TWISTED SEGRE PRODUCTS AND NONCOMMUTATIVE QUADRIC SURFACES

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Abstract. In this paper, we study twisted Segre products $A \circ_\psi B$ of noetherian Koszul AS-regular algebras A and B. We state that if $A \circ_\psi B$ is noetherian, then the noncommutative projective scheme $\mathsf{qgr}\, A \circ_\psi B$ has finite global dimension. We also state that if $A = k[u, v]$, $B = k[x, y]$, and ψ is a diagonal twisting map, then $A \circ_\psi B$ is a noncommutative quadric surface.

1. Introduction

Throughout this paper, let k be an algebraically closed field of characteristic 0. All algebras and vector spaces considered in this paper are over k, and all unadorned tensor products $\otimes$ are taken over k. For an algebra S, we write $\mathsf{Mod}\, S$ for the category of right S-modules. For a graded algebra S, we write $\mathsf{GrMod}\, S$ for the category of graded right S-modules.

Denote by $\mathbb{P}^d$ the d-dimensional projective space over k. Let $\Phi : \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \to \mathbb{P}^{nm-1}$ be the map defined by

$$((a_1, \dots, a_n), (b_1, \dots, b_m)) \mapsto (a_1b_1, a_2b_1, \dots, a_{n-1}b_m, a_nb_m).$$

Note that Φ is injective. It is called the *Segre embedding*. Let $A = \bigoplus_{i\in\mathbb{Z}} A_i$ and $B = \bigoplus_{i\in\mathbb{Z}} B_i$ be (not necessarily commutative) $\mathbb{Z}$-graded algebras. The *Segre product* of A and B is the $\mathbb{Z}$-graded algebra defined by

$$A \circ B := \bigoplus_{i\in\mathbb{Z}} (A_i \otimes B_i).$$

It is well-known that if $X \subset \mathbb{P}^{n-1}$ and $Y \subset \mathbb{P}^{m-1}$ are projective varieties with the homogeneous coordinate rings A and B, respectively, then $A \circ B$ is the homogeneous coordinate ring for the image of $X \times Y$ in $\mathbb{P}^{nm-1}$ under the Segre embedding Φ.

Let us consider the simplest case. The Segre embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3; \quad ((a_1, a_2), (b_1, b_2)) \mapsto (a_1b_1, a_2b_1, a_1b_2, a_2b_2).$$

embeds $\mathbb{P}^1 \times \mathbb{P}^1$ as a smooth quadric surface $Q = V(XW - YZ)$ in $\mathbb{P}^3$, and the Segre product

$$k[u, v] \circ k[x, y] = k[X, Y, Z, W]/(XW - YZ)$$

is the homogeneous coordinate ring of Q, where $X = u \otimes x,\ Y = v \otimes x,\ Z = u \otimes y,\ W = v \otimes y$.

This is a summary of joint work with Ji-Wei He (Hangzhou Normal University). The detailed version of this paper was published as [4].

To develop the study of noncommutative quadric surfaces (in the sense of [3]), it is natural to consider noncommutative generalizations of the Segre product $k[u,v]\circ k[x,y]$. Since noetherian Koszul AS-regular algebras are considered as nice noncommutative generalizations of polynomial algebras in noncommutative algebraic geometry, one of the natural noncommutative generalizations of $k[u,v]\circ k[x,y]$ is to replace $k[u,v]$ and $k[x,y]$ by 2-dimensional noetherian Koszul AS-regular algebras. However, this is not so interesting in the following sense.

Proposition 1 ([5, Lemma 2.12])**.** *If C and D are 2-dimensional noetherian Koszul AS-regular algebras, then we have an equivalence* $\mathsf{GrMod}\, C\circ D \cong \mathsf{GrMod}\, k[x,y]\circ k[u,v]$.

To obtain a proper noncommutative generalization (up to equivalence of graded module categories), in this paper, we discuss the notion of twisted Segre product. In particular, we focus on the study of twisted Segre products of noetherian Koszul AS-regular algebras.

2. Twisted Segre products

In this section, we give the definition of a twisted Segre product.

Definition 2. Let A, B be $\mathbb{Z}$-graded algebras. A bijective linear map $\psi : B\otimes A\to A\otimes B$ is called a *twisting map* if

(1) $\psi(B_i\otimes A_j)\subset A_j\otimes B_i$ for all $i,j\in\mathbb{Z}$,
(2) $\psi(1\otimes a)=a\otimes 1$ for all $a\in A$,
(3) $\psi(b\otimes 1)=1\otimes b$ for all $b\in B$,
(4) the following diagrams commute:

$$\begin{array}{ccccc} B\otimes B\otimes A & \xrightarrow{\mathrm{id}_B\otimes\psi} & B\otimes A\otimes B & \xrightarrow{\psi\otimes\mathrm{id}_B} & A\otimes B\otimes B \\ \downarrow{\scriptstyle m_B\otimes\mathrm{id}_A} & & & & \downarrow{\scriptstyle \mathrm{id}_A\otimes m_B} \\ B\otimes A & & \xrightarrow{\quad\psi\quad} & & A\otimes B, \end{array}$$

$$\begin{array}{ccccc} B\otimes A\otimes A & \xrightarrow{\psi\otimes\mathrm{id}_A} & A\otimes B\otimes A & \xrightarrow{\mathrm{id}_A\otimes\psi} & A\otimes A\otimes B \\ \downarrow{\scriptstyle \mathrm{id}_B\otimes m_A} & & & & \downarrow{\scriptstyle m_A\otimes\mathrm{id}_B} \\ B\otimes A & & \xrightarrow{\quad\psi\quad} & & A\otimes B, \end{array}$$

where m_A and m_B are the multiplications of A and B, respectively.

Definition 3. Let A, B be $\mathbb{Z}$-graded algebras, and let $\psi : B\otimes A\to A\otimes B$ be a twisting map. Then the *twisted Segre product* of A and B with respect to ψ, denoted by $A\circ_\psi B$, is the $\mathbb{Z}$-graded algebra defined as follows:

- $A\circ_\psi B = A\circ B$ as a graded vector space,
- the multiplication of $A\circ_\psi B$ is defined by
$$(a\otimes b)(c\otimes d)=a\psi(b\otimes c)d$$
for $a\in A_n, c\in A_m, b\in B_n, d\in B_m$.

Note that this definition is well-defined. If ψ is the flip map, i.e., $\psi(b\otimes a)=a\otimes b$ for all $a\in A$ and $b\in B$, then $A\circ_\psi B$ is the usual Segre product of A and B.

Remark 4. Let A and B be $\mathbb{Z}$-graded algebras, and let $\psi : B \otimes A \to A \otimes B$ be a twisting map. Write $A \otimes_\psi B$ for the graded vector space $A \otimes B$ with the multiplication defined by $(a \otimes b)(c \otimes d) = a\psi(b \otimes c)d$ for $a, c \in A$ and $b, d \in B$. By [2, Theorem 2.5], $A \otimes_\psi B$ is also a $\mathbb{Z}$-graded algebra. We say that $A \otimes_\psi B$ is a *twisted tensor product* of A and B. It is easy to see that $A \circ_\psi B$ is an ungraded subalgebra of $A \otimes_\psi B$.

3. Dade's theorem for densely bigraded algebras

In this section, as a preparation for discussing twisted Segre products, we give a version of Dade's theorem for densely bigraded algebras.

Definition 5. Let $S = \bigoplus_{i \in \mathbb{Z}} S_i$ be a $\mathbb{Z}$-graded algebra.

(1) S is called a *strongly graded algebra* if $S_{i+j} = S_i S_j$ for all $i, j \in \mathbb{Z}$.
(2) S is called a *densely graded algebra* if $\dim_k(S_{i+j}/S_i S_j) < \infty$ for all $i, j \in \mathbb{Z}$.

Clearly a strongly graded algebra is a densely graded algebra. If S is a strongly graded algebra, then

$$(-)_0 : \mathsf{GrMod}\, S \to \mathsf{Mod}\, S_0; \qquad M \mapsto M_0$$

is an equivalence. This is well-known as Dade's theorem.

Let $S = \bigoplus_{i,j \in \mathbb{Z}} S_{(i,j)}$ be a $\mathbb{Z}^2$-bigraded algebra. Let $S_i = \bigoplus_{j \in \mathbb{Z}} S_{(i,j)}$ for all $i \in \mathbb{Z}$. Then $S_{\mathbb{Z}} := \bigoplus_{i \in \mathbb{Z}} S_i$ is a $\mathbb{Z}$-graded algebra. Note that S and $S_{\mathbb{Z}}$ are the same as ungraded algebras. However, they have different gradings.

Definition 6. Let $S = \bigoplus_{i,j \in \mathbb{Z}} S_{(i,j)}$ be a $\mathbb{Z}^2$-bigraded algebra. Then S is called a *densely bigraded algebra* if $S_{\mathbb{Z}}$ is a densely graded algebra.

Definition 7. Let S be a noetherian $\mathbb{Z}$-graded algebra.

(1) An element $m \in M \in \mathsf{GrMod}\, S$ is called *torsion* if $\dim_k mS < \infty$.
(2) A graded module $M \in \mathsf{GrMod}\, S$ is called *torsion* if every homogeneous element of M is torsion.
(3) Let $\mathsf{Tors}\, S$ be the full subcategory of $\mathsf{GrMod}\, S$ consisting of torsion modules.
(4) Let $\mathsf{QGr}\, S$ be the Serre quotient category $\mathsf{GrMod}\, S/\,\mathsf{Tors}\, S$.
(5) Let $\mathsf{grmod}\, S$ be the full subcategory of $\mathsf{GrMod}\, S$ consisting of finitely generated modules.
(6) Let $\mathsf{tors}\, S = \mathsf{grmod}\, S \cap \mathsf{Tors}\, S$.
(7) Let $\mathsf{qgr}\, S$ be the Serre quotient category $\mathsf{grmod}\, S/\,\mathsf{tors}\, S$.

The categories $\mathsf{QGr}\, S$ and $\mathsf{qgr}\, S$ are often called *noncommutative projective schemes* and play an important role in noncommutative algebraic geometry; see [1].

Definition 8. Let S be a densely bigraded algebra satisfying the conditions

(D1) $S_0 := \bigoplus_{j \in \mathbb{Z}} S_{(0,j)}$ is a noetherian $\mathbb{Z}$-graded algebra, and
(D2) $S_i := \bigoplus_{j \in \mathbb{Z}} S_{(i,j)}$ is a finitely generated graded right and left S_0-module for every $i \in \mathbb{Z}$.

We use the following terminology and notation:

(1) Let $\mathsf{BiGrMod}\, S$ be the category of bigraded right S-modules.
(2) An element $m \in M \in \mathsf{BiGrMod}\, S$ is called *locally torsion* if $\dim_k mS_0 < \infty$.

(3) A bigraded module $M \in \mathsf{BiGrMod}\, S$ is called *locally torsion* if every homogeneous element of M is locally torsion.
(4) Let $\mathsf{BiLTors}\, S$ be the full subcategory of $\mathsf{BiGrMod}\, S$ consisting of locally torsion modules.
(5) Let $\mathsf{QBiGr_L}\, S$ be the Serre quotient category $\mathsf{BiGrMod}\, S/ \mathsf{BiLTors}\, S$.
(6) Let $\mathsf{lbigrmod}\, S$ be the full subcategory of $\mathsf{BiGrMod}\, S$ consisting of bigraded right S-modules $M = \bigoplus_{i,j\in\mathbb{Z}} M_{(i,j)}$ such that $M_i := \bigoplus_{j\in\mathbb{Z}} M_{(i,j)}$ is finitely generated as a graded right S_0-module.
(7) Let $\mathsf{lbiltors}\, S = \mathsf{lbigrmod}\, S \cap \mathsf{BiLTors}\, S$.
(8) Let $\mathsf{qlbigr_l}\, S$ be the Serre quotient category $\mathsf{lbigrmod}\, S/ \mathsf{lbiltors}\, S$.

Then we have the following theorem.

Theorem 9 ([4, Theorem 4.10])**.** *Let S be a densely bigraded algebra satisfying the conditions* (D1) *and* (D2) *in Definition 8. Then the functor*

$$(-)_0 : \mathsf{BiGrMod}\, S \to \mathsf{GrMod}\, S_0; \qquad M \mapsto M_0 \ (= \bigoplus_{j\in\mathbb{Z}} M_{(0,j)})$$

induces equivalences

$$\mathsf{QBiGr_L}\, S \cong \mathsf{QGr}\, S_0 \quad \textit{and} \quad \mathsf{qlbigr_l}\, S \cong \mathsf{qgr}\, S_0.$$

This theorem can be regarded as a version of Dade's theorem for densely bigraded algebras.

4. Twisted Segre products of noetherian Koszul AS-regular algebras

In this section, we study twisted Segre products of noetherian Koszul AS-regular algebras.

Definition 10. Let $A = \bigoplus_{i\in\mathbb{N}} A_i$ be a noetherian $\mathbb{N}$-graded algebra with $A_0 = k$.
(1) A is called *an AS-Gorenstein algebra* of dimension d if
  (a) $\operatorname{injdim}_A A = \operatorname{injdim}_{A^{\mathrm{op}}} A = d < \infty$, and
  (b) $\operatorname{Ext}^i_A(k, A) \cong \operatorname{Ext}^i_{A^{\mathrm{op}}}(k, A) \cong \begin{cases} k(\ell) \text{ for some } \ell \in \mathbb{Z} & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$
(2) A is called *an AS-regular algebra* if A is an AS-Gorenstein algebra of $\operatorname{gldim} A = d$.
(3) A is called *Koszul* if $k \in \mathsf{GrMod}\, A$ has a free resolution

$$\cdots \to A(-3)^{r_3} \to A(-2)^{r_2} \to A(-1)^{r_1} \to A \to k \to 0.$$

Let A and B be noetherian Koszul AS-regular algebras with a twisting map $\psi : B\otimes A \to A \otimes B$. Let $S = A \otimes_\psi B$ be the twisted tensor product; see Remark 4. Here we endow S with a bigraded structure as follows:

$$S_{(i,j)} = A_{i+j} \otimes B_j$$

for $i, j \in \mathbb{Z}$. Note that $S_0 = \bigoplus_{j\in\mathbb{Z}} S_{(0,j)}$ is equal to the twisted Segre product $A \circ_\psi B$ as graded algebras.

Proposition 11 ([4, Proposition 4.11]). *Let A and B be noetherian Koszul AS-regular algebras with a twisting map $\psi : B \otimes A \to A \otimes B$. Then $A \otimes_\psi B$ is a densely bigraded algebra satisfying the condition* (D2) *in Definition 8.*

By Proposition 11, if $A \circ_\psi B$ is noetherian, then $A \otimes_\psi B$ is a densely bigraded algebra satisfying the conditions (D1) and (D2) in Definition 8. Therefore, Theorem 9 yields the following result.

Theorem 12 ([4, Theorem 4.13]). *Let A and B be noetherian Koszul AS-regular algebras with a twisting map $\psi : B \otimes A \to A \otimes B$. Assume that $A \circ_\psi B$ is noetherian. Then there exist equivalences*

$$\mathsf{QBiGr_L}\, A \otimes_\psi B \cong \mathsf{QGr}\, A \circ_\psi B \quad \textit{and} \quad \mathsf{qlbigr_l}\, A \otimes_\psi B \cong \mathsf{qgr}\, A \circ_\psi B.$$

By [6, Theorem 2], it follows that $A \otimes_\psi B$ has finite global dimension, so we see that $\mathsf{qlbigr_l}\, A \otimes_\psi B$ in Theorem 12 has finite global dimension. Thus, we obtain the following consequence.

Corollary 13 ([4, Theorem 4.16]). *Let A and B be noetherian Koszul AS-regular algebras with a twisting map $\psi : B \otimes A \to A \otimes B$. Assume that $A \circ_\psi B$ is noetherian. Then $\mathsf{qgr}\, A \circ_\psi B$ has finite global dimension.*

The above corollary can be regarded as a noncommutative analogue of the fact that $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ is smooth.

5. Twisted Segre products of $k[u, v]$ and $k[x, y]$

Let $A = k[u, v], B = k[x, y]$ be standard graded polynomial rings in two variables. In this section, we consider certain twisted Segre products of $A = k[u, v]$ and $B = k[x, y]$. If $\psi : B \otimes A \to A \otimes B$ is a twisting map, then

$$\begin{aligned} \psi(x \otimes \begin{pmatrix} u \\ v \end{pmatrix}) &= N_{11} \begin{pmatrix} u \\ v \end{pmatrix} \otimes x + N_{12} \begin{pmatrix} u \\ v \end{pmatrix} \otimes y, \\ \psi(y \otimes \begin{pmatrix} u \\ v \end{pmatrix}) &= N_{21} \begin{pmatrix} u \\ v \end{pmatrix} \otimes x + N_{22} \begin{pmatrix} u \\ v \end{pmatrix} \otimes y, \end{aligned} \tag{5.1}$$

where $N_{11}, N_{12}, N_{21}, N_{22} \in M_2(k)$.

Definition 14. Let $A = k[u, v], B = k[x, y]$ be standard graded polynomial rings in two variables. A twisting map $\psi : B \otimes A \to A \otimes B$ is called *diagonal* if $N_{12} = N_{21} = 0$ in (5.1).

Then we have the following theorem.

Theorem 15 ([4, Theorem 6.4, Corollary 6.12]). *Let $A = k[u, v], B = k[x, y]$ be standard graded polynomial rings in two variables, and let $\psi : B \otimes A \to A \otimes B$ be a diagonal twisting map. Then the following statements hold.*

(1) *The twisted Segre product $A \circ_\psi B$ is a noncommutative quadric surface, i.e., there exist a 4-dimensional noetherian Koszul AS-regular algebra S with Hilbert series $H_S(t) = (1 - t)^{-4}$ and a regular normal homogeneous element $f \in S$ of degree 2 such that $A \circ_\psi B \cong S/(f)$. In particular, $A \circ_\psi B$ is a 3-dimensional noetherian*

Koszul AS-Gorenstein algebra, and hence $\mathsf{qgr}\, A \circ_\psi B$ *has finite global dimension (by Corollary 13).*

(2) *There exists an equivalence of triangulated categories*

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(A \circ_\psi B) \cong \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, k \times k),$$

where $\underline{\mathsf{CM}}^{\mathbb{Z}}(A \circ_\psi B)$ *is the stable category of graded maximal Cohen-Macaulay modules over* $A \circ_\psi B$ *and* $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, k \times k)$ *is the bounded derived category of finite dimensional modules over* $k \times k$*.*

It turns out that a twisted Segre product $A \circ_\psi B$ appearing in the above theorem has a nice property similar to the usual Segre product $A \circ B = k[u, v] \circ k[x, y] = k[X, Y, Z, W]/(XW - YZ)$, which is the homogeneous coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$. On the other hand, one can check that there exists a twisted Segre product $A \circ_\psi B$ such that $\mathsf{GrMod}\, A \circ_\psi B \not\cong \mathsf{GrMod}\, A \circ B = \mathsf{GrMod}\, k[u, v] \circ k[x, y]$. This is different from the situation in Proposition 1.

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