $\mathsf{mod}\Lambda$ *is functorially finite.*

Now we give the following result analogous to the above[1].

Theorem 8. [5, Proposition 2.10] *Let* Λ *be a finite dimensional algebra. The following are equivalent.*

(1) Λ *is τ-tilting finite.*
(2) *The set of IE-closed subcategories of* $\mathsf{mod}\Lambda$ *is a finite set.*
(3) *Every IE-closed subcategory of* $\mathsf{mod}\Lambda$ *is functorially finite.*
(4) *Every IE-closed subcategory of* $\mathsf{mod}\Lambda$ *is covariantly finite.*
(5) *Every IE-closed subcategory of* $\mathsf{mod}\Lambda$ *is contravariantly finite.*

There is an analogous result for ICE-closed subcategories, see [4, Proposition 4.20].

[1]The author would like to thank Ryo takahashi and Haruhisa Enomoto for the conversation after the author's talk which gives the equivalence between (3) and (4).

3. Classification

In this section, we give the classification result of IE-closed subcategories. We start giving the following result which classify torsion classes in $\mathsf{mod}\Lambda$.

Theorem 9. [1, Theorem 2.7] *There exists bijective correspondences between:*

(1) *The set of functorially finite torsion classes in* $\mathsf{mod}\Lambda$*,*
(2) *The set of isomorphism classes of basic support* τ*-tilting modules.*

The correspondence from (1) to (2) is given by $\mathcal{T} \mapsto \mathbf{P}(\mathcal{T})$, and well-defined by Proposition 5.

Now we aim to classify IE-closed subcategories as an analogue of the above. Unfortunately, we need the assumption that Λ is hereditary. In the rest of this note, we assume it. We start introducing the concept of twin rigid modules.

Definition 10. A pair (P, I) of Λ-modules is a *twin rigid module* if it satisfies

- P and I are rigid, that is, $\mathrm{Ext}^1_\Lambda(P, P) = 0$ and $\mathrm{Ext}^1_\Lambda(I, I) = 0$.
- There are short exact sequences

$$0 \to P \to I^0 \to I^1 \to 0$$
$$0 \to P_1 \to P_0 \to I \to 0$$

with $P_0, P_1 \in \mathsf{add}P$ and $I^0, I^1 \in \mathsf{add}I$.

Two twin rigid pairs (P_1, I_1) and (P_2, I_2) are *isomorphic* if we have $P_1 \cong P_2$ and $I_1 \cong I_2$. A twin rigid module (P, I) is *basic* if P and I are basic.

The above concept is appered as a pair of an Ext-progenerator and an Ext-injective cogenerator of an IE-closed subcategory:

Theorem 11. [5, Theorem 2.14] *Let Λ be a hereditary finite dimensional algebra. Then there exist bijective correspondences between:*

(1) *The set of functorially finite IE-closed subcategories of* $\mathsf{mod}\Lambda$*,*
(2) *The set of isomorphism classes of basic* twin rigid Λ-modules.

The correspondence from (1) to (2) is given by $\mathcal{C} \mapsto (\mathbf{P}(\mathcal{C}), \mathbf{I}(\mathcal{C}))$, and well-defined by Proposition 6 and its dual.

Next we give the property of twin rigid modules, which gives a connection between twin rigid modules and tilting modules.

Proposition 12. [5, Proposition 3.4 (1)] *Assume that Λ is hereditary. Let (P, I) be a twin rigid module and set $\Gamma_P = \mathsf{End}_\Lambda(P)$. Then*

(1) $\mathsf{Hom}_\Lambda(P, I)$ *is a tilting* Γ_P*-module.*
(2) *The equality* $|P| = |I|$ *holds.*

The equality (2) gives a partial anwer to the question raised by Auslander and Smalø in [2], see [5, Remark 3.8]. Thanks to the previous proposition, we obtatin the following bijection.

Proposition 13. [5, Proposition 3.4 (2)] *Assume that Λ is hereditary. Let (P, I) be a twin rigid module. Then the functor* $\mathrm{Hom}_\Lambda(P, -)\colon \mathsf{mod}\Lambda \to \mathsf{mod}\Gamma_P$ *induces a bijective correspondence between:*

(1) *the set of isomorphism classes of twin rigid modules* (P, I),
(2) *the set of isomorphism classes of tilting* Γ_P*-modules contatined in* $\mathsf{Sub}(\mathrm{Hom}_k(P, k))$.

In [5], the notion of completion and mutation of twin rigid modules is introduced taking advantage of that of tilting modules through the above proposition. All twin rigid modules are obtained by mutation in the case that Λ is a representation-finite hereditary algebra. By Theorem 11, we obtain all IE-closed subcategories.

References

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