DIMITROV–HAIDEN–KATZARKOV–KONTSEVICH COMPLEXITIES FOR SINGULARITY CATEGORIES

RYO TAKAHASHI

Abstract. Dimitrov, Haiden, Katzarkov and Kontsevich have introduced the notion of complexities for arbitrary triangulated categories. This paper deals with complexities for singularity categories.

1. Preliminaries

In this section, we work on a general triangulated category.

Setup 1. Throughout this section, let $\mathcal{T}$ be a triangulated category. All subcategories of $\mathcal{T}$ are assumed to be strictly full. We may omit a subscript if it is clear from the context.

We introduce the operation $\star$ for subcategories of $\mathcal{T}$, which plays a central role throughout the paper.

Definition 2. Let $\mathcal{X}$ and $\mathcal{Y}$ be subcategories of $\mathcal{T}$.

(1) We denote by $\mathcal{X}\star\mathcal{Y}$ the subcategory of $\mathcal{T}$ consisting of objects $T\in\mathcal{T}$ such that there exists an exact triangle $X\to T\to Y\rightsquigarrow$ in $\mathcal{T}$ such that $X\in\mathcal{X}$ and $Y\in\mathcal{Y}$.

(2) When $\mathcal{X},\mathcal{Y}$ consist of single objects X,Y respectively, we simply write $X\star Y$ to denote $\mathcal{X}\star\mathcal{Y}$.

In the following lemma, we make a list of several fundamental properties of the operation $\star$. The first assertion says that the operation $\star$ satisfies associativity. The second and third assertions state that the operation $\star$ is compatible with taking finite direct sums and shifts. The proof is standard.

Lemma 3. (1) *For subcategories $\mathcal{X},\mathcal{Y},\mathcal{Z}$ of $\mathcal{T}$ one has $(\mathcal{X}\star\mathcal{Y})\star\mathcal{Z}=\mathcal{X}\star(\mathcal{Y}\star\mathcal{Z})$. Hence, there is no ambiguity in writing $\bigstar_{i=1}^n\mathcal{X}_i=\mathcal{X}_1\star\cdots\star\mathcal{X}_n$ for subcategories $\mathcal{X}_1,\dots,\mathcal{X}_n$ of $\mathcal{T}$ or $\mathcal{X}^{\star n}=\underbrace{\mathcal{X}\star\cdots\star\mathcal{X}}_{n}$.*

(2) *Let $\{X_{ij}\}_{1\leqslant i\leqslant m,\,1\leqslant j\leqslant n}$ and $\{M_i\}_{1\leqslant i\leqslant m}$ be families of objects of $\mathcal{T}$. Suppose that $M_i\in\bigstar_{j=1}^nX_{ij}$ for each $1\leqslant i\leqslant m$. Then it holds that $\bigoplus_{i=1}^m M_i\in\bigstar_{j=1}^n(\bigoplus_{i=1}^m X_{ij})$.*

(3) *Let $X_1,\dots,X_n\in\mathcal{T}$. Then the following statements hold true.*

(a) *If $M\in\bigstar_{i=1}^nX_i$, then $M[s]\in\bigstar_{i=1}^nX_i[s]$ for all integers s, $M^{\oplus m}\in\bigstar_{i=1}^nX_i^{\oplus m}$ for all positive integers m, and $M\oplus(\bigoplus_{i=1}^n Y_i)\in\bigstar_{i=1}^n(X_i\oplus Y_i)$ for all objects $Y_1,\dots,Y_n\in\mathcal{T}$.*

(b) *One has the containment $\bigoplus_{i=1}^n X_i\in\bigstar_{i=1}^nX_i$.*

Here we recall the definition of split generators, which are used to define complexities and entropies.

The detailed version [10] of this paper has been submitted for publication elsewhere.

Definition 4. (1) A *thick subcategory* of $\mathcal{T}$ is by definition a triangulated subcategory of $\mathcal{T}$ closed under direct summands, i.e., a subcategory closed under shifts, mapping cones and direct summands.

(2) For an object $X \in \mathcal{T}$ we denote by $\mathsf{thick}_{\mathcal{T}} X$ the *thick closure* of T, that is to say, the smallest thick subcategory of $\mathcal{T}$ to which X belongs.

(3) A *split generator* of $\mathcal{T}$, which is also called a *thick generator* of $\mathcal{T}$, is defined to be an object of $\mathcal{T}$ whose thick closure coincides with $\mathcal{T}$.

Now we can state the definitions of complexities and entropies introduced in [5].

Definition 5 (Dimitrov–Haiden–Katzarkov–Kontsevich)**.**

(1) Let $X, Y \in \mathcal{T}$ and $t \in \mathbb{R}$. We denote by $\delta_t(X, Y)$ the infimum of the sums $\sum_{i=1}^{r} \mathrm{e}^{n_i t}$, where r runs through the nonnegative integers and n_i run through the integers such that there exist a sequence

$$\begin{array}{ccccccccccc} 0 = Y_0 & & \longrightarrow & & Y_1 \longrightarrow \cdots \longrightarrow & Y_{r-1} & & \longrightarrow & & Y_r = Y \oplus Y' \\ & \nwarrow\!\!\sim & & \swarrow & & & \nwarrow\!\!\sim & & \swarrow & \\ & & X[n_1] & & \cdots & & & X[n_r] & & \end{array}$$

of exact triangles $\{Y_{i-1} \to Y_i \to X[n_i] \rightsquigarrow\}_{i=1}^{r}$ in $\mathcal{T}$. The function $\mathbb{R} \ni t \mapsto \delta_t(X, Y) \in \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ is called the *complexity* of Y relative to X. When $Y = 0$, one can take $r = 0$, and hence $\delta_t(X, Y) = 0$.

(2) Let $F : \mathcal{T} \to \mathcal{T}$ be an exact functor and $t \in \mathbb{R}$. The *entropy* $\mathrm{h}_t(F)$ of F is defined by

$$\mathrm{h}_t(F) = \lim_{n\to\infty} \tfrac{1}{n} \log \delta_t(G, F^n(G)),$$

where G is a split generator of $\mathcal{T}$. This is independent of the choice of G; see [5, Lemma 2.6].

The following proposition gives an equivalent definition of a complexity.

Proposition 6. *Let $X, Y \in \mathcal{T}$ and $t \in \mathbb{R}$. One then has the equality*

$$\delta_t(X, Y) = \inf\{\textstyle\sum_{i=1}^{r} \mathrm{e}^{n_i t} \mid Y \oplus Y' \in \bigstar_{i=1}^{r} X[n_i] \text{ for some } Y' \in \mathcal{T}\}.$$

We give a couple of statements concerning complexities. Recall that $\mathcal{T}$ is said to be *periodic* if there exists an integer $n > 0$ such that the nth shift functor $[n]$ is isomorphic to the identity functor $\mathrm{id}_{\mathcal{T}}$ of $\mathcal{T}$.

Proposition 7. *Let X and Y be objects of $\mathcal{T}$. Then the following statements hold.*

(1) *Let $t \in \mathbb{R}$. Then $\delta_t(X, Y) < \infty$ if and only if $Y \in \mathsf{thick}_{\mathcal{T}} X$.*

(2) *There is an equality $\delta_0(X, Y) = \inf\{r \in \mathbb{Z}_{\geqslant 0} \mid Y \oplus Y' \in \bigstar_{i=1}^{r} X[n_i]$ for some $Y' \in \mathcal{T}\}$.*

(3) *Let $t \in \mathbb{R}$. Suppose that $\mathcal{T}$ is periodic and $\delta_t(X, Y) < \infty$. Then $\delta_t(X, Y) = 0$ unless $t = 0$.*

Remark 8. The equality in Proposition 7(2) may remind the reader of the notion of a *level* introduced by Avramov, Buchweitz, Iyengar and Miller [2]. Namely, $\delta_0(X, Y)$ looks closely related to the X-level $\mathrm{level}_{\mathcal{T}}^{X}(Y)$ of Y. The difference is that an X-level ignores finite direct sums of copies of X. This is similar to the difference between the lengths of a composition series and a Loewy series of a module over a ring. The complexity $\delta_t(X, Y)$ can also be regarded as a weighted version of $\delta_0(X, Y)$ with respect to shifts.

The following lemma comes from [5, Proposition 2.2]. In this proposition, neither $\delta_t(X,Y)$ nor $\delta_t(Y,Z)$ is assumed to be finite, but in its proof both $\delta_t(X,Y)$ and $\delta_t(Y,Z)$ seem to be assumed to be finite. In fact, without this assumption, we would need to clarify what $0 \cdot \infty$ and $\infty \cdot 0$ mean.

Lemma 9. *Let t be a real number. Let X, Y and Z be objects of $\mathcal{T}$. Suppose that both $\delta_t(X,Y)$ and $\delta_t(Y,Z)$ are finite. Then there is an inequality $\delta_t(X,Z) \leqslant \delta_t(X,Y) \cdot \delta_t(Y,Z)$.*

2. Main results

In this section, we shall investigate complexities and entropies for the singularity category of a commutative noetherian local ring, which is a triangulated category.

Setup 10. Throughout this section, let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. The triangulated category considered in this section is the *singularity category* $\mathsf{D}_{\mathsf{sg}}(R)$ of R, which is by definition the Verdier quotient of the bounded derived category of finitely generated R-modules by perfect complexes (i.e., bounded complexes of finitely generated projective R-modules).

We recall several fundamental notions from commutative algebra, whose details can be found in [1, 3].

Definition 11. (1) We say that R is a *singular* local ring if it is not a regular local ring. Note that R is singular if and only if the category $\mathsf{D}_{\mathsf{sg}}(R)$ is nonzero.

(2) The *codimension* and the *codepth* of R are defined by $\operatorname{codim} R = \operatorname{edim} R - \dim R$ and $\operatorname{codepth} R = \operatorname{edim} R - \operatorname{depth} R$. Here, $\operatorname{edim} R$ and $\operatorname{depth} R$ stand for the embedding dimension of R and the depth of R, respectively. Note that $\operatorname{codim} R = \operatorname{codepth} R$ if (and only if) R is Cohen–Macaulay.

(3) The local ring R is said to be a *hypersurface* provided the inequality $\operatorname{codepth} R \leqslant 1$ holds. According to Cohen's structure theorem, this condition is equivalent to saying that the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is isomorphic to the residue ring $S/(f)$ of some regular local ring S by some principal ideal (f).

(4) The local ring R is called a *complete intersection* if the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is isomorphic to the residue ring $S/(\boldsymbol{f})$ of a regular local ring $(S, \mathfrak{n})$ by the ideal $(\boldsymbol{f})$ generated by a regular sequence $\boldsymbol{f} = f_1, \dots, f_c$. One can choose $\boldsymbol{f} = f_1, \dots, f_c$ so that $c = \operatorname{codim} R$, and in this case, $f_i \in \mathfrak{n}^2$ for all i.

(5) The *Koszul complex* K^R of R is defined to be the Koszul complex $\mathrm{K}(\boldsymbol{x}, R)$ on R of a minimal system of generators $\boldsymbol{x} = x_1, \dots, x_n$ of $\mathfrak{m}$. This complex is uniquely determined up to isomorphism; see [3, the part following Remark 1.6.20]. Each homology $\mathrm{H}_i(\mathrm{K}^R)$ is a finite-dimensional k-vector space.

(6) We say that R has an *isolated singularity* if $R_\mathfrak{p}$ is a regular local ring for all $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$.

(7) Let $\mathrm{e}(R)$ and $\mathrm{r}(R)$ be the *(Hilbert–Samuel) multiplicity* and *type* of R, respectively. One has R is singular if and only if $\mathrm{e}(R) > 1$, and R is Gorenstein if and only if R is Cohen–Macaulay and $\mathrm{r}(R) = 1$.

(8) Let M be a finitely generated R-module. Let n be a nonnegative integer. Then we denote by $\Omega_R^n M$ the nth *syzygy* of M over R, that is, the image of the nth differential

map in a minimal free resolution of the R-module M. Note that the module $\Omega_R^n M$ is uniquely determined up to isomorphism. We denote by $\beta_n^R(M)$ the nth *Betti number* of M, namely, the minimal number of generators of $\Omega_R^n M$.

(9) For an R-module M, we denote by $\ell(M)$ the *length* of (a composition series of) M.

What we want to consider in this section is the following conjecture.

Conjecture 12. Let G be a split generator of $\mathsf{D_{sg}}(R)$. Then one has the equality $\delta_t(G,X)=0$ for all objects X of $\mathsf{D_{sg}}(R)$ and for all nonzero real numbers t.

In the case where R is a hypersurface, it is easy to see that Conjecture 12 holds true.

Example 13. If R is a hypersurface, then $\delta_t(G,X)=0$ for all split generators G of $\mathsf{D_{sg}}(R)$, for all $X\in\mathsf{D_{sg}}(R)$ and for all $0\neq t\in\mathbb{R}$. Indeed, in this case, there exists an isomorphism $\widehat{R}\cong S/(f)$, where S is a regular local ring and $f\in S$. The singularity category $\mathsf{D_{sg}}(\widehat{R})$ of the completion $\widehat{R}$ is equivalent as a triangulated category to the homotopy category of matrix factorizations of f over S, which is periodic of periodicity two; we refer the reader to [4, 6, 7, 8, 11] for the details. It is easy to see that $\mathsf{D_{sg}}(R)$ is also periodic of periodicity two, and the assertion follows from (1) and (3) of Proposition 7.

We introduce a condition on an object of the singularity category, which is essential in our theorems.

Definition 14. We say that an object X of $\mathsf{D_{sg}}(R)$ is *locally zero on the punctured spectrum of* R if for each nonmaximal prime ideal $\mathfrak{p}$ of R the localized complex $X_\mathfrak{p}$ is isomorphic to 0 in the singularity category $\mathsf{D_{sg}}(R_\mathfrak{p})$ of the local ring $R_\mathfrak{p}$. This condition is equivalent to saying that $X_\mathfrak{p}$ is isomorphic to a perfect complex over $R_\mathfrak{p}$ in the bounded derived category of finitely generated $R_\mathfrak{p}$-modules.

Remark 15. Suppose that R has an isolated singularity. Then every object of $\mathsf{D_{sg}}(R)$ is locally zero on the punctured spectrum of R, since $\mathsf{D_{sg}}(R_\mathfrak{p})=0$ for all nonmaximal prime ideals $\mathfrak{p}$ of R.

We establish a lemma, whose proof is done by [9, Corollary 4.3(3)], Proposition 7(1) and Lemma 9.

Lemma 16. *Let* $t\in\mathbb{R}$. *Let* X *be an object of* $\mathsf{D_{sg}}(R)$ *such that* k *belongs to* $\mathsf{thick}_{\mathsf{D_{sg}}(R)}X$. *Let* Y *be an object of* $\mathsf{D_{sg}}(R)$ *which is locally zero on the punctured spectrum of* R. *If* $\delta_t(k,k)=0$, *then* $\delta_t(X,Y)=0$.

Now we shall state three theorems, all of which support Conjecture 12. The proofs use Lemma 3, Lemma 16, [1, Theorem 8.1.2], and fundamental properties of Koszul complexes and multiplicities stated in [3]. For the details of the proofs of the theorems, we refer the reader to [10].

Theorem 17. *Let* R *be a complete intersection. Let* $X\in\mathsf{D_{sg}}(R)$ *be such that* k *belongs to* $\mathsf{thick}_{\mathsf{D_{sg}}(R)}X$. *Let* $Y\in\mathsf{D_{sg}}(R)$ *be locally zero on the punctured spectrum of* R. *Then* $\delta_t(X,Y)=0$ *for all* $t\neq 0$.

Theorem 18. *Let* R *be singular and Cohen–Macaulay. Assume that the residue field* k *is infinite. Let* X *be an object of* $\mathsf{D_{sg}}(R)$ *such that* $k\in\mathsf{thick}_{\mathsf{D_{sg}}(R)}X$. *Let* Y *be an* *object of* $\mathsf{D_{sg}}(R)$ *which is locally zero on the punctured spectrum of* R*. Put* $u = \mathrm{e}(R)$ *and* $r = \mathrm{r}(R)$*. Then* $\delta_t(X, Y) = 0$ *for all* $t < -\log(u-1)$ *and for all* $t > \log(u-r)$*. Therefore,* $\delta_t(X, Y) = 0$ *for all* $|t| > \log(u-1)$ *provided that* R *is Gorenstein.*

Theorem 19. *Suppose* R *is singular. Set* $c = \operatorname{codepth} R$ *and* $m = \max_{1 \leqslant i \leqslant c}\{\dim_k \mathrm{H}_i(\mathrm{K}^R)\}$*. Let* X *be an object of* $\mathsf{D_{sg}}(R)$ *such that* k *belongs to* $\mathsf{thick}_{\mathsf{D_{sg}}(R)} X$*, and let* Y *be an object of* $\mathsf{D_{sg}}(R)$ *which is locally zero on the punctured spectrum of* R*. Then* $\delta_t(X, Y) = 0$ *for all* $|t| > \frac{\log c + \log m}{2}$*.*

Remark 20. (1) Put $n = \operatorname{edim} R$. Cohen's structure theorem shows that there exist an n-dimensional regular local ring $(S, \mathfrak{n}, k)$ and an ideal I of S such that the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is isomorphic to the residue ring S/I. Choose a minimal system of generators $\boldsymbol{x} = x_1, \ldots, x_n$ of $\mathfrak{n}$. It holds that

$$\mathrm{H}_i(\mathrm{K}^R) = \mathrm{H}_i(\boldsymbol{x}, R) \cong \mathrm{H}_i(\boldsymbol{x}, R) \otimes_R \widehat{R} \cong \mathrm{H}_i(\boldsymbol{x}, \widehat{R}) \cong \mathrm{H}_i(\mathrm{K}(\boldsymbol{x}, S) \otimes_S \widehat{R}) \cong \operatorname{Tor}_i^S(k, \widehat{R})$$

for each integer i, where the first isomorphism holds since the R-module $\mathrm{H}_i(\boldsymbol{x}, R)$ has finite length, while the last isomorphism follows from the fact that the Koszul complex $\mathrm{K}(\boldsymbol{x}, S)$ is a free resolution of k over S. Hence, the number $\dim_k \mathrm{H}_i(\mathrm{K}^R)$ is equal to the ith Betti number $\beta_i^S(\widehat{R})$ of $\widehat{R}$ over S.

(2) Let R be a singular hypersurface. Let G be a split generator of $\mathsf{D_{sg}}(R)$, and let X be an object of $\mathsf{D_{sg}}(R)$ which is locally zero on the punctured spectrum of R. The following two statements hold.
   (a) As R is a complete intersection, Theorem 17 implies that $\delta_t(G, X) = 0$ for all $0 \neq t \in \mathbb{R}$.
   (b) Put $c = \operatorname{codepth} R$ and $m = \max_{1 \leqslant i \leqslant c}\{\dim_k \mathrm{H}_i(\mathrm{K}^R)\}$. Then $c = 1$. We have $\widehat{R} \cong S/(f)$ for some regular local ring $(S, \mathfrak{n})$ and some element $f \in \mathfrak{n}^2$. The sequence $0 \to S \xrightarrow{f} S \to \widehat{R} \to 0$ gives a minimal free resolution of the S-module $\widehat{R}$, and the equalities $\dim_k \mathrm{H}_1(\mathrm{K}^R) = \beta_1^S(\widehat{R}) = 1$ hold by (1). Hence $m = 1$. We get $\frac{\log c + \log m}{2} = 0$, and $\delta_t(G, X) = 0$ for all $t \neq 0$ by Theorem 19.

   Thus, each of Theorems 17 and 19 recovers Example 13 in the case where X is locally zero on the punctured spectrum of R (e.g., in the case where R has an isolated singularity by Remark 15).

Combining the above three theorems with Remark 15, we obtain the corollary below on entropies.

Corollary 21. *Let* R *be singular with an isolated singularity. Let* F *be an exact endofunctor of* $\mathsf{D_{sg}}(R)$*.*

(1) *Put* $c = \operatorname{codepth} R$ *and* $m = \max_{1 \leqslant i \leqslant c}\{\dim_k \mathrm{H}_i(\mathrm{K}^R)\}$*. Then* $\delta_t(G, X) = 0$ *for all split generators* $G \in \mathsf{D_{sg}}(R)$*, all* $X \in \mathsf{D_{sg}}(R)$ *and all* $|t| > \frac{\log c + \log m}{2}$*. Thus* $\mathrm{h}_t(F)$ *is not defined if* $|t| > \frac{\log c + \log m}{2}$*.*

(2) *Assume that* R *is Gorenstein and* k *is infinite. Then* $\delta_t(G, X) = 0$ *for all split generators* $G \in \mathsf{D_{sg}}(R)$*, all* $X \in \mathsf{D_{sg}}(R)$ *and all* $|t| > \log(\mathrm{e}(R) - 1)$*. Thus* $\mathrm{h}_t(F)$ *is not defined for* $|t| > \log(\mathrm{e}(R) - 1)$*.*

(3) *Suppose that R is a complete intersection. Then $\delta_t(G, X) = 0$ for all split generators $G \in \mathsf{D}_{\mathsf{sg}}(R)$, all $X \in \mathsf{D}_{\mathsf{sg}}(R)$ and all nonzero real numbers t. Therefore, the entropy $\mathrm{h}_t(F)$ is defined only for $t = 0$.*

We close this section by mentioning that examples are constructed in [10], which say that the bounds $\frac{\log c + \log m}{2}$ and $\log(\mathrm{e}(R) - 1)$ for the real numbers t given in Theorems 18, 19 and Corollary 21(1)(2) are not necessarily best possible.

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Graduate School of Mathematics
Nagoya University
Furocho, Chikusaku, Nagoya 464-8602, JAPAN

Email address: takahashi@math.nagoya-u.ac.jp