CHARACTERIZATION OF EVENTUALLY PERIODIC MODULES AND ITS APPLICATIONS

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Abstract. For a left Noetherian ring, its singularity category makes only the modules of finite projective dimension vanish among the modules. Thus the singularity categories are expected to characterize homological properties of modules with infinite projective dimension. In this paper, we focus on eventually periodic modules over a left artin ring, and we characterize them in terms of morphisms in the singularity category. As applications, we prove that, for the class of finite dimensional algebras over a field, being eventually periodic is preserved under singular equivalence of Morita type with level. Moreover, we determine which finite dimensional connected Nakayama algebras are eventually periodic when the ground field is algebraically closed.

1. Introduction

Throughout this paper, let k be a field, and we assume that all rings are associative and unital. By a module, we mean a left module.

The *singularity category* $\mathcal{D}_{\mathrm{sg}}(R)$ of a left Noetherian ring R is defined to be the Verdier quotient of the bounded derived category $\mathcal{D}^{\mathrm{b}}(R\text{-mod})$ of finitely generated R-modules by the full subcategory of perfect complexes (see [2]), and it provides a homological measure of singularity of finitely generated R-modules M in the following sense: M has finite projective dimension if and only if M is isomorphic to 0 in $\mathcal{D}_{\mathrm{sg}}(R)$. From this point of view, it is expected that the singularity categories capture homological properties of modules of infinite projective dimension.

In this paper, we work with left artin rings and focus on *eventually periodic modules*, that is, finitely generated modules whose minimal projective resolutions have infinite length and eventually become periodic. Recently, the author has proved in [12, Proposition 3.4] that a finitely generated Λ-module M is eventually periodic if and only if there exists an invertible homogeneous element of positive degree in the Tate cohomology ring of M, where Λ is a finite dimensional Gorenstein algebra over an algebraically closed field.

We first give our main result, showing that the above result also holds for any left artin rings. We then state two applications: the first is that being eventually periodic is invariant under singular equivalence of Morita type with level for the class of finite dimensional k-algebras; the second is that, when the ground field k is algebraically closed, connected Nakayama algebras that are eventually periodic are precisely those of infinite global dimension.

The detailed version of this paper will be submitted for publication elsewhere.

2. Tate cohomology rings

In this subsection, we recall from [2, 14] the definition of Tate cohomology and some related facts.

Definition 1 ([2, Definition 6.1.1] and [14, page 11])**.** Let i be an integer.

(1) Let R be a left Noetherian ring and M and N two finitely generated R-modules. We define the *i-th Tate cohomology group of M with coefficients in N* by

$$\widehat{\mathrm{Ext}}_R^i(M, N) := \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(R)}(M, N[i]).$$

(2) Let Λ be a finite dimensional k-algebra. The *i-th Tate-Hochschild cohomology group* of Λ is defined by $\widehat{\mathrm{HH}}^i(\Lambda) := \widehat{\mathrm{Ext}}_{\Lambda^{\mathrm{e}}}^i(\Lambda, \Lambda)$, where we put $\Lambda^{\mathrm{e}} := \Lambda \otimes_k \Lambda^{\mathrm{op}}$.

The graded abelian group

$$\widehat{\mathrm{Ext}}_R^\bullet(M, M) := \bigoplus_{i \in \mathbb{Z}} \widehat{\mathrm{Ext}}_R^i(M, M)$$

carries a structure of a graded ring given by the *Yoneda product*. We call this graded ring the *Tate cohomology ring* of M. It follows from the definition of singularity categories that $\widehat{\mathrm{Ext}}_R^\bullet(M, M)$ is the zero ring if and only if the projective dimension of M is finite.

The Tate cohomology ring $\widehat{\mathrm{Ext}}_{\Lambda^{\mathrm{e}}}^\bullet(\Lambda, \Lambda)$ for a finite dimensional k-algebra Λ is called the *Tate-Hochschild cohomology ring* of Λ and denoted it by $\widehat{\mathrm{HH}}^\bullet(\Lambda)$. It was proved by Wang [14, Proposition 4.7] that $\widehat{\mathrm{HH}}^\bullet(\Lambda)$ is graded commutative.

3. Main result

In this section, we give our main result. Let R be a left artin ring. We denote by $\Omega_R^n(M)$ the n-th syzygy of a finitely generated R-module M. Recall that a finitely generated R-module M is called *periodic* if $\Omega_R^p(M) \cong M$ as R-modules for some $p > 0$. The least such p is called the *period* of M. We say that M is *eventually periodic* if $\Omega_R^n(M)$ is non-zero and periodic for some $n \geq 0$. Remark that our eventually periodic modules are those of infinite projective dimension in the sense ever before. Following Küpper [7], we define *eventually periodic algebras* as finite dimensional algebras that are eventually periodic as regular bimodules.

We now characterize the existence of invertible homogeneous elements in Tate cohomology rings. In the following result, Ω_R denotes the loop space functor on the stable module category R-$\underline{\mathrm{mod}}$.

Theorem 2. *Let R be a left Noetherian ring, and let p be a positive integer. Then the following conditions are equivalent for a finitely generated R-module M.*

(1) *There exists an isomorphism $\Omega_R^{n+p}(M) \cong \Omega_R^n(M)$ in R-$\underline{\mathrm{mod}}$ for some $n \geq 0$.*

(2) *The Tate cohomology ring $\widehat{\mathrm{Ext}}_R^\bullet(M, M)$ has an invertible homogeneous element of degree p.*

The main result of this paper can be obtained from the above theorem.

Corollary 3. *Let R be a left artin ring. Then the following conditions are equivalent for a finitely generated R-module M.*

(1) *M is eventually periodic.*
(2) *The Tate cohomology ring of M has a non-zero invertible homogeneous element of positive degree.*

In this case, there exists an invertible homogeneous element in the Tate cohomology ring of M whose degree equals the period of some periodic syzygy $\Omega_R^n(M)$ with $n \geq 0$.

Remark 4. For a left Noetherian local ring R, one can define eventually periodic R-modules by using minimal free resolutions. Then Theorem 2 enables us to characterize eventually periodic R-modules as in Corollary 3.

We end this section by giving a result on Tate-Hochschild cohomology rings, which will be used in the next section.

Proposition 5. *Let Λ be a finite dimensional k-algebra. Then Λ is eventually periodic if and only if $\widehat{\mathrm{HH}}^{\bullet}(\Lambda)$ has a non-zero invertible homogeneous element of positive degree. In this case, there exists an isomorphism of graded rings*

$$\widehat{\mathrm{HH}}^{\bullet}(\Lambda) \cong \widehat{\mathrm{HH}}^{\geq 0}(\Lambda)[\chi^{-1}],$$

where we set $\widehat{\mathrm{HH}}^{\geq 0}(\Lambda) := \bigoplus_{i \geq 0} \widehat{\mathrm{HH}}^{i}(\Lambda)$, and the degree of the invertible homogeneous element χ is equal to the period of some periodic syzygy $\Omega_{\Lambda^{\mathrm{e}}}^n(\Lambda)$ with $n \geq 0$.

Note that the proposition generalizes [4, Corollary 6.4] and [12, Theorem 3.5] (in a proper sense).

4. Applications

There are two aims of this section. The first is to show that being eventually periodic is preserved under singular equivalence of Morita type with level. The second is to give a criterion for a Nakayama algebra to be eventually periodic. Throughout this section, algebras will mean finite dimensional k-algebras.

4.1. **Eventually periodic algebras and singular equivalences of Morita type with level.**

This subsection is devoted to showing that singular equivalences of Morita type with level preserve eventual periodicity of algebras. We start with the definition of *singular equivalences of Morita type with level.*

Definition 6 ([13, Definition 2.1])**.** Let Λ and Γ be two algebras, and let $l \geq 0$ be an integer. We say that a pair $({}_{\Lambda}M_{\Gamma}, {}_{\Gamma}N_{\Lambda})$ of bimodules *defines a singular equivalence of Morita type with level l* (and that Λ and Γ are *singularly equivalent of Morita type with level l*) if the following conditions are satisfied.

(1) The one-sided modules ${}_{\Lambda}M, M_{\Gamma}, {}_{\Gamma}N$ and N_{Λ} are finitely generated and projective.
(2) There exist isomorphisms $M \otimes_{\Gamma} N \cong \Omega_{\Lambda^{\mathrm{e}}}^{l}(\Lambda)$ and $N \otimes_{\Lambda} M \cong \Omega_{\Gamma^{\mathrm{e}}}^{l}(\Gamma)$ in Λ^{e}-$\underline{\mathrm{mod}}$ and Γ^{e}-$\underline{\mathrm{mod}}$, respectively.

A lot of invariants under singular equivalence of Morita type with level have been discovered by Skartsæterhagen [11], Qin [8] and Wang [13, 15]. We now give a new invariant, using Corollary 5 and Wang's result [15].

Theorem 7. *Assume that* Λ *and* Γ *are singularly equivalent of Morita type with level. If* Λ *is eventually periodic, then so is* Γ*. In particular, the periods of their periodic syzygies coincide.*

Recall that two algebras Λ and Γ are *derived equivalent* if there exists a triangle equivalence between $\mathcal{D}^{\mathrm{b}}(\Lambda\text{-mod})$ and $\mathcal{D}^{\mathrm{b}}(\Gamma\text{-mod})$. It was proved by Wang [13, Theorem 2.3] that any two derived equivalent algebras are singularly equivalent of Morita type with level. Thus Theorem 7 generalizes a result of Erdmann and Skowroński [6, Theorem 2.9].

We end this subsection with examples of eventually periodic algebras. Note that Γ and Σ below can be found in [3, Example 4.3 (2)] and [12, Example 3.2 (1)], respectively.

Example 8. Let Λ, Γ and Σ be the k-algebras given by the following quivers with relations

$$\alpha \circlearrowright 1 \qquad\qquad \alpha^2 = 0,$$

$$\alpha \circlearrowright 1 \xrightarrow{\beta} 2 \qquad\qquad \alpha^2 = 0 = \beta\alpha$$

and

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \qquad\qquad \alpha\beta\alpha = 0,$$

respectively. Λ is a self-injective Nakayama and hence a periodic algebra (see [5, Lemma in Section 4.2]). Note that Λ has period 1 if the characteristic of k is 2 and 2 otherwise. By [11, Example 7.5], Λ and Γ are singularly equivalent of Morita type with level 1, so that Γ is eventually periodic by Theorem 7. Also, using the APR-tiling Γ-module corresponding to the vertex 2, one sees that Γ is derived equivalent to Σ, and hence Σ is eventually periodic by Theorem 7 again. On the other hand, a direct computation shows that $\Omega^3_{\Gamma^{\mathrm{e}}}(\Gamma)$ and $\Omega^2_{\Sigma^{\mathrm{e}}}(\Sigma)$ are the first periodic syzygies. Thus we conclude from Theorem 7 that the periodic bimodules Λ, $\Omega^3_{\Gamma^{\mathrm{e}}}(\Gamma)$ and $\Omega^2_{\Sigma^{\mathrm{e}}}(\Sigma)$ have the same period.

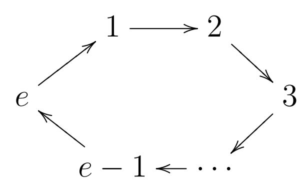
4.2. **Eventually periodic Nakayama algebras.** Throughout, we suppose that the field k is algebraically closed. Then the global dimension of an algebra is equal to the projective dimension of the algebra as a regular bimodule over itself. We denote by $\mathcal{CN}$ the class of connected Nakayama algebras. The aim of this subsection is to decide which algebras from $\mathcal{CN}$ are eventually periodic.

Let Λ be in $\mathcal{CN}$ and $J(\Lambda)$ its Jacobson radical. Then Λ is Morita equivalent to a bound quiver algebra whose ordinary quiver is given by either

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow e-1 \longrightarrow e$$

or

Z_e :



where $e \geq 1$. Note that the global dimension of Λ is finite in the first case. Moreover, Λ is non-simple and self-injective if and only if it is Morita equivalent to the bound quiver algebra kZ_e/R^N for some $e \geq 1$ and $N \geq 2$, where R denotes the arrow ideal of the path algebra kZ_e.

Using results of Asashiba [1], Qin [8] and Shen [9, 10], we can classify $\mathcal{CN}$ up to singular equivalence of Morita type with level. We note that $\mathcal{CN}$ is not closed under singular equivalence of Morita type with level (see Example 8).

Theorem 9. *The algebras k and kZ_e/R^N with $e \geq 1$ and $N \geq 2$ form a complete set of representatives of pairwise different equivalence classes of finite dimensional connected Nakayama k-algebras under singular equivalence of Morita type with level.*

It was proved by Erdmann and Holm [5, Lemma in Section 4.2] that kZ_e/R^N is periodic for all $e \geq 1$ and $N \geq 2$. Hence we obtain the following consequence of Theorems 7 and 9, which is the main result of this subsection.

Corollary 10. *Let Λ be a connected Nakayama algebra. Then Λ is eventually periodic if and only if the global dimension of Λ is infinite.*

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