Proceedings of the 54th Symposium on Ring Theory and Representation Theory

September 6 (Tue.) – 9 (Fri.), 2022
Saitama University, Japan

Edited by
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Tokyo University of Science

February, 2023
Tokyo, JAPAN

第 54 回 環論および表現論シンポジウム報告集

2022 年 9 月 6 日 (火) – 9 日 (金)
埼玉大学

編集: 切刀直子 (東京理科大学)

2023 年 2 月
東京理科大学

Organizing Committee of The Symposium on Ring Theory and Representation Theory

The Symposium on Ring Theory and Representation Theory has been held annually in Japan and the Proceedings have been published by the organizing committee. The first Symposium was organized in 1968 by H. Tominaga, H. Tachikawa, M. Harada and S. Endo. After their retirement a new committee has been formed in 1997 to manage the Symposium, and its committee members are listed in the web page

http://www.ring-theory-japan.com/ring/h-of-ringsymp.html.

The present members of the committee are S. Kawata (Nagoya City Univ.), I. Kikumasa (Yamaguchi Univ.), I. Mori (Shizuoka Univ.), and T. Nishinaka (Univ. of Hyogo).

For information on ring theory and representation theory of groups and algebras (including schedules of meetings and symposiums), as well as ring mailing list service for registered members, please refer to the following websites, which are edited by K. Yamaura (Univ. of Yamanashi):

http://www.ring-theory-japan.com/ring/ (in Japanese)

http://www.ring-theory-japan.com/ring/japan/ (in English)

The Symposium in 2023 will be held at Osaka Metropolitan University from September 5 (Tue.) to September 8 (Fri.). The program organizer is K. Ueyama (Hirosaki Univ.), and the local organizer is R. Kanda (Osaka Metropolitan Univ.).

Shigeto Kawata
Nagoya, Japan
February, 2023

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Preface

The 54th Symposium on Ring Theory and Representation Theory was held at Saitama University on September 6th – 9th, 2022. The symposium and this proceedings are financially supported by

Akihiko Hida (Saitama University)
Grant-in-Aid for Scientific Research (C) 21K03154, JSPS
Shigeto Kawata (Nagoya City University)
Grant-in-Aid for Scientific Research (C) 19K03451, JSPS
Naoko Kunugi (Tokyo University of Science)
Grant-in-Aid for Scientific Research (C) 18K03255, JSPS
Izuru Mori (Shizuoka University)
Grant-in-Aid for Scientific Research (C) 20K03510, JSPS

This volume consists of the articles presented at the symposium. We would like to thank all speakers and coauthors for their contributions.

We would also like to express our thanks to all the members of the organizing committee for their helpful suggestions concerning the symposium. Finally we would like to express our gratitude to Local organizer, Professor Akihiko Hida, and students of Shizuoka University and Tokyo University of Science who contributed in the organization of the symposium.

Naoko Kunugi
Tokyo, Japan
February, 2023

第54回環論および表現論シンポジウム

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A BIJECTION BETWEEN SILTING SUBCATEGORIES AND BOUNDED HEREDITARY COTORSION PAIRS

TAKAHIDE ADACHI AND MAYU TSUKAMOTO

Abstract. In a triangulated category, there exists a bijection between silting subcategories and bounded co-t-structures. In this article, as a generalization of this result, we give a bijection between silting subcategories and bounded hereditary cotorsion pairs in an extriangulated category. Moreover, we prove that our result recovers a bijection between basic tilting modules and contravariantly finite resolving subcategories for a finite dimensional algebra with finite global dimension.

Throughout this article, we assume that every category is skeletally small, that is, the isomorphism classes of objects form a set. In addition, all subcategories are assumed to be full and closed under isomorphisms.

The notion of silting subcategories was firstly introduced by Keller and Vossieck [5].

Definition 1. Let $\mathcal{D}$ be a triangulated category with shift functor Σ. A subcategory $\mathcal{M}$ of $\mathcal{D}$ is called a *silting subcategory* if it satisfies the following conditions.

- $\mathcal{M}$ is closed under direct summands.
- $\mathcal{D}(\mathcal{M}, \Sigma^k\mathcal{M}) = 0$ for each $k \geq 1$.
- $\mathcal{D} = \mathsf{thick}\mathcal{M}$, where $\mathsf{thick}\mathcal{M}$ is the smallest thick subcategory containing $\mathcal{M}$.

Bondarko ([3]) and Pauksztello ([8]) independently introduced co-t-structures as an analog of t-structures.

Definition 2. Let $\mathcal{D}$ be a triangulated category with shift functor Σ. A pair $(\mathcal{U}, \mathcal{V})$ of subcategories of $\mathcal{D}$ is called a *co-t-structure* on $\mathcal{D}$ if it satisfies the following conditions.

- $\mathcal{U}$ and $\mathcal{V}$ are closed under direct summands.
- For each $D \in \mathcal{D}$, there exists a triangle $\Sigma^{-1}U \to D \to V \to U$ such that $U \in \mathcal{U}$ and $V \in \mathcal{V}$.
- $\mathcal{D}(\Sigma^{-1}\mathcal{U}, \mathcal{V}) = 0$.
- $\mathcal{U}$ is closed under a negative shift, that is, $\Sigma^{-1}\mathcal{U} \subseteq \mathcal{U}$.

A co-t-structure $(\mathcal{U}, \mathcal{V})$ on $\mathcal{D}$ is said to be *bounded* if $\cup_{n\in\mathbb{Z}}\Sigma^n\mathcal{U} = \mathcal{D} = \cup_{n\in\mathbb{Z}}\Sigma^n\mathcal{V}$.

Bondarko ([3]) and Mendoza–Santiago–Sáenz–Souto ([6]) gave the following result.

Theorem 3 ([3, 6])**.** *Let $\mathcal{D}$ be a triangulated category. Then there exist mutually inverse bijections between the set of silting subcategories of $\mathcal{D}$ and the set of bounded co-t-structures on $\mathcal{D}$.*

The detailed version of this article has been published in [1].

The aim of this article is to generalize Theorem 3 to extriangulated categories introduced by Nakaoka and Palu ([7]) as a simultaneous generalization of a triangulated category and an exact category.

Let R be a commutative ring and let $\mathcal{C} := (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an R-linear extriangulated category. For definition and terminologies of extriangulated categories, see [7, 4]. A complex $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$ is called an $\mathfrak{s}$-*conflation* if there exists $\delta \in \mathbb{E}(C, A)$ such that $\mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{g} C]$, where $[A \xrightarrow{f} B \xrightarrow{g} C]$ is an equivalence class of a complex $A \xrightarrow{f} B \xrightarrow{g} C$. We write the $\mathfrak{s}$-conflation as $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$. Recently, Gorsky, Nakaoka and Palu ([4]) gave an R-bilinear functor $\mathbb{E}^n : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Mod}\,R$ for each $n \geq 2$. We recall examples of extriangulated categories (for detail, see [7, 4])

Example 4. (1) Let $\mathcal{D}$ be a triangulated category with shift functor Σ. Then $\mathcal{D}$ becomes an extriangulated category by the following data.
- $\mathbb{E}(C, A) := \mathcal{D}(C, \Sigma A)$ for all $A, C \in \mathcal{D}$.
- For $\delta \in \mathbb{E}(C, A)$, we take a triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\delta} \Sigma A$. Then we define $\mathfrak{s}(\delta) := [A \xrightarrow{f} B \xrightarrow{g} C]$.

In this case, we have $\mathbb{E}^k(C, A) = \mathcal{D}(C, \Sigma^k A)$ for all $A, C \in \mathcal{D}$ and $k \geq 1$.

(2) Let $\mathcal{E}$ be an exact category. Then $\mathcal{E}$ becomes an extriangulated category by the following data.
- $\mathbb{E}(C, A) := \mathrm{Ext}^1_{\mathcal{E}}(C, A)$, where $\mathrm{Ext}^1_{\mathcal{E}}(C, A)$ is the set of isomorphism classes of conflations in $\mathcal{E}$ of the form $0 \to A \to B \to C \to 0$ for $A, C \in \mathcal{E}$.
- $\mathfrak{s}$ is the identity.

In this case, we have $\mathbb{E}^k(C, A) = \mathrm{Ext}^k_{\mathcal{E}}(C, A)$ for all $A, C \in \mathcal{D}$ and $k \geq 1$.

For a subcategory $\mathcal{X}$ of $\mathcal{C}$, we define a subcategory ${}^{\perp}\mathcal{X}$ as

$$ {}^{\perp}\mathcal{X} := \{M \in \mathcal{C} \mid \mathbb{E}^k(M, \mathcal{X}) = 0 \text{ for each } k \geq 1\}. $$

Dually, we define a subcategory $\mathcal{X}^{\perp}$. Moreover, the following subcategories play a crucial role in this article.

Definition 5. Let $\mathcal{X}, \mathcal{Y}$ be subcategories of $\mathcal{C}$.

(1) Let $\mathcal{X} * \mathcal{Y}$ denote the subcategory of $\mathcal{C}$ consisting of $M \in \mathcal{C}$ which admits an $\mathfrak{s}$-conflation $X \to M \to Y \dashrightarrow$ in $\mathcal{C}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We say that $\mathcal{X}$ *is closed under extensions* if $\mathcal{X} * \mathcal{X} \subseteq \mathcal{X}$.

(2) Let $\mathrm{Cone}(\mathcal{X}, \mathcal{Y})$ denote the subcategory of $\mathcal{C}$ consisting of $M \in \mathcal{C}$ which admits an $\mathfrak{s}$-conflation $X \to Y \to M \dashrightarrow$ in $\mathcal{C}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We say that $\mathcal{X}$ *is closed under cones* if $\mathrm{Cone}(\mathcal{X}, \mathcal{X}) \subseteq \mathcal{X}$.

(3) Let $\mathrm{Cocone}(\mathcal{X}, \mathcal{Y})$ denote the subcategory of $\mathcal{C}$ consisting of $M \in \mathcal{C}$ which admits an $\mathfrak{s}$-conflation $M \to X \to Y \dashrightarrow$ in $\mathcal{C}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We say that $\mathcal{X}$ *is closed under cocones* if $\mathrm{Cocone}(\mathcal{X}, \mathcal{X}) \subseteq \mathcal{X}$.

(4) We call $\mathcal{X}$ a *thick subcategory* of $\mathcal{C}$ if it is closed under extensions, cones, cocones and direct summands. Let $\mathsf{thick}\mathcal{X}$ denote the smallest thick subcategory containing $\mathcal{X}$.

(5) For each $n \geq 0$, we inductively define subcategories $\mathcal{X}^{\wedge}_n$ and $\mathcal{X}^{\vee}_n$ of $\mathcal{C}$ as $\mathcal{X}^{\wedge}_n := \mathrm{Cone}(\mathcal{X}^{\wedge}_{n-1}, \mathcal{X})$ and $\mathcal{X}^{\vee}_n := \mathrm{Cocone}(\mathcal{X}, \mathcal{X}^{\vee}_{n-1})$, where $\mathcal{X}^{\wedge}_{-1} := \{0\}$ and $\mathcal{X}^{\vee}_{-1} := \{0\}$.

Put

$$\mathcal{X}^\wedge := \bigcup_{n\geq 0} \mathcal{X}_n^\wedge, \quad \mathcal{X}^\vee := \bigcup_{n\geq 0} \mathcal{X}_n^\vee.$$

When $\mathcal{C}$ is a triangulated category, descriptions of $\mathcal{X}^\wedge$ and $\mathcal{X}^\vee$ are well-known. Indeed, let $\mathcal{D}$ be a triangulated category (regarded as an extriangulated category) with shift functor Σ. For a subcategory $\mathcal{X}$ and an integer $n \geq 0$, we obtain

$$\mathcal{X}_n^\wedge = \mathcal{X} * \Sigma\mathcal{X} * \cdots * \Sigma^n\mathcal{X}.$$

If $\mathcal{X}$ is closed under extensions and a negative shift, then $\mathcal{X}_n^\wedge = \Sigma^n\mathcal{X}$ holds. Similarly, if $\mathcal{X}$ is closed under extensions and a positive shift, then $\mathcal{X}_n^\vee = \Sigma^{-n}\mathcal{X}$ holds.

We introduce the notion of silting subcategories of an extriangulated category, which is a generalization of silting subcategories of a triangulated category. For a class $\mathcal{X}$ of objects in $\mathcal{C}$, let $\mathsf{add}\mathcal{X}$ denote the smallest subcategory of $\mathcal{C}$ containing $\mathcal{X}$ and closed under finite direct sums and direct summands.

Definition 6. Let $\mathcal{C}$ be an extriangulated category and $\mathcal{M}$ a subcategory of $\mathcal{C}$. We call $\mathcal{M}$ a *silting subcategory* of $\mathcal{C}$ if it satisfies the following conditions.

(1) $\mathcal{M}$ is closed under direct summands.
(2) $\mathbb{E}^k(\mathcal{M}, \mathcal{M}) = 0$ for each $k \geq 1$.
(3) $\mathcal{C} = \mathsf{thick}\mathcal{M}$.

Let $\mathsf{silt}\mathcal{C}$ denote the set of all silting subcategories of $\mathcal{C}$. An object $M \in \mathcal{C}$ is called a *silting object* if $\mathsf{add}M$ is a silting subcategory of $\mathcal{C}$.

We give examples of silting subcategories.

Example 7. (1) Let $\mathcal{D}$ be a triangulated category. Then silting subcategories of a triangulated category $\mathcal{D}$ are exactly silting subcategories of an extriangulated category $\mathcal{D}$.
(2) Let A be an artin algebra and let $\mathcal{P}^{<\infty}(A)$ denote the category of finitely generated right A-modules of finite projective dimension. Since $\mathcal{P}^{<\infty}(A)$ is closed under extensions, it becomes an extriangulated category. We can check that silting objects of $\mathcal{P}^{<\infty}(A)$ coincide with tilting A-modules. Thus if A has finite global dimension, then silting objects of $\mathsf{mod}A$ coincide with tilting A-modules.

Example 8. Let $\mathbf{k}$ be an algebraically closed field. Consider the bounded derived category $\mathcal{D}$ of the path algebra $\mathbf{k}(1 \to 2 \to 3 \to 4)$. Then the Auslander–Reiten quiver of $\mathcal{D}$ is as follows.

$$
\begin{tikzcd}[column sep=small]
 & & & \begin{smallmatrix}1\\2\\3\\4\end{smallmatrix} \arrow[rr, dotted, no head] \arrow[dr] & & \Sigma 4 \arrow[rr, dotted, no head] \arrow[dr] & & \Sigma 3 \arrow[rr, dotted, no head] \arrow[dr] & & \Sigma 2 \\
 & & \begin{smallmatrix}2\\3\\4\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \Sigma \begin{smallmatrix}3\\4\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \Sigma \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] \arrow[dr] & \\
 & \begin{smallmatrix}3\\4\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \Sigma \begin{smallmatrix}2\\3\\4\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \Sigma \begin{smallmatrix}1\\2\\3\end{smallmatrix} \\
4 \arrow[ur] \arrow[rr, dotted, no head] & & 3 \arrow[ur] \arrow[rr, dotted, no head] & & 2 \arrow[ur] \arrow[rr, dotted, no head] & & 1 \arrow[ur] \arrow[rr, dotted, no head] & & \Sigma \begin{smallmatrix}1\\2\\3\\4\end{smallmatrix} \arrow[ur] &
\end{tikzcd}
$$

Let $\mathcal{X} := \mathsf{add}(\begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus 2 \oplus \Sigma 3)$. Since $\mathcal{X}$ is closed under extensions, it follows from [7, Remark 2.18] that $\mathcal{X}$ becomes an extriangulated category. Remark that $\mathcal{X}$ is neither an exact category nor a triangulated category. We can check that $\begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus \Sigma 3$ and $\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus 2 \oplus \Sigma 3$ are silting objects in $\mathcal{X}$.

We recall the definition of hereditary cotorsion pairs.

Definition 9. Let $\mathcal{C}$ be an extriangulated category and let $\mathcal{X}, \mathcal{Y}$ be subcategories of $\mathcal{C}$. We call a pair $(\mathcal{X}, \mathcal{Y})$ a *hereditary cotorsion pair* in $\mathcal{C}$ if it satisfies the following conditions.

(CP1) $\mathcal{X}$ and $\mathcal{Y}$ are closed under direct summands.
(CP2) $\mathbb{E}^k(\mathcal{X}, \mathcal{Y}) = 0$ for each $k \geq 1$.
(CP3) $\mathcal{C} = \mathrm{Cone}(\mathcal{Y}, \mathcal{X})$.
(CP4) $\mathcal{C} = \mathrm{Cocone}(\mathcal{Y}, \mathcal{X})$.

Let $\mathsf{hcotors}\,\mathcal{C}$ denote the set of hereditary cotorsion pairs in $\mathcal{C}$. For $(\mathcal{X}_1, \mathcal{Y}_1), (\mathcal{X}_2, \mathcal{Y}_2) \in \mathsf{hcotors}\,\mathcal{C}$, we write $(\mathcal{X}_1, \mathcal{Y}_1) \leq (\mathcal{X}_2, \mathcal{Y}_2)$ if $\mathcal{Y}_1 \subseteq \mathcal{Y}_2$. Then $(\mathsf{hcotors}\,\mathcal{C}, \leq)$ clearly becomes a partially ordered set. Remark that if $(\mathcal{X}, \mathcal{Y})$ is a hereditary cotorsion pair in $\mathcal{C}$, then $\mathcal{X}$ is closed under extensions and cocones. Similarly, $\mathcal{Y}$ is closed under extensions and cones.

The following examples show that the notion of hereditary cotorsion pairs in an extriangulated category is a common generalization of co-t-structures on a triangulated category and hereditary cotorsion pairs in an exact category.

Example 10. (1) Let $\mathcal{D}$ be a triangulated category with shift functor Σ. By regarding $\mathcal{D}$ as an extriangulated category, co-t-structures on $\mathcal{D}$ are exactly hereditary cotorsion pairs.

(2) Let $\mathcal{E}$ be an exact category. A pair $(\mathcal{X}, \mathcal{Y})$ of subcategories of $\mathcal{E}$ is called a *hereditary cotorsion pair* in $\mathcal{E}$ if it satisfies the following conditions.
- $\mathcal{X}$ and $\mathcal{Y}$ are closed under direct summands.
- $\mathrm{Ext}^k_{\mathcal{E}}(\mathcal{X}, \mathcal{Y}) = 0$ for each $k \geq 1$.
- For each $E \in \mathcal{E}$, there exists a conflation $0 \to Y_E \to X_E \to E \to 0$ such that $Y_E \in \mathcal{Y}$ and $X_E \in \mathcal{X}$.
- For each $E \in \mathcal{E}$, there exists a conflation $0 \to E \to Y^E \to X^E \to 0$ such that $Y^E \in \mathcal{Y}$ and $X^E \in \mathcal{X}$.

By regarding $\mathcal{E}$ as an extriangulated category, hereditary cotorsion pairs in the exact category $\mathcal{E}$ are exactly hereditary cotorsion pairs.

We say that a hereditary cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is *bounded* if $\mathcal{C} = \mathcal{X}^\wedge$ and $\mathcal{C} = \mathcal{Y}^\vee$. Let $\mathsf{bdd\text{-}hcotors}\,\mathcal{C}$ denote the partially ordered set of bounded hereditary cotorsion pairs in $\mathcal{C}$. The following theorem is a main result of this article.

Theorem 11 ([1, Theorem 5.7])**.** *Let $\mathcal{C}$ be an extriangulated category. Then there exist mutually inverse bijections*

$$\mathsf{bdd\text{-}hcotors}\,\mathcal{C} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathsf{silt}\mathcal{C},$$

where $\Phi(\mathcal{X}, \mathcal{Y}) := \mathcal{X} \cap \mathcal{Y}$ and $\Psi(\mathcal{M}) := (\mathcal{M}^\vee, \mathcal{M}^\wedge) = ({}^\perp\mathcal{M}, \mathcal{M}^\perp)$.

For a triangulated category $\mathcal{D}$, let $\mathsf{bdd\text{-}co\text{-}t\text{-}str}\,\mathcal{D}$ denote the set of bounded co-t-structures on $\mathcal{D}$. By regarding $\mathcal{D}$ as an extriangulated category, it follows from Example 10(1) that $\mathsf{bdd\text{-}co\text{-}t\text{-}str}\,\mathcal{D} = \mathsf{bdd\text{-}hcotors}\,\mathcal{D}$. Thus we can recover the following result by Theorem 11.

Corollary 12 ([6, Corollary 5.9])**.** *Let $\mathcal{D}$ be a triangulated category. Then there exist mutually inverse bijections*

$$\mathsf{bdd\text{-}co\text{-}t\text{-}str}\,\mathcal{D} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathsf{silt}\mathcal{D},$$

where $\Phi(\mathcal{X}, \mathcal{Y}) := \mathcal{X} \cap \mathcal{Y}$ and $\Psi(\mathcal{M}) := (\mathcal{M}^\vee, \mathcal{M}^\wedge)$.

For two subcategories $\mathcal{M}, \mathcal{N}$ of $\mathcal{C}$, we write $\mathcal{M} \geq \mathcal{N}$ if $\mathbb{E}^k(\mathcal{M}, \mathcal{N}) = 0$ for each $k \geq 1$. Since $\mathsf{bdd\text{-}hcotors}\,\mathcal{C}$ is a partially ordered set, the correspondence in Theorem 11 induces a partial order on $\mathsf{silt}\mathcal{C}$.

Corollary 13. *Let $\mathcal{M}, \mathcal{N}$ be silting subcategories of $\mathcal{C}$. Then $\mathcal{M} \geq \mathcal{N}$ if and only if $\mathcal{M}^\wedge \supseteq \mathcal{N}^\wedge$ holds. In particular, $\geq$ gives a partial order on $\mathsf{silt}\mathcal{C}$.*

In the following, we explain that Theorem 11 can recover Auslander–Reiten's result (see Corollary 14). Let $\mathsf{proj}\mathcal{C}$ denote the subcategory of $\mathcal{C}$ consisting of all projective objects in $\mathcal{C}$. We assume that an extriangulated category $\mathcal{C}$ is a Krull–Schmidt category, and has enough projective objects (i.e., $\mathcal{C} = \mathrm{Cone}(\mathcal{C}, \mathsf{proj}\mathcal{C})$) and enough injective objects. For a subcategory $\mathcal{X}$ of $\mathcal{C}$, we call $\mathcal{X}$ a resolving subcategory of $\mathcal{C}$ if $\mathsf{proj}\mathcal{C} \subseteq \mathcal{X}$ and it is closed under extensions, cocones and direct summands. Let $\mathsf{confin\text{-}resolv}\,\mathcal{C}$ denote the set of contravariantly finite resolving subcategories of $\mathcal{C}$. Then there exist mutually inverse bijections

$$\mathsf{hcotors}\,\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{confin\text{-}resolv}\,\mathcal{C},$$

where $F(\mathcal{X}, \mathcal{Y}) = \mathcal{X}$ and $G(\mathcal{X}) = (\mathcal{X}, \mathcal{X}^\perp)$. By restricting these bijections, we have

$$\mathsf{bdd\text{-}hcotors}\,\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \{\mathcal{X} \in \mathsf{confin\text{-}resolv}\,\mathcal{C} \mid \mathcal{X}^\wedge = \mathcal{C}, \mathcal{X} \subseteq (\mathsf{proj}\mathcal{C})^\wedge\}.$$

By Theorem 11, we have mutually inverse bijections

$$\mathsf{silt}\mathcal{C} \underset{\Phi\circ G}{\overset{F\circ\Psi}{\rightleftarrows}} \{\mathcal{X} \in \mathsf{confin\text{-}resolv}\,\mathcal{C} \mid \mathcal{X}^{\wedge} = \mathcal{C}, \mathcal{X} \subseteq (\mathsf{proj}\mathcal{C})^{\wedge}\}.$$

Let A be an artin algebra with finite global dimension. Applying these bijections to $\mathcal{C} = \mathsf{mod}\,A$, we obtain

$$\mathsf{silt}(\mathsf{mod}\,A) \underset{\Phi\circ G}{\overset{F\circ\Psi}{\rightleftarrows}} \mathsf{confin\text{-}resolv}(\mathsf{mod}\,A).$$

Moreover, it follows from Example 7(2) that silting objects of $\mathsf{mod}\,A$ coincide with tilting A-modules. Therefore we have the following result.

Corollary 14 ([2, Corollary 5.6])**.** *Let A be an artin algebra with finite global dimension. Then $T \mapsto {}^{\perp}T$ gives a bijection between the set of isomorphism classes of basic tilting modules and the set of contravariantly finite resolving subcategories, and $T \mapsto T^{\perp}$ gives a bijection between the set of isomorphism classes of basic tilting modules and the set of covariantly finite coresolving subcategories.*

References

[1] T. Adachi, M. Tsukamoto, *Hereditary cotorsion pairs and silting subcategories in extriangulated categories*, J. Algebra **594** (2022), 109–137.

[2] M. Auslander, I. Reiten, *Applications of contravariantly finite subcategories*, Adv. Math. **86** (1991), no. 1, 111–152.

[3] M. V. Bondarko, *Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general)*, J. K-Theory 6 (2010), no. 3, 387–504.

[4] M. Gorsky, H. Nakaoka, Y. Palu, *Positive and negative extensions in extriangulated categories*, arXiv:2103.12482.

[5] B. Keller, D. Vossieck, *Aisles in derived categories*, Bull. Soc. Math. Belg. Sér. A **40** (1988), no. 2, 239–253.

[6] O. Mendoza, V. Santiago, C. Sáenz, V. Souto, *Auslander–Buchweitz context and co-t-structures*, Appl. Categ. Structures **21** (2013), 417–440.

[7] H. Nakaoka, Y. Palu, *Extriangulated categories, Hovey twin cotorsion pairs and model structures*, Cah. Topol. Géom. Différ. Catég. **60** (2019), no. 2, 117–193.

[8] D. Pauksztello, *Compact corigid objects in triangulated categories and co-t-structures*, Cent. Eur. J. Math. 6 (2008), no. 1, 25–42.

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ON τ-TILTING FINITENESS OF GROUP ALGEBRAS

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Abstract. In this note, we explore when a group algebra is τ-tilting finite. One classifies τ-tilting finite group algebras with 2 simple modules and studies τ-tilting finite blocks of group algebras when the characteristic of the base field is 2.

1. Introduction

T-tilting theory has been introduced by Adachi–Iyama–Reiten [1] and is now one of the most important subjects in representation theory of algebras. In the theory, support τ-tilting modules play a central role and are mutatable; that is, we can make a new support τ-tilting module from a given one by replacing a direct summand. Moreover, the set of support τ-tilting modules admits a poset structure whose Hasse quiver coincides with the mutation quiver.

In this note, we discuss τ-tilting theory for group algebras, and attack the problem on "the structure of a group algebra A vs. that of the poset $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$". Here, $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ stands for the set of basic support τ-tilting modules of A.

2. Preliminaries

Let A be a finite dimensional symmetric algebra over an algebraically closed field k. Thanks to Adachi–Iyama–Reiten [1], we know that the theory of support τ-tilting modules and that of 2-term tilting complexes coincide. In this note, we use the latter. We denote by $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, A)$ the perfect derived category of A.

Let us first recall the definition of tilting complexes.

Definition 1. (1) A perfect complex T is said to be *tilting* if $\mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, A)}(T, T[i]) = 0$ for any $i \neq 0$ and it generates $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, A)$.
(2) We say that T is *2-term* provided it concentrates on degree 0 and -1.
(3) The set of basic 2-term tilting complexes of A is denoted by $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$.
(4) We call A *τ-tilting finite* if $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ is a finite set.

Remark 2. [1, Definition 0.3 and Theorem 3.2] A *support τ-tilting* module is defined to be the 0th cohomology of some 2-term tilting complex.

For perfect complexes T and U, we write $T \geq U$ if $\mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, A)}(T, U[i]) = 0$ for every $i > 0$. This actually gives a partial order on the set of basic tilting complexes [3, Theorem 2.11]. We utilize the 2-term version of this result.

Theorem 3. [1, Lemma 2.5] *The set $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ is a partially orderd set by the relation $\geq$.*

The detailed version of this paper will be submitted for publication elsewhere.

The partial order is compatible with tilting mutation, which is provided as follows:

- Let T be a tilting complex with decomposition $T = X \oplus M$. Taking a (minimal) left $\mathsf{add}\, M$-approximation $f : X \to M'$ of X, we get the new complex $\mu_X^-(T) := Y \oplus M$, where Y is the mapping cone of f.

Then, $\mu_X^-(T)$ is also tilting [3, Theorem 2.31], called the *left mutation* of T with respect to X. Dually, we have right mutations $\mu_X^+(T)$. Note that $\mu_P^\pm(A)$ are nothing but Okuyama–Rickard complexes [5], which means that tilting mutations can be obtained by repeatedly taking Okuyama–Rickard complexes (via derived equivalences).

Remark 4. The assumption of A being symmetric plays a crucial role to get tilting complexes; that is, the operation above does not necessarily give a tilting complex in general. To take away the disadvantage, we need the notion of *silting* complexes; see [3] for the details. In this note, we will consider only symmetric algebras.

Let us introduce the *(2-)tilting quiver* of A. The vertices of the quiver are basic (2-term) tilting complexes and we draw an arrow $T \to U$ if $U \simeq \mu_X^-(T)$ for an indecomposable summand X of T. Then the following result indicates a relationship between the partial order and tilting mutation.

Theorem 5. [1, Corollary 2.34] *The Hasse quiver of the poset* $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ *coincides with the 2-tilting quiver of* A.

Example 6. Let G be the dihedral group of order 6 and $\operatorname{char} k = 3$. The group algebra $A := kG$ is presented by the quiver $1 \underset{x}{\overset{x}{\rightleftarrows}} 2$ with relations $x^3 = 0$. Let P_i denote the indecomposable projective module of A corresponding to the vertex i. Then, we have the 2-tilting quiver of A:

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
\boxed{\begin{matrix} P_1 \to P_2 \\ 0 \to P_2 \end{matrix}} \arrow[d] & & \boxed{\begin{matrix} 0 \to P_1 \\ P_2 \to P_1 \end{matrix}} \arrow[d] \\
\boxed{\begin{matrix} P_1 \to P_2 \\ P_1 \to 0 \end{matrix}} \arrow[dr] & & \boxed{\begin{matrix} P_2 \to 0 \\ P_2 \to P_1 \end{matrix}} \arrow[dl] \\
& A[1] &
\end{tikzcd}$$

"hexagon"

We will observe that this is independent of the prime p of the order $2p$ and $\operatorname{char} k$ later.

3. Main results

Let G be a finite group and $p := \operatorname{char} k$; then, the group algebra kG and its blocks (i.e., summands of kG as algebras) are symmetric algebras.

The first aim is to classify τ-tilting finite group algebras kG with 2 simple modules.

We say that G is *p-perfect* if it has no normal subgroup with index p. Here is the first main theorem.

Theorem 7. *Assume that G is p-perfect.*

(1) *The following are equivalent:*
   (a) *the 2-tilting quiver is of the form:*

$$\begin{tikzcd}
& \circ \arrow[dl] \arrow[dr] & \\
\circ \arrow[d] & & \circ \arrow[d] \\
\circ \arrow[dr] & & \circ \arrow[dl] \\
& \circ &
\end{tikzcd}$$

   (b) *it is the hexagon;*
   (c) $4 < |\,\mathsf{s}\tau\text{-}\mathsf{tilt}\, kG| < 8$*;*
   (d) *p is odd and G is isomorphic to the dihedral group of order $2p^n$.*

(2) *The following are equivalent:*
   (a) *the 2-tilting quiver is of the form:*

$$\begin{tikzcd}
& \circ \arrow[dl] \arrow[dr] & \\
\circ \arrow[dr] & & \circ \arrow[dl] \\
& \circ &
\end{tikzcd}$$

   (b) *it is the tetragon;*
   (c) $2 < |\,\mathsf{s}\tau\text{-}\mathsf{tilt}\, kG| < 6$*;*
   (d) *p is odd and G is isomorphic to the cyclic group of order 2.*

(3) *There is no group G such that kG has 2 simple modules and $6 < |\,\mathsf{s}\tau\text{-}\mathsf{tilt}\, kG| < \infty$.*

Remark 8. The assumption of G being p-perfect plays an important role:

(1) For a p-group P, the group algebras $k[G \times P]$ and kG admit the same poset structure for $\mathsf{s}\tau\text{-}\mathsf{tilt}(-)$ [4, 2].
(2) There exists a non-p-perfect group G such that kG has 2 simple modules and $|\,\mathsf{s}\tau\text{-}\mathsf{tilt}\, kG| = 8$. We will see its example later.

Let A be a block of kG with defect group D; the defect group of a block is a p-subgroup of G controlling the block, for example, D is cyclic (dihedral/semidihedral/quaternion and $p = 2$) iff A is representation-finite (representation-tame).

The second aim is to study the τ-tilting finiteness of A when $p = 2$. *In the rest of this note, assume that $p = 2$.*

The cyclic group of order n is denoted by C_n. Let us start with examples of τ-tilting finite 2-blocks.

Example 9 (See [4])**.** (1) Let G be the symmetric group of degree 4. Then kG is indecomposable, so $A = kG$ (D is isomorphic to the dihedral group of order 8), which has 2 simple modules. Moreover, A admits 8 support τ-tilting modules.
(2) Let G be the alternating group of degree 4. Then we have $A = kG$ with 3 simple modules (D is isomorphic to $C_2 \times C_2$), and there are 32 support τ-tilting modules.

(3) Let G be the alternating group of degree 5. Then $kG = A \oplus \mathrm{Mat}_4(k)$ and A has 3 simple modules (D is the same as (2)). Furthermore, $|\,\mathsf{s}\tau\text{-}\mathsf{tilt}\,A| = 32$.

Denote by Λ the algebra A as in Example 9(2)(3). We now state the second main theorem of this note.

Theorem 10. *Assume that A is nonnilpotent ($\doteqdot$ nonlocal).*

(1) *Suppose that D is isomorphic to $C_{2^m} \times C_{2^n}$. Then the following are equivalent:*
  (a) *A is τ-tilting finite;*
  (b) *A is Morita equivalent to Λ;*
  (c) *$m = n = 1$.*

(2) *Suppose that D is isomorphic to $C_2 \times C_2 \times C_2$. Then A is τ-tilting finite if and only if it is Morita equivalen to $kC_2 \times \Lambda$. In the case, it admits the same poset structure for $\mathsf{s}\tau\text{-}\mathsf{tilt}(-)$ as Λ.*

At the time of writing, we find no representation-infinite 2-block A such that $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ has a different poset structure as $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$. We close this note by putting this question.

Question 11. Let $p = 2$ and A be a representation-infinite nonnilpotent block of a group algebra kG. Then does the following hold true?

- A is τ-tilting finite if and only if $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$ as posets.

(Find a representation-infinite nonnilpotent 2-block A satisfying $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A \not\simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$.)

References

[1] T. Adachi, O. Iyama and I. Reiten, τ-tilting theory. *Compos. Math.* **150**, no. 3, 415–452 (2014).
[2] T. Aihara and T. Honma, τ-tilting finite triangular matrix algebras. *J. Pure Appl. Algebra* **225** (2021), no. 12, Paper No. 106785, 10pp.
[3] T. Aihara and O. Iyama, Silting mutation in triangulated categories. *J. Lond. Math. Soc. (2)* **85** (2012), no. 3, 633–668.
[4] F. Eisele, G. Janssens and T. Raedschelders, A reduction theorem for τ-rigid modules. *Math. Z.* **290** (2018), no. 3–4, 1377–1413.
[5] T. Okuyama, Some examples of derived equivalent blocks of finite groups. Preprint (1998).

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APPROXIMATION BY INTERVAL-DECOMPOSABLES AND INTERVAL RESOLUTIONS OF 2D PERSISTENCE MODULES

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Abstract. In topological data analysis, in contrast to the case of one-parameter persistent homology, two-parameter persistent homology presents algebraic difficulties due to its wild representation type. We consider approximations of two-parameter persistence modules: (1) In a previous work, we defined interval approximations using "compression" to essential vertices of intervals together with Möbius inversion. (2) Another idea is to consider homological approximations of persistence modules using interval resolutions. In this work, we first study (2) in the general setting of finite posets and show the following: the interval resolution global dimension is finite for finite posets, and that it can be computed using the Auslander-Reiten translates of the interval representations. Then, in the commutative ladder case, we provide a formula linking the two notions of approximation by a suitable modification of (1). This is an extended abstract summarizing the results of the detailed version [arXiv:2207.03663].

Keywords: Representation, Relative homological algebra, Multiparameter persistence
2020 *Mathematics Subject Classification*: 16G20, 55N31

1. Introduction

In the field of data analysis, one recent development is the rapidly growing subfield called "topological data analysis", which applies ideas from (algebraic) topology for data analysis. One of its main tools is persistent homology [6], which has found applications in various fields of study. Persistent homology is able to describe the topological features (connected components, holes, voids, etc.) of data, and in a multiscale way by providing information of "birth" and "death" parameter values, with respect to one parameter of the data. Algebraically, persistent homology is described as a persistence module, which can be formalized as a representation of an A_n-type quiver, and the topological features are encoded as a choice of generators for an indecomposable decomposition of the persistence module. The indecomposable decomposition is given by interval representations, with the endpoints giving the "birth" and "death" parameter values.

However, coming from motivations in data analysis, there is a need to deal with multiple parameters, leading to multiparameter persistent homology [5]. In this case, the

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The detailed version of this paper will be submitted for publication elsewhere. A preprint is available at [2].

H.A. is supported by JSPS Grant-in-Aid for Scientific Research (C) 18K03207, and JSPS Grant-in-Aid for Transformative Research Areas (A) (22A201). E.G.E. is supported by JSPS Grant-in-Aid for Transformative Research Areas (A) (22H05105). K.N. is supported by JSPS Grant-in-Aid for Transformative Research Areas (A) (20H05884). M.Y. is supported by JSPS Grant-in-Aid for Scientific Research (C) (20K03760). H.A. and M.Y. are partially supported by Osaka Central Advanced Mathematical Institute (JPMXP0619217849).

underlying parameter space is an n-dimensional commutative grid for n parameters, and the corresponding algebra is of wild representation type for large enough parameter space.

Thus there many attempts to overcome the difficulties in the multiparameter setting, such as by using a suitable generalization of the intervals. The full version [2] of this work studies relative homological algebra with respect to the interval modules, and also gives a more detailed review of the literature. Here, we summarize the results of [2].

2. Background

Throughout, k is a field, vect_k is the category of finite dimensional k-vector spaces, and $\mathcal{P}$ is a finite poset. We first recall the following definitions.

Definition 1.

(1) The *segment* from $p \in \mathcal{P}$ to $q \in \mathcal{P}$ is $[p, q] := \{x \in \mathcal{P} \mid p \leq x \leq q\}$
(2) $\mathcal{P}$ is said to be *connected* if for any $p, q \in \mathcal{P}$, there exists a sequence $p = r_0, r_1, \ldots, r_\ell = q$ of elements with r_{i-1} and r_i comparable for each $i \in \{1, \ldots, \ell\}$.
(3) A subset $S \subseteq \mathcal{P}$ is *convex* if $[p, q] \subseteq S$ for any $p, q \in S$.
(4) A subset $S \subseteq \mathcal{P}$ is an *interval* if it is convex and the subposet induced by S is connected.

We denote by $A := k\mathcal{P}$ the incidence algebra of $\mathcal{P}$ over k. Alternatively, we can consider the Hasse diagram of $\mathcal{P}$ as a quiver Q, and let R be the two-sided ideal of the path algebra of kQ generated by all commutativity relations. With this, we can identify the incidence algebra with the path algebra of the bound quiver (Q, R). We let $\mathrm{mod}\, A$ be the category of finitely generated left A-modules.

Note that a poset $\mathcal{P}$ can be considered as a category with a unique morphism $p \to q$ whenever $p \leq q$. A pointwise finite dimensional (pfd) persistence module over $\mathcal{P}$ is a functor $M : \mathcal{P} \to \mathrm{vect}_k$. Furthermore, $\mathrm{mod}\, k\mathcal{P}$ can be identified with the category of pfd persistence modules over $\mathcal{P}$. We freely identify persistence modules over $\mathcal{P}$, modules over the incidence algebra $k\mathcal{P}$, and representations of the bound quiver (Q, R). In what follows, by persistence module we mean pfd persistence module.

Definition 2. Let $\mathcal{P}$ be a poset and $A = k\mathcal{P}$.

(1) For an interval I of $\mathcal{P}$, the A-module V_I is defined by $V_I(i) = k \ \ (i \in I_0)$, $V_I(i \leq j) = 1_k \ \ (i, j \in I_0)$ and 0 otherwise, is called an *interval module* over A. An A-module M is called *interval-decomposable* if M is isomorphic to a finite direct sum of interval modules.
(2) We denote by $\mathbb{I}(P)$ a set of representatives of all interval modules, with one representative chosen from each isomorphism class. If P is clear we write $\mathbb{I}$.
(3) We denote by $\mathcal{I}(P)$ the set of all interval-decompoable modules. If P is clear we omit it and write $\mathcal{I}$.

We introduce our main poset of interest.

Definition 3 (2D commutative grid)**.** For $m, n \in \mathbb{N} := \{1, 2, \ldots\}$, the poset $\vec{G}_{m,n}$ is defined by

$$\vec{G}_{m,n} = (\{1, \ldots, m\}, \leq) \times (\{1, \ldots, n\}, \leq)$$

and called the $m \times n$ commutative 2D grid. That is, the partial order defined by $(i,j) \leq (k,\ell)$ if and only if $i \leq k$ and $j \leq \ell$.

In topological data analysis, the interval(-decomposable) modules play a central role in one-parameter persistent homology, as they are used to express the "birth" and "death" of topological features. In case of $\mathcal{P} = \vec{G}_{m,n}$, A-modules are called 2-parameter (or 2D) *persistence modules*, and can be used to study the evolution of topological features varying across two parameters. We are interested in approximating 2D persistence modules using interval-decomposable persistence modules.

It is known that each interval I of $\vec{G}_{m,n}$ has a "staircase" form (see the discussion in Section 4.1 of [1]): a full subposet induced by a set of the form

$$I = \{(j,i) \mid i \in \{s, s+1, \ldots, t\}, j \in \{b_i, b_i+1, \ldots, d_i\}\}$$

for some $1 \leq s \leq t \leq n$ and some $1 \leq b_i \leq d_i \leq m$ for each $s \leq i \leq t$ such that

$$b_{i+1} \leq b_i \leq d_{i+1} \leq d_i$$

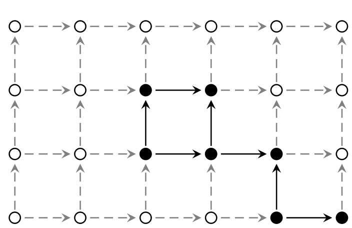
for all $i \in \{s, \ldots, t-1\}$. We adopt the notation of [1] writing

$$I = \bigsqcup_{i=s}^{t} [b_i, d_i]_i$$

to denote the interval above. In this notation, each $[b_i, d_i]_i$ is the "slice" of the staircase at height i.

Example 4. Below is an example of an interval I (filled-in points and arrows) of $\vec{G}_{6,4}$, displaying posets using their Hasse diagrams. This interval is denoted as $[5,6]_1 \sqcup [3,5]_2 \sqcup [3,4]_3$. The corresponding interval module V_I is given to its right.

(2.1) $I:$



$$V_I : \begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & k \arrow[r, "1"] \arrow[u] & k \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & k \arrow[r, "1"] \arrow[u, "1"] & k \arrow[r, "1"] \arrow[u, "1"] & k \arrow[r] \arrow[u] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & k \arrow[r, "1"] \arrow[u, "1"] & k \arrow[u]
\end{tikzcd}$$

Next, we recall some definitions from relative homological algebra. Throughout the definitions below, $\mathcal{P}$ is a finite poset, $A = k\mathcal{P}$ and M is a persistence module over $\mathcal{P}$.

(1) A *right interval approximation* of M is a morphism $f \in \mathrm{Hom}_A(X, M)$ with $X \in \mathcal{I}$ such that for any $g \in \mathrm{Hom}_A(Y, M)$ with $Y \in \mathcal{I}$ there exists some $h \in \mathrm{Hom}_A(Y, X)$ such that $g = fh$. This is equivalent to saying that f induces an epimorphism

$$\mathrm{Hom}_A(-, f) : \mathrm{Hom}_A(-, X)|_{\mathcal{I}} \to \mathrm{Hom}_A(-, M)|_{\mathcal{I}}.$$

Note that since $\mathcal{I}$ contains all finitely generated projectives, it can be checked that a right interval approximation is guaranteed to be surjective.

(2) An *interval resolution* of M is an exact sequence

$$\cdots \to X_n \xrightarrow{f_n} \cdots \to X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \to 0$$

such that f_0 is a right interval approximation of M, and for each $i \geq 1$, f_i is a right interval approximation of $\operatorname{Im} f_i = \operatorname{Ker} f_{i-1}$.

(3) A morphism $f \in \operatorname{Hom}_A(X, M)$ is said to be *right minimal* if every morphism $g\colon X \to X$ with $f = fg$ is an automorphism.

(4) A morphism $f\colon X \to M$ is said to be a *right minimal interval approximation* of M if it is a right interval approximation that is right minimal.

(5) A *minimal interval resolution* of M is an exact sequence

$$\cdots \to X_n \xrightarrow{f_n} \cdots \to X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \to 0$$

such that f_0 is a right minimal interval approximation of M, and for each $i \geq 1$, f_i is a right minimal interval approximation of $\operatorname{Im} f_i = \operatorname{Ker} f_{i-1}$.

(6) If there exists an interval resolution of M of the form

$$0 \to X_n \xrightarrow{f_n} \cdots \to X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \to 0$$

for some $n \geq 0$, we say that *interval resolution dimension* of M is at most n, and write $\text{int-dim}\, M \leq n$. Otherwise we say that interval resolution dimension of M is infinity.

(7) If $\text{int-dim}\, M \leq n$ and $\text{int-dim}\, M \not\leq n-1$, then we say that interval resolution dimension of M is equal to n, and denote it by $\text{int-dim}\, M = n$.

(8) Finally, we define

$$\text{int-gldim}\, A := \sup\{\text{int-dim}\, M \mid M \in \operatorname{mod} A\}$$

and call it the *interval resolution global dimension* of A.

Now, we let

$$G := \bigoplus_{I \in \mathbb{I}(\mathcal{P})} I, \text{ and } \Lambda := \operatorname{End}(G).$$

Since each indecomposable projective module and each indecomposable injective module is isomorphic to some interval in $\mathbb{I}(\mathcal{P})$, G is a generator-cogenerator. Since G is a generator, it is well-known (see for example [4, Proposition 4.17(1)(2)]) that

$$\text{int-dim}\, M = \operatorname{pd}_\Lambda \operatorname{Hom}_A(G, M) \tag{2.2}$$

where $\operatorname{pd}_\Lambda$ is the projective dimension of Λ-modules. Since G is a generator-cogenerator, we obtain the following equality from Erdmann–Holm–Iyama–Schröer [7, Lemma 2.1]:

$$\text{int-gldim}\, k\mathcal{P} = \operatorname{gldim} \Lambda - 2. \tag{2.3}$$

3. Results

Proposition 5. *For any finite poset $\mathcal{P}$,* $\text{int-gldim}\, k\mathcal{P} < \infty$.

The proof of Proposition 5 is provided below. First, the following can be shown.

Lemma 6. *Let M be an interval module of $A = k\mathcal{P}$, and N a submodule of M. Then, N is interval-decomposable.*

Then, we extract the following immediate corollary from known results.

Corollary 7 (Corollary of [11, Theorem in §5], cf. [8, Lemma 2.2])**.** *Let B be an artin algebra, and X a finitely generated B-module. Assume that for each indecomposable direct summand X' of X, all submodules of X' are in* $\operatorname{add} X$*, then* $\operatorname{End}_B(X)$ *is left strongly quasi-hereditary, and its global dimension is finite.*

We remark that the finiteness of the global dimension of $\operatorname{End}_B(X)$ also follows from Corollary 2.4.1(1) together with Theorem 3.3 of [9]. The fact that it is left strongly quasi-hereditary can be shown using [9, Corollary 2.4.1(1)] with [13, Theorem 3.22].

Proof of Proposition 5. This follows immediately by applying Corollary 7 with

$$X := G = \bigoplus_{I \in \mathbb{I}(\mathcal{P})} I, \text{ and } B := \Lambda = \operatorname{End}(G).$$

Note that each indecomposable direct summand X' of X is simply an interval module, and by Lemma 6 each submodule of X' is interval-decomposable and thus is in $\operatorname{add} X$. □

Proposition 8. *For any finite poset $\mathcal{P}$,*

$$\text{int-gldim}\, k\mathcal{P} = \max_{I \in \mathbb{I}} \text{int-dim}(\tau V_I),$$

where τ is the Auslander–Reiten translation.

Proof. Since Λ is not semisimple, we have

$$\begin{aligned}
\text{int-gldim}\, k\mathcal{P} &= \operatorname{gldim} \Lambda - 2 \\
&= \max\{\operatorname{pd}(\operatorname{Hom}_A(G, V_I)/\operatorname{rad}(G, V_I)) \mid I \in \mathbb{I}, \operatorname{rad}(G, V_I) \neq 0\} - 2 \\
&= \max\{\operatorname{pd} \operatorname{rad}(G, V_I) + 1 \mid I \in \mathbb{I}, \operatorname{rad}(G, V_I) \neq 0\} - 2 \\
&= \max\{\operatorname{pd} \operatorname{rad}(G, V_I) - 1 \mid I \in \mathbb{I}, \operatorname{rad}(G, V_I) \neq 0\}
\end{aligned}$$

where the first equality is Eq. (2.3). For V_I is not projective, there exists an almost split sequence of the form $0 \to \tau V_I \to E_I \to V_I \to 0$, yielding an exact sequence $0 \to \operatorname{Hom}_A(G, \tau V_I) \to \operatorname{Hom}_A(G, E_I) \to \operatorname{rad}(G, V_I) \to 0$ of Λ-modules, showing that

$$\operatorname{pd} \operatorname{rad}(G, V_I) \leq \max\{\operatorname{pd} \operatorname{Hom}_A(G, E_I), \operatorname{pd} \operatorname{Hom}_A(G, \tau V_I) + 1\}.$$

Applying, we obtain

$$\begin{aligned}
\text{int-gldim}\, k\mathcal{P} &\leq \max\{\operatorname{pd} \operatorname{Hom}_A(G, E_I) - 1, \operatorname{pd} \operatorname{Hom}_A(G, \tau V_I)\} \mid I \in \mathbb{I}\} \\
&= \max\{\text{int-dim}\, E_I - 1, \text{int-dim}\, \tau V_I \mid I \in \mathbb{I}\} \\
&\leq \text{int-gldim}\, k\mathcal{P}.
\end{aligned}$$

where the second line follows from Eq. (2.2).

By Proposition 5, int-gldim $k\mathcal{P} = d$ for some positive integer d, and hence there exists some $I \in \mathbb{I}$ such that either int-dim $E_I - 1 = d$ or int-dim $\tau V_I = d$. In the former case, we have int-dim $E_I = d + 1 >$ int-gldim $k\mathcal{P} = \max\{\text{int-dim}\, X \mid X \in \operatorname{mod} k\mathcal{P}\}$, a contradiction. Therefore, d is the maximum of $\{\text{int-dim}\, \tau V_I \mid I \in \mathbb{I}\}$. □

We use Proposition 8 in computational experiments, and obtain some conjectures about the value of int-gldim for the 2D commutative grids.

Example 9. Let $k = \mathbb{F}_2$, the finite field with 2 elements, and $A = k\vec{G}_{m,n}$ $(m, n \geq 2)$. In the table below, the row labelled n and column labelled m contains the value (or a lower bound) of int-gldim $k\vec{G}_{m,n}$ obtained by numerical computation.

	2	3	4	5	6	7	8	9	10
2	0	1	2	2	2	2	2	2	2
3	1	2	3	4	4	4			
4	2	3	4	5	≥6	≥6			
5	2	4	5						
6	2	4	≥6						
7	2	4	≥6						

We conjecture that for the row $n = 2$, the value of int-gldim $k\vec{G}_{m,2}$ is 2 for all $m \geq 4$. We further conjecture that for each fixed row n, the value of int-gldim $k\vec{G}_{m,n}$ eventually stabilizes to some fixed constant $C(n)$, and that this happens for $m \geq n + 2$.

The commutative ladder $\vec{G}_{m,2}$ case

From here on, we consider only the commutative ladder, that is, the $m \times 2$ commutative grid $\vec{G}_{m,2}$ (or symmetrically, $\vec{G}_{2,n}$).

Let us fix some notation and let $I = [x_i, x_j]_1 \sqcup [y_k, y_l]_2$ be an interval of $\vec{G}_{m,2}$. Thus, $1 \leq k \leq i \leq l \leq j \leq n$, and I is illustrated by its Hasse diagram

$$\begin{tikzcd} y_k \arrow[r] & \cdots \arrow[r] & y_i \arrow[r] & \cdots \arrow[r] & y_l & & \\ & & x_i \arrow[u] \arrow[r] & \cdots \arrow[r] & x_l \arrow[u] \arrow[r] & \cdots \arrow[r] & x_j \end{tikzcd} \tag{3.1}$$

Given a persistence module M of $\vec{G}_{2,n}$, we "compress" M using I in the following way. Let S_I be the subposet of I with the following Hasse diagram:

$$\begin{tikzcd} y_k \arrow[r] \arrow[rr, bend left] & y_i & y_l & \\ & x_i \arrow[u] \arrow[rr] & & x_j \end{tikzcd} \tag{3.2}$$

Note that S_I is *not* a full subposet of I, since for example $y_i < y_l$ in I but $y_i \not< y_l$ in S_I. Viewing posets as categories, we define the inclusion functor $\xi(I) : S_I \to \vec{G}_{2,n}$, one for each $I \in \mathbb{I}$. We then define $R_{\xi(I)}(M) := M \circ \xi(I)$, which is M restricted ("compressed") to S_I.

Definition 10 (Compressed multiplicity)**.** The *compressed multiplicity with respect to* ξ of V_I in M is

$$c_M^{\xi}(I) := d_{R_{\xi(I)}(M)}(R_{\xi(I)}(V_I)),$$

the multiplicity of $R_{\xi(I)}(V_I)$ as a direct summand of $R_{\xi(I)}(M)$.

Note that there are other ways of "compressing" M, which can be set by changing the choice of the functor $\xi(I)$ (and S_I). In fact, the definition here is a modification of the compressed multiplicity introduced in [3], where the functor is defined as an inclusion of the full subposet of a set of "essential vertices" of I. As another example, when the functor is defined using the inclusion of $I \hookrightarrow \vec{G}_{2,n}$ as is, one recovers the generalized rank invariant of Kim and Memoli [10] (see [3] for a detailed discussion).

The following theorem relates the compressed multiplicity c_M^ξ in terms of a formula involving only the multiplicities of the intervals in an interval resolution of M.

Theorem 11. *Let M be a persistence module over $\vec{G}_{2,n}$ with an interval resolution*

$$0 \to X_r \xrightarrow{f_r} \cdots \to X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \to 0, \tag{3.3}$$

with each term X_i a direct sum of interval modules V_J as $X_i \cong \bigoplus_{J\in\mathbb{I}} V_J^{d_J^{(i)}}$. Then,

$$c_M^\xi(I) = \sum_{I\subseteq J\in\mathbb{I}} \sum_{i=0}^{r} (-1)^i d_J^{(i)}.$$

Proof. See the detailed version [2] for a proof. □

The set of intervals $\mathbb{I}(\vec{G}_{m,2})$ can be given a poset structure with partial order defined by $I \leq J$ if and only if I is a subposet of J. Following previous works [10, 3], we use Möbius inversion [12] of c_M^ξ viewed as a function on the elements of $\mathbb{I}(\vec{G}_{m,2})$, to obtain another invariant for persistence modules.

Definition 12 (Interval approximation)**.** The *interval approximation δ_M^ξ with respect to ξ of M* is the Möbius inversion of c_M^ξ:

$$\delta_M^\xi(J) := \sum_{S\subseteq \mathrm{Cov}(J)} (-1)^{\#S} c_M^\xi(\bigvee S)$$

for all $J \in \mathbb{I}$, where $\mathrm{Cov}(J)$ is the set of "cover elements" of J, and $\bigvee S$ is the join of the elements of S (see [3] for a detailed discussion of the poset structure of $\mathbb{I}(\vec{G}_{m,n})$).

Applying Möbius inversion to Theorem 11, we immediately obtain the following.

Corollary 13. *Let M be a persistence module over $\vec{G}_{2,n}$ with an interval resolution as in Theorem 11. Then we have*

$$\delta_M^\xi(J) = \sum_{i=0}^{r} (-1)^i d_J^{(i)}$$

for all $J \in \mathbb{I}$.

This links the two notions of "approximation": one is combinatorial via Möbius inversion (δ_M^ξ), and the other is coming from relative homological algebra (the multiplicities of the intervals in an interval resolution of M).

References

[1] Hideto Asashiba, Mickaël Buchet, Emerson G Escolar, Ken Nakashima, and Michio Yoshiwaki. On interval decomposability of 2d persistence modules. *Computational Geometry*, 101879, 2022.

[2] Hideto Asashiba, Emerson G Escolar, Ken Nakashima, and Michio Yoshiwaki. Approximation by interval-decomposables and interval resolutions of persistence modules *arXiv preprint arXiv:2207.03663*, 2022.

[3] ———, On approximation of 2d persistence modules by interval-decomposables. *arXiv preprint arXiv:1911.01637*, 2019.

[4] Benjamin Blanchette, Thomas Brüstle, and Eric J Hanson. Homological approximations in persistence theory. *arXiv preprint arXiv:2112.07632*, 2021.

[5] Gunnar Carlsson and Afra Zomorodian. The theory of multidimensional persistence. *Discrete & Computational Geometry*, 42(1):71–93, 2009.

[6] Herbert Edelsbrunner, David Letscher, and Afra Zomorodian. Topological persistence and simplification. *Discrete Comput. Geom.*, 28(4):511–533, November 2002.

[7] Karin Erdmann, Thorsten Holm, Osamu Iyama, and Jan Schröer. Radical embeddings and representation dimension. *Advances in mathematics*, 185(1):159–177, 2004.

[8] Osamu Iyama. Finiteness of representation dimension. *Proceedings of the american mathematical society*, 131(4):1011–1014, 2003.

[9] Osamu Iyama. Rejective subcategories of artin algebras and orders. arXiv:math/0311281, 2003.

[10] Woojin Kim and Facundo Mémoli. Generalized persistence diagrams for persistence modules over posets. *Journal of Applied and Computational Topology*, 5(4):533–581, 2021.

[11] Claus Michael Ringel. Iyama's finiteness theorem via strongly quasi-hereditary algebras. *Journal of Pure and Applied Algebra*, 214(9):1687–1692, 2010.

[12] Gian-Carlo Rota. On the foundations of combinatorial theory i. theory of möbius functions. *Probability theory and related fields*, 2(4):340–368, 1964.

[13] Mayu Tsukamoto. Strongly quasi-hereditary algebras and rejective subcategories. *Nagoya Math. J.* 237, 10–38, 2020.

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A CHARACTERIZATION OF STANDARD DERIVED EQUIVALENCES OF DIAGRAMS OF DG CATEGORIES AND THEIR GLUING

HIDETO ASASHIBA AND SHENGYONG PAN

Abstract. A diagram consisting of differential graded (dg for short) categories and dg functors is formulated as a colax functor X from a small category I to the 2-category $\mathbb{k}$-**dgCat** of small dg categories, dg functors and dg natural transformations for a fixed commutative ring $\mathbb{k}$. If I is a group regarded as a category with only one object $*$, then X is nothing but a colax action of the group I on the dg category $X(*)$. In this sense, this X can be regarded as a generalization of a dg category with a colax action of a group. We define a notion of standard derived equivalence between such colax functors by generalizing the corresponding notion between dg categories with a group action. Our first main result gives some characterizations of this notion without an assumption of $\mathbb{k}$-flatness (or $\mathbb{k}$-projectivity) on X, one of which is given in terms of generalized versions of a tilting object and a quasi-equivalence. On the other hand, for such a colax functor X, the dg categories $X(i)$ with i objects of I can be glued together to have a single dg category $\int X$, called the Grothendieck construction of X. Our second main result insists that for such colax functors X and X', the Grothendieck construction $\int X'$ is derived equivalent to $\int X$ if there exists a standard derived equivalence from X' to X. These results generalize the main results of [3] and [4] to the dg case, respectively. These are new even for dg categories with a group action. In particular, the second result gives a new tool to show the derived equivalence between the orbit categories of dg categories with a group action (see [6] for such examples).

Key Words: derived equivalence, dg category, Grothendieck construction, 2-category, colax functor
2020 *Mathematics Subject Classification*: 18G35, 16E35, 16E45, 16W22, 16W50

1. Introduction

Throughout this note $\mathbb{k}$ is a commutative ring, and I is a small category. In [2], when $\mathbb{k}$ is an algebraically closed field, we classified (basic, connected) representation-finite selfinjective algebras up to derived equivalences. They are divided into two classes: the class sRFS of standard algebras and the class nRFS of nonstandard algebras. The sRFS forms a major part. We denote by sRFS$'$ to be the subclass of sRFS consisting of algebras not isomorphic to $\mathbb{k}$. Then sRFS is a disjoint union of sRFS$'$ and the derived equivalence class of $\mathbb{k}$ that coincides with the isoclass of $\mathbb{k}$. We here review how sRFS$'$ was classified. Each member A of sRFS$'$ has the form $\hat{B}/G$ of the orbit category, where B is a tilted

The detailed version of this paper will be submitted for publication elsewhere.
A preprint is available at [6]. H.A. is supported by JSPS Grant-in-Aid for Scientific Research (C) 18K03207, and JSPS Grant-in-Aid for Transformative Research Areas (A) (22A201), and partially supported by Osaka Central Advanced Mathematical Institute (JPMXP0619217849). S.P. is supported by China Scholarship Council.

algebra of Dynkin type Δ, and $\hat{B}$ is the repetitive category of B having a G-action with G an infinite cyclic group. Thus there exists a G-covering $P\colon \hat{B} \to A$. Then we defined the *derived equivalence type* $\operatorname{typ}(A) := (\Delta, f, t)$ ($f \in \mathbb{Q}, t \in \{1, 2, 3\}$) of A, where f, t were derived from the information of the action of a generator of G, and the $\operatorname{typ}(A)$ was shown to be derived invariant of A. In addition, from each type T in the list of all possible types, the normal form $\Lambda(T)$ was constructed. Let A' be another member of sRFS$'$ with a G-covering $P'\colon \hat{B}' \to A'$, $A' \cong \hat{B}/G$, and $\operatorname{typ}(A') = (\Delta', f', t')$. To classify sRFS$'$, it is enough to show that A and A' are derived equivalent if and only if $\operatorname{typ}(A) = \operatorname{typ}(A')$. The only if part means that $\operatorname{typ}(A)$ is derived invariant. The if part is proved by showing that A is derived equivalent to $\Lambda(\operatorname{typ}(A))$. (Thus we may assume that $A' = \Lambda(\operatorname{typ}(A))$.) Note that if $\operatorname{typ}(A) = \operatorname{typ}(A')$, then since $\Delta = \Delta'$, both B and B' are derived equivalent to the hereditary algebra $\Bbbk Q$ with Q a Dynkin quiver of type Δ, and hence B and B' are derived equivalent. Then the main tools for the proof of if part were as follows given in [1]:

(1) If B and B' are derived equivalent, then so are $\hat{B}$ and $\hat{B}'$.
(2) If $\hat{B}$ and $\hat{B}'$ are derived equivalent satisfying an additional compatibility condition with P, P', then $\hat{B}/G$ and $\hat{B}'/G$ are derived equivalent.

The tool (2) is generalized in [3, 4] as follows. First, the setting is changed as follows. The algebraically closed field $\Bbbk$ is changed to any commutative ring. The cyclic group G is regarded as a category with single object $*$, and is changed to a small category I. $\hat{B}$ is changed to any small $\Bbbk$-category $\mathscr{C}$. The G-action $G \to \operatorname{Aut}(\mathscr{C})$ on $\mathscr{C}$ is regarded as a functor X from G as a category with single object $*$ to the category of small $\Bbbk$-categories with $X(*) = \mathscr{C}$, and the G-action on $\mathscr{C}$ is changed to a colax functor $X\colon I \to \Bbbk\text{-}\mathbf{Cat}$ (see Example 5). The "derived module categoy" $\mathscr{D}(\operatorname{Mod} X)$ is defined as a colax functor from I to the 2-category $\Bbbk\text{-}\underline{\mathbf{TRI}}^2$ of triangulated 2-moderate[1] categories, and X is defined to be derived equivalent to another colax functor $X'\colon I \to \Bbbk\text{-}\mathbf{Cat}$ if $\mathscr{D}(\operatorname{Mod} X)$ and $\mathscr{D}(\operatorname{Mod} X')$ are equivalent in the 2-category of colax functors $I \to \Bbbk\text{-}\underline{\mathbf{TRI}}^2$. The orbit category $\mathscr{C}/G$ for a category $\mathscr{C}$ with a G-action $X \in \operatorname{Aut}(\mathscr{C})$ is changed to the Grothendieck construction $\int X$. In this general setting, the following two questions arise to generalize the tool (2).

Q 1. Characterize derived equivalence for X and X'.
Q 2. When $\int X$ and $\int X'$ are derived equivalent?

Answers are given as the following two theorems.

Theorem 1. *Let $X, X'\colon I \to \Bbbk\text{-}\mathbf{Cat}$ be colax functors. Then* (1) *implies* (2)*:*

(1) *X' is derived equivalent to X.*
(2) *X' is equivalent to a tilting colax functor*[2] *$\mathscr{T}$ for X.*

If X is $\Bbbk$-flat[3]*, then the converse holds.*

Theorem 2. *If* (2) *above holds, then $\int X'$ is derived equivalent to $\int X$.*

Since characterization of derived equivalences of $\Bbbk$-categories are well controlled in the setting of dg categories as in Keller [8], it is interesting to generalize these theorems to

[1] See Definition 3.
[2] This is defined in a way similar to Definition 14(2).
[3] The $\Bbbk$-modules $X(i)(x, y)$ are flat for all objects i of I and objects x, y of $X(i)$.

dg categories. In this connection, the purpose of the talk is to give a characterization of standard derived equivalences of colax functors from I to the 2-category of dg categories, and to extend Theorem 2 in this setting.

2. Preliminaries

In this section, we collect necessary terminologies.

2.1. **A set theory for the foundation of category theory.** First of all, we remark the set theoretical foundation that we use here, which is needed because we collect many categories. We refer the reader to [5, Appendix A] for details. To avoid the set theoretic paradox, it is usually enough to consider three kinds of collections: sets, classes, and conglomerates. However, to construct mathematical theory only within the scope of sets, one considers a (Grothendieck) universe $\mathfrak{U}$, and assume the *axiom of universes* stating that any set is an element of a universe. We note that the class of all universes is well-ordered. An element of $\mathfrak{U}$ is called a $\mathfrak{U}$-small set, and a subset of $\mathfrak{U}$ is called a $\mathfrak{U}$-class. If a collection S constructed from $\mathfrak{U}$-sets and $\mathfrak{U}$-classes cannot be a $\mathfrak{U}$-class, for example a set of the form $\mathscr{D}(\operatorname{Mod} A)(X, Y)$, where A is a $\mathfrak{U}$-small algebra, X, Y are objects of the derived category of the $\mathfrak{U}$-small modules over A, then we take the smallest universe $\mathfrak{U}'$ having S as its element, and we next consider $\mathfrak{U}'$-small sets, and $\mathfrak{U}'$-classes. If we repeat this procedure, we need more and more universes. To avoid this repetition, we adopt the hierarchy proposed by Levy [9]. First we fix a universe $\mathfrak{U}$ once for all, and we construct mathematical theory within $\mathfrak{U}'$-small sets, where $\mathfrak{U}'$ is the smallest universe having the power set of $\mathfrak{U}$ as its element. In particular, all categories discussed here are small categories in the usual sense. For a category $\mathscr{C}$, the set of all objects of $\mathscr{C}$ is denoted by $\mathscr{C}_0$. Levy's hierarchy defines a $\mathfrak{U}'$-small set $\mathbf{Class}_0^k$ of the k-classes for each non-negative integer k, and we have a sequence of strict inclusions

$$\mathbf{Class}_0^0 \subset \mathbf{Class}_0^1 \subset \mathbf{Class}_0^2 \subset \cdots,$$

where 0-classes are nothing but $\mathfrak{U}$-small sets, usually called just as small sets, and 1-classes are nothing but $\mathfrak{U}$-classes. See [5, Definition A.2.5] for definition of $\mathbf{Class}_0^k$ for $k \geq 2$.

Definition 3. Let $\mathscr{C}$ be a category.

(1) $\mathscr{C}$ is called a *small* category[4] if $\mathscr{C}_0$ and $\mathscr{C}(x, y)$ are small for all $x, y \in \mathscr{C}_0$.
(2) $\mathscr{C}$ is called a *light* category if $\mathscr{C}_0$ is a 1-class, and $\mathscr{C}(x, y)$ are small for all $x, y \in \mathscr{C}_0$.
(3) For each $k \geq 1$, $\mathscr{C}$ is called a k-*moderate* category if $\mathscr{C}_0$ and $\mathscr{C}(x, y)$ are k-classes for all $x, y \in \mathscr{C}_0$.

2.2. **2-categories and colax functors.**

Definition 4. A 2-*category* $\mathbf{C}$ is a sequence of data:

(1) a non-empty set $\mathbf{C}_0$,
(2) a family of categories $(\mathbf{C}(x, y))_{x,y\in\mathscr{C}_0}$,
(3) a family of functors $\circ := (\circ_{x,y,z} : \mathbf{C}(y, z) \times \mathbf{C}(x, y) \to \mathbf{C}(x, z))_{x,y,z\in\mathbf{C}_0}$, and
(4) a family of functors $(u_x : \mathbf{1} \to \mathbf{C}(x, x))_{x\in\mathbf{C}_0}$, where $\mathbf{1}$ is the category consisting of one object $*$ and one morphism 1_* (we set $\mathbb{1}_x := u_x(*)$, $\mathbb{1}_{\mathbb{1}_x} := u_x(\mathbb{1}_*)$)

[4] abbreviation of a $\mathfrak{U}$-small category

that satisfies associativity and unitality.

Elements of $\mathbf{C}_0$ are called *objects* of $\mathbf{C}$, elements of $\mathbf{C}_1 := \bigcup_{x,y\in\mathbf{C}_0}\mathbf{C}(x,y)_0$ are called 1-*morphisms* of $\mathbf{C}$, and elements of $\mathbf{C}_2 := \bigcup_{x,y\in\mathbf{C}_0}\mathbf{C}(x,y)_1$ are called 2-*morphisms* of $\mathbf{C}$. The compositions in $\mathbf{C}(x,y)$ with $x,y\in\mathbf{C}_0$ are called *vertical compositions* of 2-morphisms, and the composition $\circ$ for 2-morphisms are called *horizontal compositions*. Sometimes 2-categories are defined by giving the set of objects, 1-morphisms and 2-morphisms, and by omitting the definitions of vertical and horizontal compositions and identities, when they are obvious.

Example 5 (2-categories)**.**

(1) We denote by $\Bbbk$-**Cat** the 2-category of small categories, functors between them, and natural transformations between those functors. Similarly, $\Bbbk$-**FRB**, $\Bbbk$-**TRI** and $\Bbbk$-$\underline{\mathbf{TRI}}^k$ denote the 2-category of light Frobenius $\Bbbk$-categories, of light triangulated $\Bbbk$-categories and of k-moderate triangulated $\Bbbk$-categories ($k\ge 1$), respectively.

(2) Any category $\mathscr{C}$ can be regarded as a 2-category $\mathscr{C}'$ defined as follows, and we identify $\mathscr{C}$ with $\mathscr{C}'$ below, especially for $\mathscr{C}=I$: Objects of $\mathscr{C}'$ are the objects of $\mathscr{C}$; 1-morphisms in $\mathscr{C}'$ are the morphisms in $\mathscr{C}$; and 2-morphisms in $\mathscr{C}'$ are the identities $\mathbb{1}_f$ with $f\in\mathscr{C}(x,y)$ for all $x,y\in\mathscr{C}_0$.

Definition 6. Let $\mathbf{A}$ and $\mathbf{B}$ be 2-categories. A 2-*functor* $X:\mathbf{A}\to\mathbf{B}$ is a pair of data:

(1) a map $X_0:\mathbf{A}_0\to\mathbf{B}_0$ (we set $X(x):=X_0(x)$ for all $x\in\mathbf{A}_0$ for short), and

(2) a family of functors $(X_{(x,y)}:\mathbf{A}(x,y)\to\mathbf{B}(X(x),X(y)))_{x,y\in\mathbf{A}_0}$ (we set $X(f):=X_{(x,y)}(f)$ for all $f\in\mathbf{A}(x,y)$ for short)

that preserves compositions and identities.

Definition 7. Let $\mathbf{A}$ and $\mathbf{B}$ be 2-categories. A *colax functor* from $\mathbf{A}$ to $\mathbf{B}$ is a quadruple of data:

(1) , (2) as above,

(3) a family $(X_i)_{i\in\mathbf{A}_0}$ of 2-morphisms $X_i\colon X(\mathbb{1}_i)\Rightarrow\mathbb{1}_{X(i)}$ in $\mathbf{B}$ indexed by $i\in\mathbf{A}_0$, and

(4) a family $(X_{b,a})_{(b,a)}$ of 2-morphisms $X_{b,a}\colon X(ba)\Rightarrow X(b)X(a)$ in $\mathbf{B}$ indexed by $(b,a)\in\operatorname{com}(\mathbf{A}):=\{(b,a)\in\mathbf{A}_1\times\mathbf{A}_1\mid ba\text{ is defined}\}$

that satisfies the axioms

(a) Counitality: For each $a\colon i\to j$ in $\mathbf{A}_1$ the following are commutative:

$$\begin{tikzcd} X(a\mathbb{1}_i) \arrow[r, Rightarrow, "X_{a,\mathbb{1}_i}"] \arrow[dr, equal] & X(a)X(\mathbb{1}_i) \arrow[d, Rightarrow, "X(a)X_i"] \\ & X(a)\mathbb{1}_{X(i)} \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} X(\mathbb{1}_j a) \arrow[r, Rightarrow, "X_{\mathbb{1}_j,a}"] \arrow[dr, equal] & X(\mathbb{1}_j)X(a) \arrow[d, Rightarrow, "X_jX(a)"] \\ & \mathbb{1}_{X(j)}X(a) \end{tikzcd} \ ;\ \text{and}$$

(b) Coassociativity: For each $i\xrightarrow{a}j\xrightarrow{b}k\xrightarrow{c}l$ in $\mathbf{A}_1$ the following is commutative:

$$\begin{tikzcd} X(cba) \arrow[r, Rightarrow, "X_{c,ba}"] \arrow[d, Rightarrow, "X_{cb,a}"'] & X(c)X(ba) \arrow[d, Rightarrow, "X(c)\theta_{b,a}"] \\ X(cb)X(a) \arrow[r, Rightarrow, "X_{c,b}X(a)"'] & X(c)X(b)X(a). \end{tikzcd}$$

A *pseudofunctor* is a colax functor with all X_i and $X_{b,a}$ 2-isomorphisms. A 2-*functor* is nothing but a colax functor with all X_i and $X_{b,a}$ identities.

2.3. **Dg categories.** We now review necessary terminologies for dg categories.

Definition 8 (Dg categories and dg functors)**.**

(1) A *dg category* (a short form of *differential graded category*) is a $\Bbbk$-category $\mathscr{A}$ whose morphism spaces $\mathscr{A}(x,y)$ are (cochain) complexes of $\Bbbk$-modules for all $x,y \in \mathscr{A}_0$, and whose compositions

$$\mathscr{A}(y,z) \otimes_{\Bbbk} \mathscr{A}(x,y) \to \mathscr{A}(x,z)$$

are chain maps of complexes for all $x,y,z \in \mathscr{A}_0$. Note that the Leibniz rule is automatically satisfied.

(2) Let $\mathscr{A}, \mathscr{B}$ be dg categories. Then a *dg functor* $F\colon \mathscr{A} \to \mathscr{B}$ is a sequence of data
  (a) a map $F_0\colon \mathscr{A}_0 \to \mathscr{B}_0$, where we set $F(x) := F_0(x)$ for all $x \in \mathscr{A}_0$ fir short; and
  (b) a family $(F_{(x,y)}\colon \mathscr{A}(x,y) \to \mathscr{B}(F(x),F(y)))_{(x,y)\in\mathscr{A}_0\times\mathscr{A}_0}$ of chain maps, where we set $F(f) := F_{(x,y)}(f)$ for all $f \in \mathscr{A}(x,y)$ for short;

  that preserves compositions and identities.

Definition 9. Let $\mathscr{A}, \mathscr{B}$ be dg categories, $E, F\colon \mathscr{A} \to \mathscr{B}$ dg functors, and $n \in \mathbb{Z}$. Then we set $\mathscr{H}om(E,F)^n$ to be the set of all $(\alpha_x^n)_{x\in\mathscr{A}_0} \in \prod_{x\in\mathscr{A}_0} \mathscr{B}(E(x),F(x))^n$ such that $F(f)\alpha_x^n = (-1)^{mn}\alpha_y^n E(f)$ for all $f \in \mathscr{A}(x,y)^m, m \in \mathbb{Z}, x,y \in \mathscr{A}_0$. Using this we define a complex $\mathscr{H}om(E,F) := \bigoplus_{n\in\mathbb{Z}} \mathscr{H}om(F,G)^n$ of $\Bbbk$-modules with the differential d given by $\mathscr{H}om(E,F)^n \to \mathscr{H}om(E,F)^{n+1}, (\alpha_x^n)_x \mapsto (d_{\mathscr{B}}(\alpha_x^n))_x$. Then $\alpha^n := (\alpha_x^n)_{x\in\mathscr{A}_0}$ is called a *derived transformation of degree* n, and $\alpha := (\alpha^n)_{n\in\mathbb{Z}}$ is called a *derived transformation*. An element α of $Z^0(\mathscr{H}om(E,F))$ is called a *dg natural transformation*, which is identified with the family $\alpha := (\alpha_x)_x \in \prod_{x\in\mathscr{A}_0} \mathscr{B}(E(x),F(x))^0$ with $d(\alpha) = 0$, and $F(f)\alpha_x = \alpha_y E(f)$ for all $f \in \mathscr{A}(x,y)$.

Definition 10. By $\mathscr{C}_{\mathrm{dg}}(\Bbbk)$ we denote the category of (co)chain complexes of $\Bbbk$-modules, where for any complexes M, N the morphism space is given by

$$\mathscr{C}_{\mathrm{dg}}(\Bbbk)(M,N) := \bigoplus_{n\in\mathbb{Z}} \prod_{p\in\mathbb{Z}} \mathrm{Hom}_{\Bbbk}(M^p, N^{p+n})$$

with the differential d defined by $d(f) := (d_N^{p+n} f^p - (-1)^n f^{p+1} d_M^d)_{p\in\mathbb{Z}}$ for all $f = (f^p)_{p\in\mathbb{Z}} \in \mathscr{C}_{\mathrm{dg}}(M,N)^n$. Then $\mathscr{C}_{\mathrm{dg}}(\Bbbk)$ is a light dg category.

We denote by $\Bbbk$-**dgCat** the 2-category of *small* dg categories, dg functors between them, and *dg natural transformations* between those dg functors. By changing small/light or dg natural/derived transformations, we have the following four variants:

	dg natural	derived
small	$\Bbbk$-**dgCat**	$\Bbbk$-**DGCat**
light	$\Bbbk$-**dgCAT**	$\Bbbk$-**DGCAT**

Let $\mathscr{A} \in \Bbbk\text{-}\mathbf{dgCat}_0 = \Bbbk\text{-}\mathbf{DGCat}_0 \ni \mathscr{C}_{\mathrm{dg}}(\Bbbk)$. Then we define the dg category

$$\mathscr{C}_{\mathrm{dg}}(\mathscr{A}) := \Bbbk\text{-}\mathbf{DGCat}(\mathscr{A}^{\mathrm{op}}, \mathscr{C}_{\mathrm{dg}}(\Bbbk)) \in \Bbbk\text{-}\mathbf{dgCAT}_0 = \Bbbk\text{-}\mathbf{DGCAT}_0,$$

of dg $\mathscr{A}$-modules, which is a light category. By taking the 0-cocycles, this defines the category $\mathscr{C}(\mathscr{A}) := Z^0(\mathscr{C}_{\mathrm{dg}}(\mathscr{A}))$ of dg $\mathscr{A}$-modules, which is a light Frobenius category, the homotopy category $\mathscr{H}(\mathscr{A}) := H^0(\mathscr{C}_{\mathrm{dg}}(\mathscr{A}))$ of $\mathscr{A}$, which is equal to the stable category $\underline{\mathscr{C}(\mathscr{A})}$ of $\mathscr{C}(\mathscr{A})$ that is a light triangulated category, and the derived category $\mathscr{D}(\mathscr{A}) := \mathscr{H}(\mathscr{A})[\mathrm{qis}^{-1}]$ of $\mathscr{A}$ as a quotient category of $\mathscr{H}(\mathscr{A})$ with respect to the quasi-isomorphisms (see Definition 11), which is known to be a 2-moderate triangulated category. Then we have $\mathscr{C}_{\mathrm{dg}}(\mathscr{A})_0 = \mathscr{C}(\mathscr{A})_0 = \mathscr{H}(\mathscr{A})_0 = \mathscr{D}(\mathscr{A})_0$.

Definition 11. Let $M \in \mathscr{C}_{\mathrm{dg}}(\mathscr{A})_0$. A morpism $f\colon M \to N$ in $\mathscr{C}(\mathscr{A})$ is called *quasi-isomorphism* (qis for short) if $H^n(f)\colon H^n(M) \to H^n(N)$ is an isomorphism for all $n \in \mathbb{Z}$. M is said to be *acyclic* if $H^n(M) = 0$ for all $n \in \mathbb{Z}$. M is said to be *homotopically projective* if $\mathscr{H}(\mathscr{A})(M, A) = 0$ for all acyclic complexes $A \in \mathscr{H}(\mathscr{A})$. We set $\mathscr{H}_{\mathrm{p}}(\mathscr{A})$ to be the full subcategoryy of $\mathscr{H}(\mathscr{A})$ consisting of homotopically projective objects.

We formulate a diagram of dg categories and dg functors as a colax functor X from I to $\Bbbk$-**dgCat**. We can also regard X as a set of dg categories $X(i)$'s with an action of I, hence as a generalization of a dg category with a group action when I is a group viewed as a category with only one object $*$. For a 2-category $\mathbf{C}$, the colax functors from I to $\mathbf{C}$ also form a 2-category $\mathrm{Colax}(I, \mathbf{C})$ with suitably defined 1-morphisms and 2-morphisms, where a 1-morphism is a pair $(F, \phi)\colon X' \to X$ of a family $F = (F(i)\colon X'(i) \to X(i))_{i\in I_0}$ of 1-morphisms in $\mathbf{C}$, and a family $\phi = (\phi_a\colon X(a)F(i) \Rightarrow F(j)X'(a))_{(a\colon i\to j)\in I_1}$ of 2-morphisms in $\mathbf{C}$.

For a colax functor X in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$, a dg category $\int X$ is constructed in [6] by "gluing" all dg categories $X(i)$'s together, which is called the *Grothendieck construction* of X, which is nothing but the orbit category $X(*)/G$ when I is a group G.

The correspondence $\mathscr{A} \mapsto \mathscr{C}_{dg}(\mathscr{A})$ can be extended to a pseudofunctor $\mathscr{C}_{dg}\colon \Bbbk\text{-}\mathbf{DGCat} \to \Bbbk\text{-}\mathbf{DGCAT}$. Similarly, we obtain pseudofunctors $\mathscr{C}\colon \Bbbk\text{-}\mathbf{dgCAT} \to \Bbbk\text{-}\mathbf{FRB}$, $\mathscr{H}\colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{TRI}$, and $\mathscr{D}\colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\underline{\underline{\mathbf{TRI}}}^2$. For a colax functor $X\colon I \to \Bbbk\text{-}\mathbf{dgCat}$, we can define its dg category of dg modules $\mathscr{C}_{dg}(X)$, category of dg modules $\mathscr{C}(X)$, homotopy category $\mathscr{H}(X)$, and derived category $\mathscr{D}(X)$ as the composite $\mathscr{C}_{dg}(X) := \mathscr{C}_{dg} \circ X$ and so on. The relationship of these constructions can be illustrated by the following strict commutative diagram on the left.

$$
\begin{tikzcd}
 & \mathscr{C}_{\mathrm{dg}}(\Bbbk\text{-}\mathbf{dgCat}) \arrow[d, "Z^0"] \arrow[dd, bend left, "H^0"] & \\
\Bbbk\text{-}\mathbf{dgCat} \arrow[ur, "\mathscr{C}_{\mathrm{dg}}"] \arrow[r, "\mathscr{C}"] \arrow[dr, "\mathscr{H}"] \arrow[ddr, "\mathscr{D}"'] & \mathscr{C}(\Bbbk\text{-}\mathbf{dgCat}) \arrow[r, hook] \arrow[d, "\mathrm{st}"] & \Bbbk\text{-}\mathbf{FRB} \\
 & \mathscr{H}(\Bbbk\text{-}\mathbf{dgCat}) \arrow[r, hook] \arrow[d, "\mathbf{L}"] & \Bbbk\text{-}\mathbf{TRI} \\
 & \Bbbk\text{-}\underline{\underline{\mathbf{TRI}}}^2 &
\end{tikzcd}
,\qquad
\begin{tikzcd}
\mathscr{H}(\mathscr{A}) \arrow[r, "H^0(F)"] & \mathscr{H}(\mathscr{B}) \arrow[dd, bend left, "Q_{\mathscr{B}}"] \\
\mathscr{H}_{\mathrm{p}}(\mathscr{A}) \arrow[u, hook] \arrow[r, "H^0(F)|"] & \mathscr{H}_{\mathrm{p}}(\mathscr{B}) \arrow[u, hook] \arrow[d, "\mathbf{j}_{\mathscr{B}}"] \\
\mathscr{D}(\mathscr{A}) \arrow[u, "\mathbf{p}_{\mathscr{A}}"] \arrow[r, dashed, "\mathbf{L}(F)"'] & \mathscr{D}(\mathscr{B})
\end{tikzcd}
.
$$

Here, in $\mathscr{C}_{\mathrm{dg}}(\Bbbk\text{-}\mathbf{dgCat})$, 1-morphisms $F\colon \mathscr{C}_{\mathrm{dg}}(\mathscr{A}) \to \mathscr{C}_{\mathrm{dg}}(\mathscr{B})$ are required to preserve homotopically projective objects: $F(\mathscr{H}_{\mathrm{p}}(\mathscr{A})_0) \subseteq \mathscr{H}_{\mathrm{p}}(\mathscr{B})_0$ (similar for $\mathscr{C}(\Bbbk\text{-}\mathbf{dgCat})$ and $\mathscr{H}(\Bbbk\text{-}\mathbf{dgCat})$), which enables us to define a pseudofunctor $\mathbf{L}$ defined by $\mathbf{L}(F) := \mathbf{L}(H^0(F)) = \mathbf{j}_{\mathscr{B}} \circ H^0(F)| \circ \mathbf{p}_{\mathscr{A}}$ in the diagram above, where $Q_{\mathscr{B}}$ is the quotient functor, $\mathbf{j}_{\mathscr{B}}$ is the restriction of $Q_{\mathscr{B}}$ to $\mathscr{H}_{\mathrm{p}}(\mathscr{B})$, and $\mathbf{p}_{\mathscr{A}}$ is given by a "projective resolution" with $\mathbf{p}_{\mathscr{A}} \circ \mathbf{j}_{\mathscr{A}} = \mathbb{1}_{\mathscr{H}_{\mathrm{p}}(\mathscr{A})}$.

Let $\alpha\colon E \Rightarrow F$ be a dg natural transformation of small dg functors $E, F\colon \mathscr{A} \to \mathscr{B}$ of dg categories. Thus α is a 2-morphism in $\Bbbk$-**dgCat**. We here observe how this α is sent by the pseudofunctors $\mathscr{C}_{\mathrm{dg}}, H^0$ and $\mathbf{L}$. By applying $\mathscr{C}_{\mathrm{dg}}$, we obtain a dg natural transformation $\text{-}\otimes_{\mathscr{A}}\overline{\alpha}\colon \text{-}\otimes_{\mathscr{A}}\overline{E} \Rightarrow \text{-}\otimes_{\mathscr{A}}\overline{F}$ of dg functors $\text{-}\otimes_{\mathscr{A}}\overline{E}, \text{-}\otimes_{\mathscr{A}}\overline{F}\colon \mathscr{C}_{\mathrm{dg}}(\mathscr{A}) \to \mathscr{C}_{\mathrm{dg}}(\mathscr{B})$, where we set $\overline{E}$ to be the $\mathscr{A}$-$\mathscr{B}$-bimodule $\mathscr{B}(\text{-}, E(?))$ (similar for $\overline{F}$), and $\overline{\alpha}$ to be the morphism $\mathscr{B}(\text{-}, \alpha_{(?)})$ of bimodules. This is sent by $\mathbf{L} \circ H^0$ to the natural transformation $\text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{\alpha}\colon \text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{E} \Rightarrow \text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{F}$ of triangle functors $\text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{E}, \text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{F}\colon \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{B})$ of derived categories of $\mathscr{A}$ and $\mathscr{B}$, respectively.

3. Results

In this section we state our main results. To state them we need the following three definitions.

Definition 12. Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. Then X' is said to be *standardly derived equivalent* to X if there exists a 1-morphism $(F, \psi)\colon \mathscr{C}_{\mathrm{dg}}(X') \to \mathscr{C}_{\mathrm{dg}}(X)$ in the 2-category $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCAT})$ such that $\mathbf{L}(F, \psi)\colon\ \mathscr{D}(X') \to \mathscr{D}(X)$ is an equivalence in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$. Here, this F is said to *preserve homotopically projective objects* if $F(i)(\mathscr{H}_{\mathrm{p}}(X'(i))_0) \subseteq \mathscr{H}_{\mathrm{p}}(X(i))_0$ for all $i \in I_0$.

Remark 13. It is possible to state this sentence using a derived tensor such as: "There exists an X'-X-bimodule Z such that $\text{-}\overset{\mathbf{L}}{\otimes}_{X'}Z\colon \mathscr{D}(X') \to \mathscr{D}(X)$ is an equivalence in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$." See [6] for details.

Definition 14. Let $\mathscr{A}$ be a small dg category, and $X \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$.

(1) A dg subcategory $\mathscr{T}$ of $\mathscr{C}_{\mathrm{dg}}(\mathscr{A})$ is called a *tilting dg subcategory* for $\mathscr{A}$ if all $T \in \mathscr{T}_0$ is compact and the smallest localizing subcategory of $\mathscr{D}(\mathscr{A})$ containing $\mathscr{T}_0$ coincides with $\mathscr{D}(\mathscr{A})$.
(2) A colax subfunctor $\mathscr{T}$ of $\mathscr{C}_{\mathrm{dg}}(X)$ is called a *tilting colax subfunctor* for X if there exists a 1-morphism $(\sigma, \rho)\colon \mathscr{T} \to \mathscr{C}_{\mathrm{dg}}(X)$ such that $\sigma(i)\colon \mathscr{T}(i) \to \mathscr{C}_{\mathrm{dg}}(X(i))$ is the inclusion, and $\mathscr{T}(i)$ is a tilting dg subcategory for $X(i)$ for all $i \in I_0$.

Definition 15 (Quasi-equivalence 1-morphisms)**.**

(1) A dg functor $F\colon \mathscr{A} \to \mathscr{B}$ of dg categories is said to be *quasi-equivalence* if $H^n(F)\colon H^n(\mathscr{A}) \to H^n(\mathscr{B})$ is fully faithful for all $n \in \mathbb{Z}$, and $H^0(F)\colon H^0(\mathscr{A}) \to H^0(\mathscr{B})$ is an equivalence.
(2) A dg natural transformation $\alpha\colon E \Rightarrow F$ of dg functors $E, F\colon \mathscr{A} \to \mathscr{B}$ of dg categories is called a 2-*quasi-isomorphism* if $\text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{\alpha}\colon \text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{E} \to \text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{F}$ is an isomorphism.
(3) A 1-morphism $(F, \phi)\colon X \to \mathscr{T}$ in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$ is said to be *quasi-equivalence* if $F(i)$ is quasi-equivalence for all $i \in I_0$ and $\phi(a)$ is 2-quasi-isomorphism for all $a \in I_1$.

We obtained the following characterization of standard derived equivalences of diagrams of dg categories, where we do not need $\Bbbk$-flatness assumption unlike a result by Keller [8].

Theorem 16. *Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. Then the following are equivalent.*

(1) *There exists a* 1*-morphism* $(F, \psi)\colon \mathscr{C}_{\rm dg}(X') \to \mathscr{C}_{\rm dg}(X)$ *in* $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCAT})$ *such that* F preserves homotopically projective objects *and* $\mathbf{L}(F, \psi)\colon\ \mathscr{D}(X') \to \mathscr{D}(X)$ *is an equivalence in* $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$.
(2) X' *is standardly derived equivalent to* X.
(3) *There exists a quasi-equivalence* $(E, \phi)\colon X' \to \mathscr{T}$ *for some tilting colax functor* $\mathscr{T}$ *for* X.

Remark 17. The statement (1) guarantees that the relation to be standardly derived equivalent is transitive. But we do not know whether this relation is reflexive.

Remark 18. We do not need $\Bbbk$-flatness assumption on X. It is possible to remove this assumption also from Keller's theorem [8, Corollary 9.2] for dg categories. In connection with this, we mention that Dugger–Shipley [7] proved Rickard's theorem [10, Proposition 5.1] (it needed $\Bbbk$-projectivity) without this assumption.

The following gives a sufficient condition for the Grothendieck constructions to be derived equivalent.

Theorem 19. *Let* $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. *Assume that* X' *is standardly derived equivalent to* X*, or equivalently, there exists a quasi-equivalence from* X' *to a tilting colax functor* $\mathscr{T}$ *for* X *(cf. Theorem 16). Then* $\int X'$ *is derived equivalent to* $\int X$.

References

[1] Asashiba, H.: *A covering technique for derived equivalence*, J. Algebra., **191** (1997), 382–415.
[2] Asashiba, H.: *The derived equivalence classification of representation-finite selfinjective algebras*, J. Algebra, **214** (1999), 182–221.
[3] Asashiba, H.: *Derived equivalences of actions of a category*, Appl. Categor. Struct.**21** (2013), no. 6, 811-836.
[4] Asashiba, H.: *Gluing derived equivalences together*, Adv. Math. **235** (2013), 134-160.
[5] Asashiba, H.: *Categories and representation theory – with a focus on 2-categorical covering theory.* Mathematical Surveys and Monographs, **271**. American Mathematical Society, Providence, RI, 2022. xviii+240 pp. (Original version: SGC library series **155**, Saiensu-sha, 2019, in Japanese.)
[6] Asashiba, H.; Pan, S.: *Characterizations of standard derived equivalences of diagrams of dg categories and their gluings*. arXiv:2201.10760, 2022.
[7] Dugger, D., and Shipley, B.: *K-theory and derived equivalences*, Duke Math. J., **124** (2004), 587–617.
[8] Keller, B.: *Deriving DG categories*, Ann. scient. Éc. Norm. Sup., 4^e série, t. **27** (1994), 63–102.
[9] Levy, P.B.: *Formulating categorical concepts using classes*, arXiv.1801.08528.
[10] Rickard, J.: *Derived equivalences as derived functors*, J. London Math. Soc. **43** (1991), 37–48.

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CONNECTEDNESS OF QUASI-HEREDITARY STRUCTURES

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Abstract. Dlab and Ringel showed that algebras being quasi-hereditary in all orders for indices of primitive idempotents becomes hereditary. So, we are interested in for which orders a given quasi-hereditary algebra is again quasi-hereditary. As a matter of fact, we consider permutations of indices, and if the algebra with permuted indices is quasi-hereditary, then we say that this permutation gives a quasi-hereditary structure.

In this paper, we first give a criterion for adjacent transpositions giving quasi-hereditary structures, in terms of homological conditions of standard or costandard modules over a given quasi-hereditary algebra. Next, we consider those which we call connectedness of quasi-hereditary structures. The definition of connectedness can be found in Definition 4. We then show that any two quasi-hereditary structures are connected, which is our main result. By this result, once we know that there are two quasi-hereditary structures, then permutations in some sense lying between them give also quasi-hereditary structures.

1. Introduction

Quasi-hereditary algebras, introduced by Cline, Parshall and Scott, generalize hereditary algebras. Moreover Dlab and Ringel showed in Theorem 1 of [2] that if an algebra is quasi-hereditary in all orders, it becomes hereditary, and vice versa. From this point of view, we study quasi-hereditary structures for a given algebra. Recently, there are two results on quasi-hereditary structures. Coulembier showed in [1] that a quasi-hereditary algebra with simple preserving duality has only one quasi-hereditary structure. Flores, Kimura and Rognerud gave a method of counting the number of quasi-hereditary structures for a path algebras of Dynkin types in [3]. In their papers, the quasi-hereditary structure was defined by an equivalent class of partial orders with some relations. However in this paper, we define it by using a total order without using equivalent classes. Thus, our results are in the nature different from them and can not be derived from their results. Moreover we will use permutations instead of total orders when considering quasi-hereditary structures.

Throughout this paper, let K be an algebraically closed field, A a finite dimensional K-algebra with pairwise orthogonal primitive idempotents $e_1, \dots, e_n$, and let $\Lambda = \{1, \dots, n\}$. For $i \in \Lambda$, we denote $P(i) = e_iA$ the indecomposable projective module, $S(i)$ the top of $P(i)$, and $I(i)$ the injective envelope of $S(i)$. The standard K-dual $\mathrm{Hom}_K(-, K)$ is denoted by D. For an A-module M, we write the isomorphism class of M by $[M]$ and the Jordan-Hölder multiplicity of $S(i)$ in M by $[M : S(i)]$. Let $\mathfrak{S}_n$ be the symmetric group on n letters, $e \in \mathfrak{S}_n$ the trivial permutation and $\sigma_i = (i, i+1) \in \mathfrak{S}_n$ adjacent transpositions for $1 \le i \le n-1$.

The detailed version of this paper will be submitted for publication elsewhere.

First, we recall the definition of quasi-hereditary algebras and quasi-hereditary structures.

Definition 1. Let A be an algebra as above and $\sigma \in \mathfrak{S}_n$.

(1) The total order
$$\sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(n)$$
over Λ is called the $\boldsymbol{\sigma}$**-order**.

(2) For each $i \in \Lambda$, the A-module $\Delta^\sigma(i)$, called the **standard module** with respect to the σ-order, is defined by the maximal factor module of $P(i)$ having only composition factors $S(j)$ with $\sigma(j) \leq \sigma(i)$. Moreover we will write the set $\{\Delta^\sigma(1), \ldots, \Delta^\sigma(n)\}$ by Δ^σ.

(3) Dually, the A-module $\nabla^\sigma(i)$, called the **costandard module** with respect to the σ-order, is defined by the maximal submodule of $I(i)$ having only composition factors $S(j)$ with $\sigma(j) \leq \sigma(i)$. Denote the set $\{\nabla^\sigma(1), \ldots, \nabla^\sigma(n)\}$ by ∇^σ.

(4) We say that an A-module M has a Δ^σ**-filtration** (resp. a ∇^σ**-filtration**) if there is a sequence of submodules
$$0 = M_{m+1} \subset \cdots \subset M_2 \subset M_1 = M$$
such that for each $1 \leq k \leq m$, $M_k/M_{k+1} \cong \Delta^\sigma(j)$ (resp. $M_k/M_{k+1} \cong \nabla^\sigma(j)$) for some $j \in \Lambda$.

(5) A pair (A, σ) is said to be a **quasi-hereditary algebra** provided that the following conditions are satisfied:
  (a) $[\Delta^\sigma(i) : S(i)] = 1$ for all $i \in \Lambda$.
  (b) A_A has a Δ^σ-filtration.

If this is the case, we say that the permutation σ gives a **quasi-hereditary structure** of A.

Next, we show some properties which every pair of neighbor standard modules has.

Lemma 2 ([5] Lemma 2.)**.** *Assume that (A, e) is a quasi-hereditary algebra. Then we have the following equalities. For $1 \leq i \leq n-1$,*

(1) $\dim \mathrm{Hom}_A(\Delta(i), \Delta(i+1)) = \dim \mathrm{Hom}_A(P(i), \Delta(i+1)) = [\Delta(i+1) : S(i)]$,
(2) $\dim \mathrm{Hom}_A(\nabla(i+1), \nabla(i)) = \dim \mathrm{Hom}_A(\nabla(i+1), I(i)) = [\nabla(i+1) : S(i)]$,
(3) $\dim \mathrm{Ext}^1_A(\Delta(i), \Delta(i+1)) = \dim \mathrm{Ext}^1_A(\Delta(i), S(i+1)) = [P(i) : \Delta(i+1)]$,
(4) $\dim \mathrm{Ext}^1_A(\nabla(i+1), \nabla(i)) = \dim \mathrm{Ext}^1_A(S(i+1), \nabla(i)) = [I(i) : \nabla(i+1)]$.

We will denote
$$H_i = \dim \mathrm{Hom}_A(\Delta(i), \Delta(i+1)),\ E_i = \dim \mathrm{Ext}^1_A(\Delta(i), \Delta(i+1)),$$
$$\overline{H_i} = \dim \mathrm{Hom}_A(\nabla(i+1), \nabla(i)),\ \text{and}\ \overline{E_i} = \dim \mathrm{Ext}^1_A(\nabla(i+1), \nabla(i)).$$

Lemma 3 ([5] Corollary 1)**.** *Assume that (A, e) is a quasi-hereditary algebra. Then the followings hold. For $1 \leq i \leq n-1$,*

(1) $\mathrm{Hom}_A(\Delta(i), \Delta(i+1)) \cong D\mathrm{Ext}^1_A(\nabla(i+1), \nabla(i))$,
(2) $\mathrm{Ext}^1_A(\Delta(i), \Delta(i+1)) \cong D\mathrm{Hom}_A(\nabla(i+1), \nabla(i))$.

In particular, we have $H_i = \overline{E_i}$ and $E_i = \overline{H_i}$.

Finally, we define the connectedness of quasi-hereditary structures, which is the main topic of this paper.

Definition 4. Two permutations σ and τ giving quasi-hereditary structures are said to be **connected** if the following condition holds: There is a decomposition $\tau\sigma^{-1} = \sigma_{i_l} \cdots \sigma_{i_1}$ into the product of adjacent transpositions such that all $\sigma_{i_k} \cdots \sigma_{i_1}\sigma$ for $1 \leq k \leq l$ also give quasi-hereditary structures. Moreover, if any two permutations giving quasi-hereditary structures are connected, we also say that quasi-hereditary structures are **connected**.

Our aim in this paper is to claim that quasi-hereditary structures are connected.

2. Twistability

Let (A, σ) be a quasi-hereditary algebra. If $(A, \sigma_i\sigma)$ is also quasi-hereditary, then we call the original quasi-hereditary algebra (A, σ) to be i**th-twistable**. In this section, we will give the condition on standard or costandard modules equivalent to the ith-twistability for a quasi-hereditary algebra.

Lemma 5. *Let (A, e) be quasi-hereditary. Then $[\Delta^{\sigma_i}(k) : S(k)] = 1$ for all $k \in \Lambda$ if and only if $E_i\overline{E_i} = 0$.*

By using this lemma, we get a criterion for the ith-twistability.

Theorem 6. *Let (A, e) be quasi-hereditary. Then (A, σ_i) is quasi-hereditary if and only if one of the following conditions holds:*

- **$(\mathcal{E}_i)$:** *$E_i = 0$ and $\Delta(i+1)$ has a submodule isomorphic to $\Delta(i)^{H_i}$.*
- **$(\overline{\mathcal{E}_i})$:** *$\overline{E_i} = 0$ and $\nabla(i+1)$ has a factor module isomorphic to $\nabla(i)^{\overline{H_i}}$.*

In particular, if a quasi-hereditary algebra (A, e) satisfies $E_i = \overline{E_i} = 0$, then (A, σ_i) is also quasi-hereditary with $\Delta^{\sigma_i} = \Delta$ and $\nabla^{\sigma_i} = \nabla$.

3. Connectedness

In this section, we will argue about "connectivity" of quasi-hereditary structures. In general, we can obtain all permutations giving quasi-hereditary structures from one by checking repeatedly whether each quasi-hereditary algebra satisfies the condition $(\mathcal{E}_i)$ or $(\overline{\mathcal{E}_i})$. To show the connectedness of quasi-hereditary structures, we first claim that e and another are connected in Theorem 10.

Lemma 7. *Let e, σ give quasi-hereditary structures with $e \neq \sigma$. Then for the minimum element $i \in \Lambda$ satisfying $\sigma(i+1) < \sigma(i)$, it holds that $E_iH_i = 0$.*

Proposition 8. *Let e, σ give quasi-hereditary structures. Then there is a minimal decomposition $\sigma = \sigma_{i_l} \cdots \sigma_{i_1}$ into the product of adjacent transpositions such that σ_{i_1} gives a quasi-hereditary structure. Here, this i_1 is the element i given in Lemma 7.*

Corollary 9. *Let e, σ give quasi-hereditary structures. Then there is a minimal decomposition $\sigma = \sigma_{i_l} \cdots \sigma_{i_1}$ into the product of adjacent transpositions such that $\sigma_{i_l}\sigma$ gives a quasi-hereditary structure.*

The next theorem is followed from the above corollary and the induction on the length of σ.

Theorem 10. *Let e, σ give quasi-hereditary structures. Then they are connected.*

Finally, by retaking the indices of primitive idempotents, we get the following result.

Theorem 11. *Any two permutations giving quasi-hereditary structures are connected.*

Moreover, for two permutations giving quasi-hereditary structures, we get a sequence of adjacent transpositions which induce the connectedness of them, by Proposition 8. In particular, this sequence is determined by only the permutations and does not depend on the algebra.

Corollary 12. *Let σ, τ give quasi-hereditary structures with $\sigma \neq \tau$. For $k = 1, 2, \dots$, inductively take a minimal element i_k with respect to the $(\sigma_{i_{k-1}} \cdots \sigma_{i_1}\sigma)$-order satisfying $\sigma_{i_{k-1}} \cdots \sigma_{i_1}\sigma(i_k) \neq n$ and*

$$\tau(i_k) > \tau\sigma^{-1}\sigma_{i_1} \cdots \sigma_{i_{k-1}}(\sigma_{i_{k-1}} \cdots \sigma_{i_1}\sigma(i_k) + 1).$$

We take $i_1, i_2, \dots, i_k$ until those elements satisfying the above exist. If there is no i_{k+1} satisfying the above, then we do not take i_{k+1} and put $l = k$. Then the product $\sigma_{i_l} \cdots \sigma_{i_1}$ is a decomposition of $\tau\sigma^{-1}$ inducing the connectedness of σ and τ.

Example 13. Consider a quiver $\begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta"'] & 2 \arrow[r, "\gamma"] & 3 \arrow[r, "\delta"] & 4 \end{tikzcd}$ and an ideal $I = \langle \alpha\gamma\delta - \beta\delta \rangle$ of KQ, and put $A = KQ/I$. Then all indecomposable projective modules are as follows:

$$P(1) : \begin{smallmatrix} & 1 & \\ 2 & & 3 \\ 3 & & \\ & 4 & \end{smallmatrix},\ P(2) : \begin{smallmatrix} 2 \\ 3 \\ 4 \end{smallmatrix},\ P(3) : \begin{smallmatrix} 3 \\ 4 \end{smallmatrix},\ P(4) : \begin{smallmatrix} 4 \end{smallmatrix}.$$

Now we have 24 permutations on $\Lambda = \{1, 2, 3, 4\}$. In the following, we will write $\sigma_{(i_l \cdots i_2 i_1)} = \sigma_{(i_l, \dots, i_2, i_1)}$ as the product $\sigma_{i_l} \cdots \sigma_{i_2}\sigma_{i_1}$, where $i_k \in \{1, 2, 3\}$ for $1 \leq k \leq l$. For example, the $\sigma_{(21)}$-order is $2 < 3 < 1 < 4$. Let $\lambda = (i_l, \dots, i_2, i_1)$ be a sequence of elements of $\{1, 2, 3\}$ and $\Delta^{(\lambda)}$ be standard modules with respect to the $\sigma_{(\lambda)}$-order.

Clearly $(A, \sigma_{(123121)})$ is quasi-hereditary since all standard modules are projective and satisfy $[P(i) : S(i)] = 1$ for all $i \in \Lambda$. By using $(\mathcal{E}_i)$ in Theorem 6, we have the following diagram which shows that if the source of an arrow gives a quasi-hereditary structure, then so does the target.

$$\begin{tikzcd}
 & \sigma_{(12321)} \arrow[r, "\sigma_1"] & \sigma_{(2321)} \arrow[dr, "\sigma_2"] & & & & \\
\sigma_{(123121)} \arrow[ur, "\sigma_2"] \arrow[r, "\sigma_1"] \arrow[dr, "\sigma_3"'] & \sigma_{(23121)} \arrow[r, "\sigma_2"] \arrow[dr, "\sigma_3"] & \sigma_{(3121)} \arrow[r, "\sigma_1"] \arrow[dr, "\sigma_3"] & \sigma_{(321)} \arrow[r, "\sigma_3"] & \sigma_{(21)} \arrow[r, "\sigma_2"] & \sigma_{(1)} \arrow[r, "\sigma_1"] & e \\
 & \sigma_{(12312)} \arrow[r, "\sigma_1"] & \sigma_{(2312)} \arrow[dr, "\sigma_2"] & \sigma_{(121)} \arrow[ur, "\sigma_1"] \arrow[r, "\sigma_2"] & \sigma_{(12)} \arrow[r, "\sigma_1"] & \sigma_{(2)} \arrow[ur, "\sigma_2"] & \\
 & & & \sigma_{(312)} \arrow[ur, "\sigma_3"] \arrow[r, "\sigma_1"] & \sigma_{(32)} \arrow[ur, "\sigma_3"] & &
\end{tikzcd}$$

However applying Theorem 6 to the quasi-hereditary algebra $(A, \sigma_{(12321)})$, we recognize that it is not 3rd-twistable, i.e., $\sigma_{(1231)}$ does not give a quasi-hereditary structure of

A. Similarly, $(A, \sigma_{(12312)})$ is not 2nd-twistable, and hence $\sigma_{(1232)}$ does not give a quasi-hereditary structure of A. Focus on the two permutations $\sigma_{(1231)}$ and $\sigma_{(1232)}$. Then we finally show that all the other permutations do not give quasi-hereditary structures.

$$
\begin{tikzcd}
\sigma_{(1231)} & \sigma_{(231)} \arrow[l, "\sigma_1"'] & \sigma_{(31)} \arrow[l, "\sigma_2"'] & \sigma_{(3)} \arrow[l, "\sigma_1"'] \\
 & \sigma_{(123)} \arrow[r, "\sigma_1"'] & \sigma_{(23)} \arrow[ur, "\sigma_2"'] & \\
\sigma_{(1232)} & \sigma_{(232)} \arrow[l, "\sigma_1"'] & &
\end{tikzcd}
$$

In fact, if some permutations in this diagram give quasi-hereditary structures, then $\sigma_{(1231)}$ or $\sigma_{(1232)}$ also does, a contradiction. Hence the permutations in this diagram do not give quasi-hereditary structures of A. Now we complete checking whether each permutation gives a quasi-hereditary structure or not for the algebra A.

References

[1] K. Coulembier, *The classification of blocks in BGG category O*, Mathematische Zeitschrift **295(1)** (2020), 821–837.

[2] V. Dlab, and C.M. Ringel, *Quasi-hereditary algebras*, Illinois Journal of Mathematics **33(2)** (1989), 280–291.

[3] M. Flores, Y. Kimura, and B. Rognerud, *Combinatorics of quasi-hereditary structures*, Journal of Combinatorial Theory, Series A, **187** (2022), 105559.

[4] Y. Goto, *Connectedness of quasi-hereditary structures*, arXiv:2210.17104.

[5] V. Mazorchuk, and S. Ovsienko, *A pairing in homology and the category of linear complexes of tilting modules for a quasi-hereditary algebra*, Journal of Mathematics of Kyoto University **45(4)** (2005), 711–741.

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NONCOMMUTATIVE CONICS IN CALABI-YAU QUANTUM PROJECTIVE PLANES I, II

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Abstract. In noncommutative algebraic geometry, noncommutative quadric hypersurfaces are major objects of study. In this paper, we focus on studying the homogeneous coordinate algebras A of noncommutative conics $\operatorname{Proj}_{\mathrm{nc}} A$ embedded into Calabi-Yau quantum projective planes. We give a complete classification of A up to isomorphism of graded algebras. As a consequence, we show that there are exactly 9 isomorphism classes of noncommutative conics $\operatorname{Proj}_{\mathrm{nc}} A$ in Calabi-Yau quantum projective planes.

1. Motivation

Throughout this paper, we fix an algebraically closed field k of characteristic 0. By Sylvester's theorem, it is elementary to classify (commutative) quadric hypersurfaces in $\mathbb{P}^{d-1}$, namely, they are isomorphic to

$$\operatorname{Proj} k[x_1, \dots, x_d]/(x_1^2 + \cdots + x_j^2) \subset \mathbb{P}^{d-1}$$

for some $j = 1, \dots, d$. When $d = 3$, we see that there are exactly 3 isomorphism classes of conics, exactly 1 of them is smooth and exactly 1 of them is irreducible (the same one).

The ultimate goal of our project is to classify noncommutative quadric hypersurfaces in quantum $\mathbb{P}^{d-1}$'s. As a first step to this ultimate goal, we define and classify noncommutative conics in quantum $\mathbb{P}^2$'s.

2. Quantum polynomial algebras

Definition 1 ([1])**.** A *d-dimensional quantum polynomial algebra* is a connected graded algebra S such that

1. $\operatorname{gldim} S = d < \infty$,
2. $\operatorname{Ext}^q_S(k, S) = 0$ if $q \neq d$, and $\operatorname{Ext}^d_S(k, S) \cong k$, and
3. $H_S(t) = 1/(1-t)^d$.

A d-dimensional quantum polynomial algebra S is a noncommutative analogue of the commutative polynomial algebra $k[x_1, \dots, x_d]$, so the noncommutative projective scheme $\operatorname{Proj}_{\mathrm{nc}} S$ associated to S in the sense of [3] is regarded as a quantum $\mathbb{P}^{d-1}$ (see Section 3 for details).

Next, we recall a notion of geometric algebra for a quadratic algebra.

Definition 2. Let $A = T(V)/(R)$ be a quadratic algebra where V is a finite dimensional vector space and $R \subset V \otimes V$ is a subspace.

The detailed version of this paper will be submitted for publication elsewhere.

(1) A geometric pair (E, σ) consists of a projective scheme $E \subset \mathbb{P}(V^*)$ and an automorphism $\sigma \in \operatorname{Aut} E$.
(2) We say that A satisfies (G1) if there exists a geometric pair (E, σ) such that
$$\mathcal{V}(R) = \{(p, \sigma(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in E\}.$$
In this case, we write $\mathcal{P}(A) = (E, \sigma)$.
(3) We say that A satisfies (G2) if there exists a geometric pair (E, σ) such that
$$R = \{f \in V \otimes V \mid f(p, \sigma(p)) = 0 \ \forall\, p \in E\}.$$
In this case, we write $A = \mathcal{A}(E, \sigma)$.
(4) We say that A is geometric if it satisfies both (G1) and (G2) such that $\mathcal{A}(\mathcal{P}(A)) = A$.

Theorem 3 ([2])**.** *Every 3-dimensional quantum polynomial algebra $A = \mathcal{A}(E, \sigma)$ is geometric where either $E = \mathbb{P}^2$ or $E \subset \mathbb{P}^2$ is a cubic divisor.*

Example 4. A typical example of a 3-dimensional quadratic AS-regular algebra is a 3-dimensional Sklyanin algebra
$$k\langle x, y, z\rangle/(ayz + bzy + cx^2, azx + bxz + cy^2, axy + byx + cz^2) = \mathcal{A}(E, \sigma),$$
where $E = \mathcal{V}((a^3+b^3+c^3)xyz - abc(x^3+y^3+z^3)) \subset \mathbb{P}^2$ is an elliptic curve, and $\sigma \in \operatorname{Aut} E$ is the translation by a point $(a, b, c) \in E$.

3. Quantum projective spaces

Artin and Zhang introduced a notion of noncommutative schemes.

Definition 5 ([3])**.** A *noncommutative scheme* is a pair $X = (\operatorname{mod} X, \mathcal{O}_X)$ consisting of a k-linear abelian category $\operatorname{mod} X$ and an object $\mathcal{O}_X \in \operatorname{mod} X$. We say that two noncommutative schemes X and Y are isomorphic, denoted by $X \cong Y$, if there exists an equivalence functor $F : \operatorname{mod} X \to \operatorname{mod} Y$ such that $F(\mathcal{O}_X) \cong \mathcal{O}_Y$.

We give some examples of noncommutative schemes.

Example 6. If X is a commutative noetherian scheme, then we view X as a noncommutative scheme by $X = (\operatorname{coh} X, \mathcal{O}_X)$.

Example 7. The *noncommutative affine scheme* associated to a right noetherian algebra R is a noncommutative scheme defined by $\operatorname{Spec}_{\mathrm{nc}} R := (\operatorname{mod} R, R)$. If R is commutative, then $\operatorname{Spec}_{\mathrm{nc}} R \cong \operatorname{Spec} R$.

Example 8. Let A be a right noetherian connected graded algebra. We define the quotient category $\operatorname{tails} A := \operatorname{grmod} A / \operatorname{tors} A$ where $\operatorname{tors} A$ is the full subcategory of $\operatorname{grmod} A$ consisting of finite dimensional modules over k. The *noncommutative projective scheme* associated to A is a noncommutative scheme defined by $\operatorname{Proj}_{\mathrm{nc}} A := (\operatorname{tails} A, \pi A)$ where $\pi : \operatorname{grmod} A \to \operatorname{tails} A$ is the quotient functor. If A is commutative and generated in degree 1 over k, then $\operatorname{Proj}_{\mathrm{nc}} A \cong \operatorname{Proj} A$.

Definition 9. A *quantum* $\mathbb{P}^{d-1}$ is a noncommutative projective scheme $\operatorname{Proj}_{\mathrm{nc}} S$ for some d-dimensional quantum polynomial algebra S.

4. Twisted superpotentials

Definition 10. Let V be a finite dimensional vector space and $m \in \mathbb{N}^+$. Define a linear map $\phi : V^{\otimes m} \to V^{\otimes m}$ by $\phi(v_1 \otimes v_2 \otimes \cdots \otimes v_{m-1} \otimes v_m) = v_m \otimes v_1 \otimes \cdots \otimes v_{m-2} \otimes v_{m-1}$.

(1) $w \in V^{\otimes m}$ is called *superpotential* if $\phi(w) = w$.
(2) $w \in V^{\otimes m}$ is called *twisted superpotential* if $(\tau \otimes \mathrm{id}^{\otimes m-1})\phi(w) = w$ for some $\tau \in \mathrm{GL}(V)$.
(3) The i-th derivation quotient algebra of $w \in V^{\otimes m}$ is defined by $D(w, i) := T(V)/(\partial^i w)$ where $\partial^i w$ is the "i-th left partial derivatives" of w.

The next theorem plays a key role to classify quantum polynomial algebras.

Theorem 11 ([4, Theorem 11]). *For every d-dimensional quantum polynomial algebra S, there exists a unique twisted superpotential w such that $S = D(w, d-2)$.*

Example 12. $w = a(xyz + yzx + zxy) + b(xzy + yxz + zyx) + c(x^3 + y^3 + z^3)$ is a superpotential such that

$$D(w, 1) = k\langle x, y, z\rangle/(ayz + bzy + cx^2, azx + bxz + cy^2, axy + byx + cz^2)$$

is a 3-dimensional Sklyanin algebra.

The next theorem is a characterization of "Calabi-Yau" algebras by using twisted superpotentials.

Theorem 13 ([9, Corollary 4.5]). *Let $S = D(w, d-2)$ be a d-dimensional quantum polynomial algebra where w is a twisted superpotential. Then S is "Calabi-Yau" if and only if w is $(-1)^{d+1}$ twisted superpotential.*

Example 14. Every 3-dimensional Sklyanin algebra is Calabi-Yau.

A classification of twisted superpotentials whose derivation-quotient algebras are 3-dimensional quantum polynomial algebras is completed.

Theorem 15 ([10]). *Superpotentials w such that $D(w, 1)$ are 3-dimensional quantum polynomial algebras are classified.*

Theorem 16 ([7, Theorem 3.4], [8, Theorem 4.2]). *Twisted superpotentials w such that $D(w, 1)$ are 3-dimensional quantum polynomial algebras are classified.*

By the above theorem, we have finally completed the Artin-Schelter's project in the quadratic case proposed in [1]. As an application, we have the following.

Theorem 17 ([7, Theorem 4.4]). *For every 3-dimensional quantum polynomial algebra S, there exists a 3-dimensional Calabi-Yau quantum polynomial algebra $\tilde{S}$ such that $\mathrm{Proj_{nc}}\, S \cong \mathrm{Proj_{nc}}\, \tilde{S}$.*

The above theorem tells that every quantum $\mathbb{P}^2$ is isomorphic to a "Calabi-Yau" quantum $\mathbb{P}^2$.

5. Noncommutative conics

In this section, we define a notion of noncommutative quadric hypersurface in a quantum $\mathbb{P}^{d-1}$.

Definition 18. A *noncommutative quadric hypersurface in a (Calabi-Yau) quantum* $\mathbb{P}^{d-1}$ is the noncommutative projective scheme $\mathrm{Proj}_{\mathrm{nc}} S/(f)$ for some d-dimensional (Calabi-Yau) quantum polynomial algebra S and for some regular central element $f \in Z(S)_2$. In particular, when $d = 3$ (resp. $d = 4$), we say that $\mathrm{Proj}_{\mathrm{nc}} S/(f)$ is a *noncommutative conic (resp. quadric).*

Let Sym(3) be the symmetric group of degree 3 and

$$\mathrm{Sym}^3 V = \{w \in V^{\otimes 3} \mid \theta \cdot w = w \ \forall \theta \in \mathrm{Sym}(3)\}.$$

The following is one of the main results.

Theorem 19 ([5, Proposition 3.4, Lemma 3.5], [6, Corollary 3.8])**.** *Let $S = \mathcal{A}(E, \sigma)$ be a 3-dimensional Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$.*

(1) *If A is commutative, then*
   (a) *either $E = \mathbb{P}^2$ or $E \subset \mathbb{P}^2$ is a triple line, and*
   (b) *A is isomorphic to one of the following algebras:*

$$k[x, y, z]/(x^2),\ k[x, y, z]/(x^2 + y^2),\ k[x, y, z]/(x^2 + y^2 + z^2).$$

(2) *If A is not commutative, then*
   (a) *$|\sigma| = 2$, and*
   (b) *$S = \mathcal{D}(w, 1)$ for some $w \in \mathrm{Sym}^3 V$, and*

$$A \cong k\langle x, y, z\rangle/(yz + zy + ax^2, zx + xz + by^2, xy + yx + cz^2)/(\alpha x^2 + \beta y^2 + \gamma z^2)$$

for some $(a, b, c) \in k^3$ and $(\alpha, \beta, \gamma) \in \mathbb{P}^2$.

6. Classification of A

Lemma 20 ([6, Corollary 3.8])**.** *Let $S = \mathcal{A}(E, \sigma)$ be a 3-dimensional Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$. If A is not commutative, then the quadratic dual algebra $A^! \cong k[X, Y, Z]/(F_1, F_2)$ is a complete intersection where $F_1, F_2 \in k[X, Y, Z]_2$.*

Lemma 21. *There are exactly 6 isomorphism classes of complete intersections of the form $k[X, Y, Z]/(F_1, F_2)$ where $F_1, F_2 \in k[X, Y, Z]_2$. (Classification of pencils of conics, see Table 1.)*

Table 1. List of $k[X, Y, Z]/(F_1, F_2)$

$k[X,Y,Z]/(X^2,Y^2)$,	$k[X,Y,Z]/(X^2-YZ,Z^2)$,
$k[X,Y,Z]/(XZ+Y^2,YZ)$,	$k[X,Y,Z]/(X^2-Y^2,Z^2)$,
$k[X,Y,Z]/(X^2-YZ,Y^2-XZ)$,	$k[X,Y,Z]/(X^2-Y^2,X^2-Z^2)$.

Corollary 22 ([6, Corollary 3.9]). *Let S be a 3-dimensional Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$. There are exactly 9 isomorphism classes of A (3 of them are commutative, and 6 of them are not commutative, see Table 2).*

TABLE 2. List of A

$k[x,y,z]/(x^2)$,	$k[x,y,z]/(x^2+y^2)$,	$k[x,y,z]/(x^2+y^2+z^2)$,
$S^{(0,0,0)}/(x^2)$,	$S^{(0,0,0)}/(x^2+y^2)$,	$S^{(0,0,0)}/(x^2+y^2+z^2)$,
$S^{(1,1,0)}/(x^2)$,	$S^{(1,1,0)}/(3x^2+3y^2+4z^2)$,	$S^{(1,1,0)}/(x^2+y^2-4z^2)$.

$$S^{(a,b,c)} := k\langle x,y,z\rangle/(yz+zy+ax^2, zx+xz+by^2, xy+yx+cz^2).$$

7. Classification of E_A and $C(A)$

If S is a d-dimensional quantum polynomial algebra, $f \in Z(S)_2$ is a regular central element, and $A = S/(f)$, then there exists a unique regular central element $f^! \in Z(A^!)_2$ such that $S^! = A^!/(f^!)$. We define $C(A) := A^![(f^!)^{-1}]_0$.

Theorem 23 ([12, Proposition 5.2]). *If S is a d-dimensional quantum polynomial algebra, $f \in Z(S)_2$ is a regular central element, and $A = S/(f)$, then $\underline{\mathrm{CM}}^{\mathbb{Z}}(A) \cong \mathcal{D}^b(\operatorname{mod} C(A))$.*

Theorem 24 ([5, Lemma 2.6], [6, Proposition 4.3, Lemma 4.4]). *Let $S = \mathcal{A}(E,\sigma)$ be a 3-dimensional Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$. If A is not commutative, then the following holds:*

1. *$C(A)$ is a 4-dimensional commutative Frobenius algebra.*
2. *$Z(S)_2 = \{g^2 \mid g \in S_1\}$ (every $0 \neq f \in Z(S)_2$ is reducible!)*
3. *A satisfies (G1). In fact, if $f = g^2$ for $g \in S_1$, then $\mathcal{P}(A) = (E_A, \sigma_A)$ where*

$$E_A = (E \cap \mathcal{V}(g)) \cup \sigma(E \cap \mathcal{V}(g)),\ \sigma_A = \sigma|_{E_A}.$$

Lemma 25 ([6, Proposition 4.3]). *If*

$$S = k\langle x,y,z\rangle/(yz+zy+ax^2, zx+xz+by^2, xy+yx+cz^2)$$

is a 3-dimensional Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$, then A is not commutative, and

$$\operatorname{Spec} C(A) \cong \{(\alpha,\beta,\gamma) \in \mathbb{A}^3 \mid (\alpha x+\beta y+\gamma z)^2 = f \text{ in } S\}/\sim$$

where $(\alpha,\beta,\gamma) \sim (-\alpha,-\beta,-\gamma)$.

Example 26. If $S = \mathcal{A}(E,\sigma) = k\langle x,y,z\rangle/(yz+zy, zx+xz, xy+yx)$ is a 3-dimensional Calabi-Yau quantum polynomial algebra, then

$$E = \mathcal{V}(xyz),$$
$$\begin{cases} \sigma(0,b,c) = (0,b,-c), \\ \sigma(a,0,c) = (-a,0,c), \\ \sigma(a,b,0) = (a,-b,0). \end{cases}$$

Further, if $f = x^2 + y^2 + z^2 \in Z(S)_2$, and $A = S/(f)$, then

$$(x + y + z)^2 = (x + y - z)^2 = (x - y + z)^2 = (x - y - z)^2 = f$$

in S and $C(A) \cong k^4$, so

$$\operatorname{Spec} C(A) \cong \{(1, 1, 1), (1, 1, -1), (1, -1, 1), (1, -1, -1)\} \subset \mathbb{A}^3.$$

Further, if $g = x + y + z$ so that $g^2 = f$, then

$$\begin{aligned} E_A &= \{(0, 1, -1), (-1, 0, 1), (1, -1, 0) \cup \sigma(\{(0, 1, -1), (-1, 0, 1), (1, -1, 0)\}) \\ &= \{(0, 1, -1), (-1, 0, 1), (1, -1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\} \subset \mathbb{P}^2. \end{aligned}$$

Theorem 27 ([6, Theorem 4.14])**.** *Let S, S' be 3-dimensional Calabi-Yau quantum polynomial algebras, $0 \neq f \in Z(S)_2, f' \in Z(S')_2$, and $A = S/(f), A' = S'/(f')$ such that A, A' are not commutative. Then $E_A \cong E_{A'}$ if and only if $C(A) \cong C(A')$. There are exactly 6 isomorphism classes of E_A (see Table 3), so there are exactly 6 isomorphism classes of $C(A)$ (see Table 4). Moreover, every 4-dimensional commutative Frobenius algebra appears as $C(A)$.*

TABLE 3. Pictures of E_A when A is not commutative

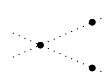
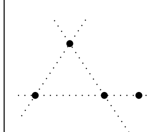
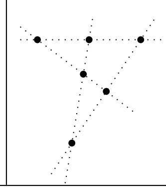
1 line	1 point	2 points	3 points	4 points	6 points
					

TABLE 4. List of $C(A)$ when A is not commutative

$k[u, v]/(u^2, v^2)$,	$k[u]/(u^4)$,	$k[u]/(u^3) \times k$,
$k[u]/(u^2) \times k[u]/(u^2)$,	$k[u]/(u^2) \times k^2$,	k^4.

Corollary 28. *Let S be a 3-dimensional Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$. There are exactly 9 isomorphism classes of $C(A)$.*

8. CLASSIFICATION OF $\operatorname{Proj}_{\text{nc}} A$

It is not easy to classify $\operatorname{Proj}_{\text{nc}} A$ directly. Thanks to the classification of A and that of $C(A)$, we can complete the classification of $\operatorname{Proj}_{\text{nc}} A$.

Theorem 29 ([6, Theorem 5.10])**.** *Let S, S' be 3-dimensional Calabi-Yau quantum polynomial algebras, $0 \neq f \in Z(S)_2, 0 \neq f' \in Z(S')_2$, and $A = S/(f), A' = S'/(f')$. Then*

$$A \cong A' \Rightarrow \operatorname{Proj}_{\text{nc}} A \cong \operatorname{Proj}_{\text{nc}} A' \Rightarrow C(A) \cong C(A').$$

Corollary 30 ([6, Theorem 5.11])**.** *There are exactly 9 isomorphism classes of noncommutative conics in Calabi-Yau quantum $\mathbb{P}^2$'s.*

Finally, we focus on studying noncommutative smooth conics.

Definition 31. We say that $\operatorname{Proj}_{\rm nc} A$ is *smooth* if $\operatorname{gldim}(\operatorname{tails} A) < \infty$.

Theorem 32 ([12, Theorem 5.6], [11, Theorem 5.5])**.** *Let S be a d-dimensional quantum polynomial algebra, $f \in Z(S)_2$ a regular central element, and $A = S/(f)$. Then* $\operatorname{Proj}_{\rm nc} A$ *is smooth if and only if $C(A)$ is semisimple.*

Theorem 33 ([6, Theorem 5.15])**.** *Let S be a 3-dimensional Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$. If* $\operatorname{Proj}_{\rm nc} A$ *is smooth, then exactly one of the following two cases occur:*

(1) (a) *A is commutative.*
  (b) *f is irreducible.*
  (c) *$C(A) \cong M_2(k)$.*
  (d) $\mathcal{D}^b(\operatorname{tails} A) \cong \mathcal{D}^b(\operatorname{mod} k\widetilde{A_1})$*, where $k\widetilde{A}_1$ is the path algebra of the quiver*

$$1 \rightrightarrows 2 \qquad (\widetilde{A_1} \textit{ type}).$$

(2) (a) *A is not commutative.*
  (b) *f is reducible.*
  (c) *$C(A) \cong k^4$.*
  (d) $\mathcal{D}^b(\operatorname{tails} A) \cong \mathcal{D}^b(\operatorname{mod} k\widetilde{D_4})$*, where $k\widetilde{D}_4$ is the path algebra of the quiver*

$$\begin{tikzcd}[column sep=small, row sep=small] 1 \arrow[dr] & & 2 \arrow[dl] \\ & 5 & \\ 3 \arrow[ur] & & 4 \arrow[ul] \end{tikzcd} \qquad (\widetilde{D_4} \textit{ type}).$$

It is known that there are infinitely many Calabi-Yau quantum $\mathbb{P}^2$'s, so it is rather surprising that there are only 9 noncommutative conics in Calabi-Yau quantum $\mathbb{P}^2$'s up to isomorphism of noncommutative schemes, exactly two of them are smooth, and exactly one of them is irreducible.

References

[1] M. Artin and W. Schelter, *Graded algebras of global dimension* 3, Adv. Math., **66** (1987), 171–216.

[2] M. Artin, J. Tate and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, The Grothendieck Festschrift, vol. 1, Progress in Mathematics vol. 86 (Birkhäuser, Basel, 1990) 33–85.

[3] M. Artin and J. J. Zhang, *Noncommutative projective schemes*, Adv. Math. **109** (1994), 228–287.

[4] M. Dubois-Violette, *Multilinear forms and graded algebras*, J. Algebra **317** (2007), 198–215, MR2360146 (2010e:16011).

[5] H. Hu, *Classification of noncommutative conics associated to symmetric regular superpotentials*, preprint (arXiv:2005.03918).

[6] H. Hu, M. Matsuno and I. Mori, *Noncommutative conics in Calabi-Yau quantum projective planes*, preprint (arXiv:2104.00221).

[7] A. Itaba and M. Matsuno, *AS-regularity of geometric algebras of plane cubic curves*, J. Aust. Math. Soc. **112**, no. 2 (2022), 193–217.

[8] M. Matsuno, *A complete classification of* 3*-dimensional quadratic AS-regular algebras of Type EC*, Canad. Math. Bull. **64** (1) (2021), 123–141.

[9] I. Mori and S. P. Smith, *m-Koszul Artin-Schelter regular algebras*, J. Algebra **446** (2006), 373–399.

[10] I. Mori and S. P. Smith, *The classification of* 3*-Calabi-Yau algebras with* 3 *generators and* 3 *quadratic relations*, Math. Z. **287** (2017), no. 1-2, 215–241.
[11] I. Mori and K. Ueyama, *Noncommutative Knörrer's periodicity theorem and noncommutative quadric hypersurfaces*, to appear in Algebra Number Theory, (arXiv:1905.12266).
[12] S. P. Smith, M. Van den Bergh, *Noncommutative quadric surfaces*, J. Noncommut. Geom. **7** (2013) 817–856.

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SYMMETRIC COHOMOLOGY AND SYMMETRIC HOCHSCHILD COHOMOLOGY OF COCOMMUTATIVE HOPF ALGEBRAS

AYAKO ITABA, YUTA SHIBA AND KATSUNORI SANADA

Abstract. Motivated by topological geometry, Staic defined the symmetric cohomology of groups by constructing an action of the symmetric group on the standard resolution which gives the group cohomology. In this paper, our aim is to construct the symmetric cohomology and the symmetric Hochschild cohomology for cocommutative Hopf algebras as a generalization of group algebras. In details, we will investigate the relationships between the symmetric cohomology and the symmetric Hochschild cohomology for cocommutative Hopf algebras. Also, we will investigate the relationships between the cohomology and the symmetric cohomology for cocommutative Hopf algebras.

1. Introduction

This paper is based on [4]. Let G be a group and X a G-module. For $n \geq 0$, we set $C^n(G,X) = \{f : G^n \to X\}$. Motivated by topological geometry, Staic [5] defined the symmetric cohomology $\mathrm{HS}^\bullet(G,X)$ of a group G by constructing an action of the symmetric group $S_{\bullet+1}$ on the standard resolution $\mathrm{C}^\bullet(G,X)$ which gives the group cohomology $\mathrm{H}^\bullet(G,X)$. Also, Staic [6] studied the injectivity of the canonical map

$$\mathrm{HS}^\bullet(G,X) \to \mathrm{H}^\bullet(G,X)$$

induced by the inclusion $\mathrm{CS}^\bullet(G,X) \hookrightarrow \mathrm{C}^\bullet(G,X)$, where $\mathrm{CS}^\bullet(G,X) := \mathrm{C}^\bullet(G,X)^{S_{\bullet+1}}$ is the subcomplex of $\mathrm{C}^\bullet(G,X)$. Moreover, Staic [6] proved that the secondary symmetric cohomology group $\mathrm{HS}^2(G,X)$ is corresponding to extensions of groups which satisfies some conditions.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{C}^0(G,X) \arrow[r] & \mathrm{C}^1(G,X) \arrow[r] & \mathrm{C}^2(G,X) \arrow[r] & \cdots \arrow[r, Rightarrow] & \mathrm{H}^\bullet(G,X) \\
0 \arrow[r] & \mathrm{CS}^0(G,X) \arrow[r] \arrow[u, hook] & \mathrm{CS}^1(G,X) \arrow[r] \arrow[u, hook] & \mathrm{CS}^2(G,X) \arrow[r] \arrow[u, hook] & \cdots \arrow[r, Rightarrow] & \mathrm{HS}^\bullet(G,X) \arrow[u]
\end{tikzcd}$$

In general, the cohomology of groups can be seen as the cohomology of group algebras. Recently, Coconet-Todea [1] defined the symmetric Hochschild cohomology $\mathrm{HHS}^\bullet(A,M)$ of twisted group algebras A which is a generalization of group algebras, where M is an A-bimodule.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{C}_\mathrm{e}^0(A,M) \arrow[r] & \mathrm{C}_\mathrm{e}^1(A,M) \arrow[r] & \mathrm{C}_\mathrm{e}^2(A,M) \arrow[r] & \cdots \arrow[r, Rightarrow] & \mathrm{HH}^\bullet(A,M) \\
0 \arrow[r] & \mathrm{CS}_\mathrm{e}^0(A,M) \arrow[r] \arrow[u, hook] & \mathrm{CS}_\mathrm{e}^1(A,M) \arrow[r] \arrow[u, hook] & \mathrm{CS}_\mathrm{e}^2(A,M) \arrow[r] \arrow[u, hook] & \cdots \arrow[r, Rightarrow] & \mathrm{HHS}^\bullet(A,M) \arrow[u]
\end{tikzcd}$$

The detailed version of this paper has been submitted for publication elsewhere.

In this paper, our aim is to construct the symmetric cohomology and the symmetric Hochschild cohomology for cocommutative Hopf algebras as another generalization of group algebras. In details, we investigate the relationships between the symmetric cohomology and the symmetric Hochschild cohomology for cocommutative Hopf algebras (Theorem 6). Also, we investigate the relationships between the cohomology and the symmetric cohomology for cocommutative Hopf algebras (Theorem 8).

2. Symmetric cohomology and symmetric Hochschild cohomology for cocomutative Hopf algebras

In the rest of this paper, let k be a field. For simplicity, we put $\otimes = \otimes_k$.

A k-algebra A is called *a Hopf algebra* if A is a k-algebra and a k-coalgebra satisfying

$$\pi \circ (\mathrm{id}_A \otimes S) \circ \Delta = \eta \circ \varepsilon = \pi \circ (S \otimes \mathrm{id}_A) \circ \Delta,$$

where the structure morphisms are as follows:

- $\pi : A \otimes A \to A$: product; $a \otimes b \mapsto ab$,
- $\eta : k \to A$: unit; $x \mapsto x \cdot 1_A$,
- $\Delta : A \to A \otimes A$: coproduct,
- $\varepsilon : A \to k$: counit,
- $S : A \to A$: antipode.

A Hopf algebra A is *cocommutative* if $a^{(1)} \otimes a^{(2)} = a^{(2)} \otimes a^{(1)}$ holds. Note that we use some standard notation for the coproduct, so called *Sweedler notation*; we write $\Delta(a) = \sum a^{(1)} \otimes a^{(2)}$, where the notation $a^{(1)}, a^{(2)}$ for tensor factors is symbolic. Throughout the paper, we omit the summation symbol $\sum$ of Sweedler notation when no confusion occurs (for details, see [7]).

Example 1. (1) Let G be a group, $A = kG$ a group algebra. For $g \in G$, we set
- coproduct $\Delta(g) := g \otimes g$,
- counit $\varepsilon(g) := 1$,
- antipode $S(g) := g^{-1}$,

then A is a cocommutive Hopf algebra.

(2) Let $A = k[X]$ be a polynomial ring. We set
- coproduct $\Delta(X) := 1 \otimes X + X \otimes 1$,
- counit $\varepsilon(X) := 0$,
- antipode $S(X) := -X$,

then A is a cocommutive Hopf algebra.

(3) Let A be a (cocommutaive) Hopf algebra. Then the opposite algebra A^{op} of A is a (cocommutative) Hopf algebra.

(4) Let A and B be (cocommutaive) Hopf algebras. Then $A \otimes B$ is a (cocommutative) Hopf algebra. In particular, if A is a (cocommutative) Hopf algebra, then the enveloping algebra $A^{\mathrm{e}} := A \otimes A^{\mathrm{op}}$ of A is a (cocommutative) Hopf algebra.

We recall the definition of a module over a Hopf algebra.

Definition 2 (cf. [9, Section 9.2])**.** Let A be a Hopf algebra and M, N left A-modules.

(1) For $a \in A$, $m \in M$ and $n \in N$,

$$a \cdot (m \otimes n) := a^{(1)} m \otimes a^{(2)} n. \text{ Then } M \otimes N \text{ is a left } A\text{-module.}$$

(2) For $a \in A$, $f \in \mathrm{Hom}_k(M, N)$ and $m \in M$,

$$(a \cdot f)(m) := a^{(1)} f(S(a^{(2)})m). \text{ Then } \mathrm{Hom}_k(M, N) \text{ is a left } A\text{-module.}$$

(3) A submodule ${}^A M$ of M is defined by ${}^A M := \{m \in M \mid a \cdot m = \varepsilon(a)m\}$, which is called an A-invariant submodule of M. For a right A-module M, M^A is defined similarly.

(4) Let M an A-bimodule. For $a \in A$ and $m \in M$, $a \cdot m := a^{(1)} m S(a^{(2)})$, which is called a left adjoint action. Using this action, we denote the left A-module by ${}^{\mathrm{ad}}M$. Similarly, we define a right adjoint action and M^{ad}.

Let A be a Hopf algebra, and M and N left A-modules. Then there is an isomorphism $\mathrm{Hom}_A(M, N) \cong {}^A(\mathrm{Hom}_k(M, N))$ as k-vector spaces (cf. [9, Lemma 9.2.2]). We define the cohomology of a Hopf algebra $\mathrm{H}^n(A, M) := \mathrm{Ext}^n_A(k, M)$.

Here, we construct the projective resolution of k as left A-modules as follows.

- $\widetilde{\mathrm{T}}_n(A) = A^{\otimes n+1}$; $\forall b \in A$,

$$b \cdot (a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}) = b^{(1)} a_1 \otimes b^{(2)} a_2 \otimes \cdots \otimes b^{(n+1)} a_{n+1}.$$

- $\cdots \longrightarrow \widetilde{\mathrm{T}}_n(A) \xrightarrow{d^{\widetilde{\mathrm{T}}}_n} \widetilde{\mathrm{T}}_{n-1}(A) \longrightarrow \cdots \longrightarrow \widetilde{\mathrm{T}}_0(A) \xrightarrow{d^{\widetilde{\mathrm{T}}}_0} k \longrightarrow 0,$

$$d^{\widetilde{\mathrm{T}}}_n(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i-1} a_1 \otimes \cdots \otimes \varepsilon(a_i) a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

We set the complex $\mathrm{K}^\bullet(A, M) := \mathrm{Hom}_A(\widetilde{\mathrm{T}}_\bullet(A), M)$.

Let A be a cocommutative Hopf algebra and M a left A-module. The n-th symmetric group is denoted by S_n. We define an action $\sigma_i = (i, i+1) \in S_{n+1}$ on $\mathrm{K}^n(A, M)$. For $f \in \mathrm{K}^n(A, M)$ $(1 \le \forall i \le n)$,

$$(\sigma_i \cdot f)(a_1 \otimes \cdots \otimes a_{n+1}) := -f(a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_{n+1}).$$

We set the subcomplex $\mathrm{KS}^\bullet(A, M) := \mathrm{K}^\bullet(A, M)^{S_{\bullet+1}}$ of $\mathrm{K}^\bullet(A, M)$.

Definition 3 ([4, Definition 3.3])**.** We define *the symmetric cohomology of a cocommutative Hopf algebra*

$$\mathrm{HS}^n(A, M) := \mathrm{H}^n(\mathrm{KS}^\bullet(A, M)),$$

Let A be a Hopf algebra and M an A-bimodule, where $A^{\mathrm{e}} = A \otimes A^{\mathrm{op}}$ is the enveloping algebra of A. We define Hochschild cohomology $\mathrm{HH}^n(A, M) = \mathrm{Ext}^n_{A^{\mathrm{e}}}(A, M)$ of A. We construct the projective resolution of A as A-bimodules as follows.

- $\widetilde{\mathrm{T}}^{\mathrm{e}}_n(A) = A^{\otimes n+2}$; for $b \otimes c^{\mathrm{op}} \in A^{\mathrm{e}}$,

$$(b \otimes c^{\mathrm{op}}) \cdot (a_1 \otimes a_2 \otimes \cdots \otimes a_{n+2}) = b^{(1)} a_1 \otimes b^{(2)} a_2 \otimes \cdots \otimes b^{(n+2)} a_{n+2} c.$$

- $\cdots \longrightarrow \widetilde{\mathrm{T}}^{\mathrm{e}}_n(A) \xrightarrow{d^{\widetilde{\mathrm{T}}^{\mathrm{e}}}_n} \widetilde{\mathrm{T}}^{\mathrm{e}}_{n-1}(A) \longrightarrow \cdots \longrightarrow \widetilde{\mathrm{T}}^{\mathrm{e}}_0(A) \xrightarrow{d^{\widetilde{\mathrm{T}}^{\mathrm{e}}}_0} A \longrightarrow 0,$

$$d^{\widetilde{\mathrm{T}}^{\mathrm{e}}}_n(a_1 \otimes \cdots \otimes a_{n+2}) = \sum_{i=1}^{n+1} (-1)^{i-1} a_1 \otimes \cdots \otimes \varepsilon(a_i) a_{i+1} \otimes \cdots \otimes a_{n+2}.$$

We set the complex $\mathrm{K}^{\bullet}_{\mathrm{e}}(A, M) := \mathrm{Hom}_{A^{\mathrm{e}}}(\widetilde{\mathrm{T}}^{\mathrm{e}}_{\bullet}(A), M)$.

Let A be a cocommutative Hopf algebra and M an A-bimodule. The n-th symmetric group is denoted by S_n. We define an action $\sigma_i = (i,\, i+1) \in S_{n+1}$ on $\mathrm{K}^n_{\mathrm{e}}(A, M)$. For $f \in \mathrm{K}^n_{\mathrm{e}}(A, M)$ $(1 \leq \forall i \leq n)$,

$$(\sigma_i \cdot f)(a_1 \otimes \cdots \otimes a_{n+2}) := -f(a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_{n+2}).$$

We set the subcomplex $\mathrm{KS}^{\bullet}_{\mathrm{e}}(A, M) := \mathrm{K}^{\bullet}_{\mathrm{e}}(A, M)^{S_{\bullet+1}}$ of $\mathrm{K}^{\bullet}_{\mathrm{e}}(A, M)$.

Definition 4 ([4, Definition 3.8])**.** We define *the symmetric Hochschild cohomology of a cocommutative Hopf algebra*

$$\mathrm{HHS}^n(A, M) := \mathrm{H}^n(\mathrm{KS}^{\bullet}_{\mathrm{e}}(A, M)).$$

3. Main results

First, our aim is to investigate the relationships between the symmetric cohomology and the symmetric Hochschild cohomology for cocommutative Hopf algebras.

Theorem 5 ([2, Section 5])**.** *Let G be a group and X a G-bimodule. Then, for each $n \geq 0$, there is an isomorphism*

$$\mathrm{HH}^n(\mathbb{Z}G, X) \cong \mathrm{H}^n(G, {}^{\mathrm{ad}}X)$$

as $\mathbb{Z}$-modules, where ${}^{\mathrm{ad}}X$ is a left G-module by $g \cdot x = gxg^{-1}$ for $g \in G$ and $x \in X$.

Theorem 5 is generalized to the case of Hopf algebras by Ginzburg-Kumar [3, Section 5].

For a cocomutative Hopf algebra, we have the following result which is a symmetric version of the classical results by Eilenberg-MacLane and Ginzburg-Kumar.

Theorem 6 ([4, Theorem 4.5])**.** *Let A be a cocommutative Hopf algebra and M an A-bimodule. Then, for each $n \geq 0$, there is an isomorphism*

$$\mathrm{HHS}^n(A, M) \cong \mathrm{HS}^n(A, {}^{\mathrm{ad}}M)$$

as k-vector spaces, where ${}^{\mathrm{ad}}M$ is a left A-module acting by the left adjoint action, that is, $a \cdot m = a^{(1)} m S(a^{(2)})$ for $m \in {}^{\mathrm{ad}}M$ and $a \in A$.

As a byproduct of Theorem 6, we have the following assertion.

Corollary 7 ([4, Corollary 4.6])**.** *Let A be a finite dimensional, commutative and cocommutative Hopf algebra. Then, for each $n \geq 0$, there is an isomorphism*

$$\mathrm{HHS}^n(A, A) \cong A \otimes \mathrm{HS}^n(A, k)$$

as k-vector spaces.

Secondly, our aim is to investigate the relationships between the cohomology and the symmetric cohomology for cocommutative Hopf algebras.

In [6] and [8], the following consequences were proved for the lower degree.

- $\mathrm{HS}^0(G, X) \cong \mathrm{H}^0(G, X)$.
- $\mathrm{HS}^1(G, X) \cong \mathrm{H}^1(G, X)$.
- $\mathrm{HS}^2(G, X) \hookrightarrow \mathrm{H}^2(G, X)$.
  Moreover, if G has no elements of order 2, then $\mathrm{HS}^2(G, X) \cong \mathrm{H}^2(G, X)$.

We consider the resolution of k;

- k is a trivial left kS_{n+1}-module; $\tau \cdot x = \varepsilon(\tau)x = x \quad (\tau \in S_{n+1},\ x \in k)$.
- $\widetilde{\mathrm{T}}_n(A)$ is a right kS_{n+1}-module; for $\sigma_i \in S_{n+1}$ $(1 \le \forall i \le n)$

$$(a_1 \otimes \cdots \otimes a_{n+1}) \cdot \sigma_i = -a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_{n+1}.$$

- $\widetilde{\mathrm{S}}_n(A) := \widetilde{\mathrm{T}}_n(A) \otimes_{kS_{n+1}} k$.
- $\cdots \longrightarrow \widetilde{\mathrm{S}}_n(A) \xrightarrow{d_n^{\widetilde{\mathrm{S}}}} \cdots \longrightarrow \widetilde{\mathrm{S}}_0(A) \xrightarrow{d_0^{\widetilde{\mathrm{S}}}} k \longrightarrow 0,$

$$d_n^{\widetilde{\mathrm{S}}}((a_1 \otimes \cdots \otimes a_{n+1}) \otimes_{kS_{n+1}} x) = d_n^{\widetilde{\mathrm{T}}}(a_1 \otimes \cdots \otimes a_{n+1}) \otimes_{kS_n} x.$$

Then we have the following isomorphism as complexes $\mathrm{KS}^\bullet(A, M) \cong \mathrm{Hom}_A(\widetilde{\mathrm{S}}_\bullet(A), M)$. Therefore, we have $\mathrm{HS}^n(A, M) \cong \mathrm{H}^n(\mathrm{Hom}_A(\widetilde{\mathrm{S}}_\bullet(A), M))$.

Theorem 8 ([4, Theorem 4.9, Remark 4.10])**.** *Let A be a cocommutative Hopf algebra. For each $n \ge 1$, if $\mathrm{ch}\, k \nmid n+1$, then $\widetilde{\mathrm{S}}_n(A)$ is projective as a left A-module.*

Therefore, if $\mathrm{ch}\, k \nmid (n+1)!$, then, for each $0 \le m \le n$, there is an isomorphism

$$\mathrm{H}^m(A, M) \cong \mathrm{HS}^m(A, M)$$

as k-vector spaces.

Remark 9. (1) By Theorem 8, if $\mathrm{ch}\, k = 0$, then $\widetilde{\mathrm{S}}_\bullet(A)$ is a projective resolution of k, and hence there is an isomorphism $\mathrm{H}^\bullet(A, M) \cong \mathrm{HS}^\bullet(A, M)$ as k-vector spaces.
(2) Moreover, by Theorem 6 and Theorem 8, if $\mathrm{ch}\, k = 0$, then there is an isomorphism $\mathrm{H}^\bullet(A, {}^{\mathrm{ad}}M) \cong \mathrm{HS}^\bullet(A, {}^{\mathrm{ad}}M) \cong \mathrm{HHS}^\bullet(A, M)$ as k-vector spaces, where ${}^{\mathrm{ad}}M$ is a left A-module acting by the left adjoint action.

Finally, we give an example of the resolution which gives symmetric cohomology. Let p be an odd prime number, k a field of characteristic p and C_p a cyclic group of order p. Then we calculate the symmetric cohomology of $A = kC_p$.

Proposition 10 ([4, Proposition 4.11])**.** *Let p be an odd prime number, $\mathrm{ch}\, k = p$ and $A = kC_p$. Then $\widetilde{\mathrm{S}}_n(A)$ is a free A-module with rank $\dfrac{{}_pC_{n+1}}{p}$ for each $1 \le n \le p-2$.*

Since $\widetilde{\mathrm{S}}_{p-1}(A)$ is isomorphic to k as a left A-module, the resolution of k is the following exact sequence

$$0 \to k \xrightarrow{d_{p-1}^{\widetilde{\mathrm{S}}}} \widetilde{\mathrm{S}}_{p-2}(A) \to \cdots \to \widetilde{\mathrm{S}}_1(A) \xrightarrow{d_1^{\widetilde{\mathrm{S}}}} \widetilde{\mathrm{S}}_0(A) \xrightarrow{d_0^{\widetilde{\mathrm{S}}}} k \to 0,$$

where $\widetilde{\mathrm{S}}_i(A)$ is a free A-module for each $0 \le i \le p-2$. This implies that there is an isomorphism

$$\mathrm{H}^n(A, M) \cong \mathrm{HS}^n(A, M)$$

for any left A-module M and each $0 \le n \le p-2$. Also, in the case of $n = p-1$, the above isomorphism is obtained by simple calculation. Note that the period of the cohomology group $\mathrm{H}^n(A, M)$ of A is 2.

Summarizing the above, we have

$$\mathrm{HS}^n(A, M) \cong \begin{cases} \mathrm{H}^n(A, M) & (0 \leq n \leq p-1), \\ 0 & (p \leq n). \end{cases}$$

References

[1] T. Coconet and C.-C. Todea, *Symmetric Hochschild cohomology of twisted group algebras*, arXiv:2103.13695 (2021).
[2] S. Eilenberg and S. MacLane, *Cohomology theory in abstract groups I*, Ann. of Math. **48**, 51–78, (1947).
[3] V. Ginzburg and S. Kumar, *Cohomology of quantum groups at roots of unity*, Duke Math. J. **69** (1993), no.1, 179–198.
[4] Y. Shiba, K. Sanada and A. Itaba, *Symmetric cohomology and symmetric Hochschild cohomology of cocommutative Hopf algebras*, arXiv: 2203.17043 (2022).
[5] M. D. Staic, *From* 3*-algebras to* Δ*-groups and symmetric cohomology*, J. Algebra **322** (2009), no.4, 1360–1378.
[6] ———, *Symmetric cohomology of groups in low dimension*, Arch. Math. **93** (2009), no.3, 205–211.
[7] M. E. Sweedler, *Hopf algebras*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York, (1969) pp. vii+336.
[8] C.-C. Todea, *Symmetric cohomology of groups as a Mackey functor*, Bull. Belg. Math. Soc. **22** (2015) 49–58.
[9] S. J. Witherspoon, *Hochschild cohomology for algebras*, Graduate Studies in Mathematics, vol. 204, Amer. Math. Soc., (2019).

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PROJECTIVE OBJECTS IN THE CATEGORY OF DISCRETE MODULES OVER A PROFINITE GROUP

RYO KANDA

Abstract. This is a summary of a joint work with Alexandru Chirvasitu. We showed that the category of discrete modules over an infinite profinite group has no non-zero projective objects and does not satisfy Ab4*.

Key Words: Profinite group; discrete module; projective object.

2010 *Mathematics Subject Classification*: Primary 18G05; Secondary 20E18, 16D40, 18E15, 18A30.

1. Introduction

This is a summary of a joint work with Alexandru Chirvasitu [CK19].

It is known that $\operatorname{Mod} R$ and $\operatorname{QCoh} X$ are both Grothendieck categories, where $\operatorname{Mod} R$ is the category of (left) modules over a ring R and $\operatorname{QCoh} X$ is the category of quasi-coherent sheaves on a scheme X. In particular, they both have exact direct limits (and hence exact direct sums) and enough injectives.

$\operatorname{Mod} R$ also has exact direct products (this property is called Grothendieck's Ab4* condition) and enough projectives, while it is known that none of these holds for $\operatorname{QCoh} X$ when X is a non-affine divisorial noetherian scheme:

Theorem 1 ([Kan19])**.** *Let X be a divisorial noetherian scheme. Then the following conditions are equivalent:*

(1) $\operatorname{QCoh} X$ *has enough projectives.*
(2) $\operatorname{QCoh} X$ *has exact direct products.*
(3) X *is an affine scheme.*

2. Main result

We consider a similar question concerning the category of discrete modules over a profinite group.

Definition 2. Let G be a topological group.

(1) G is called a *profinite group* if G is an inverse limit of finite discrete groups in the category of topological groups.

This is a summary of [CK19]. The detailed version of this paper has been submitted for publication elsewhere.

Ryo Kanda was supported by JSPS KAKENHI Grant Numbers JP16H06337, JP17K14164, JP20K14288, and JP21H04994, Leading Initiative for Excellent Young Researchers, MEXT, Japan, and Osaka Central Advanced Mathematical Institute: MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849.

(2) A *discrete G-module* is a topological G-module that is discrete as a topological space.

Let G be a profinite group. Considering a discrete G-module is equivalent to considering a (non-topological) G-module M (that is, left $\mathbb{Z}G$-module) such that the action $G \times M \to M$ is continuous if we endow M with the discrete topology. The forgetful functor gives an embedding of the category of discrete G-modules into $\operatorname{Mod} \mathbb{Z}G$ as a full subcategory. The essential image of the functor consists of all $M \in \operatorname{Mod} \mathbb{Z}G$ such that

$$M = \bigcup_{H} M^H,$$

where H runs over all open normal subgroups of G and

$$M^H := \{\, x \in M \mid hx = x \text{ for all } h \in H \,\}.$$

Our main result is the following:

Theorem 3 ([CK19]). *Let G be a profinite group. Then the following conditions are equivalent:*

(1) *The category of discrete G-modules has enough projectives.*
(2) *The category of discrete G-modules has a nonzero projective object.*
(3) *The category of discrete G-modules has exact direct products.*
(4) *G is a finite group.*

References

[CK19] Alexandru Chirvasitu and Ryo Kanda, *Projective discrete modules over profinite groups*, arXiv:1907.12496v1.

[Kan19] Ryo Kanda, *Non-exactness of direct products of quasi-coherent sheaves*, Doc. Math. **24** (2019), 2037–2056. MR 4033823

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ON THE OPENNESS OF LOCI OVER NOETHERIAN RINGS

KAITO KIMURA

Abstract. In this article, we consider openness of loci of modules over commutative noetherian rings. One of the main theorems asserts that the finite injective dimension loci over an acceptable ring are open. We give a module version of the Nagata criterion, and confirm that it holds for some properties of modules.

Key Words: openness of loci, Nagata criterion, finite injective dimension, Gorenstein, Cohen–Macaulay.

2000 *Mathematics Subject Classification*: 13D05, 13C14.

1. Introduction

We refer the reader to [3] (`arXiv:2201.11955`) for details on the contents of this article. Throughout this article, we assume that R is a commutative noetherian ring and that M is a finitely generated R-module.

Let $\mathbb{P}$ be a property of modules over a commutative local ring. The set of prime ideals $\mathfrak{p}$ of R such that the module $M_\mathfrak{p}$ over the local ring $R_\mathfrak{p}$ satisfies $\mathbb{P}$ is called the $\mathbb{P}$-locus of M (over R). There is a topology on $\operatorname{Spec}(R)$, which is called the Zariski topology. It is a natural question to ask when the $\mathbb{P}$-locus is open in the Zariski topology for a given $\mathbb{P}$. There are a lot of study studies about this question. The Cohen–Macaulay locus of a module over an excellent ring is open [2]. Furthermore, the Gorenstein locus of a module over an acceptable ring in the sense of Sharp [7] is open [4], and so is the finite injective dimension locus of a module over an excellent ring [8].

In this article, we consider the openness of the finite injective dimension locus of a module. The first main theorem of this article is the following theorem concerning the finite injective dimension locus over an acceptable ring.

Theorem 1. *The finite injective dimension locus of a module over an acceptable ring is open in the Zariski topology. In particular, the finite injective dimension locus of a module over a homomorphic image of a Gorenstein ring is open.*

For a property $\mathbb{P}$ of commutative local rings, the set of prime ideals $\mathfrak{p}$ of R such that the local ring $R_\mathfrak{p}$ satisfies $\mathbb{P}$ is called the $\mathbb{P}$-locus of R. Nagata [6] produced the following condition, which is called the Nagata criterion: if the $\mathbb{P}$-locus of $R/\mathfrak{p}$ contains a nonempty open subset of $\operatorname{Spec}(R/\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ of R, then the $\mathbb{P}$-locus of R is an open subset of $\operatorname{Spec}(R)$. This statement holds for the regular, complete intersection, Gorenstein, and Cohen–Macaulay properties and Serre's conditions; see [1, 5, 6]. We give a module version of the Nagata criterion for properties of modules, and show that it holds for the finite injective dimension property.

The detailed version [3] of this article has been submitted for publication elsewhere.

Theorem 2. *If the finite injective dimension locus of $M/\mathfrak{p}M$ over $R/\mathfrak{p}$ contains a nonempty open subset of* $\mathrm{Spec}(R/\mathfrak{p})$ *for all* $\mathfrak{p} \in \mathrm{Supp}_R(M)$*, then the finite injective dimension locus of M over R is an open subset of* $\mathrm{Spec}(R)$*.*

It is seen that some results on the finite injective dimension property hold on other properties of modules; see [3].

2. Comments on Theorem 1

We begin with proving our key proposition. For an ideal I of R, we set $\mathrm{V}(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I \subseteq \mathfrak{p}\}$. Below is called the topological Nagata criterion.

Lemma 3. *Let U be a subset of* $\mathrm{Spec}(R)$*. Then U is open if and only if the following two statements hold true.*
(1) *If $\mathfrak{p} \in U$ and $\mathfrak{q} \in \mathrm{Spec}(R)$ with $\mathfrak{q} \subseteq \mathfrak{p}$, then $\mathfrak{q} \in U$.*
(2) *U contains a nonempty open subset of $\mathrm{V}(\mathfrak{p})$ for all $\mathfrak{p} \in U$.*

Denote by $\mathsf{FID}_R(M)$ the finite injective dimension locus of M over R. The Gorenstein locus of R is denoted by $\mathsf{Gor}(R)$. Note that $\mathsf{FID}_R(M)$ satisfies (1) in the above lemma for any R-module M. Therefore, in order to show that the finite injective dimension locus is open, it suffices to verify that it satisfies (2) in the above lemma. The key role is played by the proposition below.

Proposition 4. *Let $\mathfrak{p} \in \mathrm{Supp}_R(M) \cap \mathsf{FID}_R(M)$. The following conditions are equivalent.*
(1) *$\mathsf{FID}_R(M)$ contains a nonempty open subset of $\mathrm{V}(\mathfrak{p})$.*
(2) *$\mathsf{Gor}(R/\mathfrak{p})$ contains a nonempty open subset of $\mathrm{Spec}(R/\mathfrak{p})$.*

Proof. We may assume that for any integer $i \geq 0$, $\mathrm{Ext}^i_R(R/\mathfrak{p}, M)$ is free as an $R/\mathfrak{p}$-module. Let $I : 0 \to I^0 \to I^1 \to \cdots$ be a minimal injective resolution of M. The complex

$$0 \to \mathrm{Hom}_R(R/\mathfrak{p}, I^n) \xrightarrow{d^n} \mathrm{Hom}_R(R/\mathfrak{p}, I^{n+1}) \xrightarrow{d^{n+1}} \mathrm{Hom}_R(R/\mathfrak{p}, I^{n+2}) \xrightarrow{d^{n+2}} \cdots$$

is an injective resolution of $\mathrm{Ker}\, d^n$ as an $R/\mathfrak{p}$-module. For $\mathfrak{q} \in \mathrm{V}(\mathfrak{p})$, we see that $\mathfrak{q} \in \mathsf{FID}_R(M)$ if and only if $\mathfrak{q}/\mathfrak{p} \in \mathsf{FID}_{R/\mathfrak{p}}(\mathrm{Ker}\, d^n)$. It is easy to see that the latter holds if and only if $\mathfrak{q}/\mathfrak{p} \in \mathsf{Gor}(R/\mathfrak{p})$. This means that the equivalence holds. $\square$

The result below can be obtained from Proposition 4.

Corollary 5. *Suppose that $\mathsf{Gor}(R/\mathfrak{p})$ contains a nonempty open subset of $\mathrm{Spec}(R/\mathfrak{p})$ for any $\mathfrak{p} \in \mathrm{Supp}_R(M)$. Then $\mathsf{FID}_R(M)$ is an open subset of $\mathrm{Spec}(R)$.*

Corollary 5 states that the finite injective dimension locus of M over R is open if the ring R satisfies the assumption of Nagata criterion for the Gorensteinness. Hence this corollary yields Theorem 1.

Remark 6. Theorem 1 recovers [8, Corollary 2.6] since any excellent ring is acceptable.

3. Comments on Theorem 2

Let $\mathbb{P}$ be a property of modules over a commutative local ring. In this section, we consider the following statement, which is the module version of the Nagata criterion: if the $\mathbb{P}$-locus of $M/\mathfrak{p}M$ over $R/\mathfrak{p}$ contains a nonempty open subset of $\operatorname{Spec}(R/\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Supp}_R(M)$, then the $\mathbb{P}$-locus of M over R is an open subset of $\operatorname{Spec}(R)$.

We denote by $\mathsf{Free}_R(M)$ the free locus of M over R. It is well-known fact that $\mathsf{Free}_R(M)$ is always open. We prepare the following lemma to state Theorem 2.

Lemma 7. *Let $\mathfrak{p} \in \operatorname{Supp}_R(M)$. The following are equivalent.*
(1) $\mathsf{Gor}(R/\mathfrak{p})$ *contains a nonempty open subset of* $\operatorname{Spec}(R/\mathfrak{p})$.
(2) $\mathsf{FID}_{R/\mathfrak{p}}(M/\mathfrak{p}M)$ *contains a nonempty open subset of* $\operatorname{Spec}(R/\mathfrak{p})$.

Proof. We obtain $\operatorname{Supp}_R(M/\mathfrak{p}M) = \mathrm{V}(\mathfrak{p})$, and thus $\operatorname{Supp}_{R/\mathfrak{p}}(M/\mathfrak{p}M) = \operatorname{Spec}(R/\mathfrak{p})$. Hence, we have

$$\mathsf{Gor}(R/\mathfrak{p}) \cap \mathsf{Free}_{R/\mathfrak{p}}(M/\mathfrak{p}M) = \mathsf{FID}_{R/\mathfrak{p}}(M/\mathfrak{p}M) \cap \mathsf{Free}_{R/\mathfrak{p}}(M/\mathfrak{p}M).$$

Since the set $\mathsf{Free}_{R/\mathfrak{p}}(M/\mathfrak{p}M)$ is a nonempty open subset of $\operatorname{Spec}(R/\mathfrak{p})$, we see that the equivalence holds. □

Theorem 2 follows from this lemma and Corollary 5. Theorem 2 asserts that the module version of the Nagata criterion holds for the finite injective dimension property.

4. Other properties of modules

Some results on the finite injective dimension property as we gave in the previous sections hold on other properties.

The Cohen–Macaulay locus of R is denoted by $\mathsf{CM}(R)$. Denote by $\mathsf{CM}_R(M)$ (resp. $\mathsf{MCM}_R(M)$) the Cohen–Macaulay (resp. the maximal Cohen–Macaulay) locus of M over R. The same assertion as Proposition 4 holds for the (maximal) Cohen–Macaulay property.

Proposition 8. *Let $\mathfrak{p} \in \operatorname{Supp}_R(M) \cap \mathsf{CM}_R(M)$. The following conditions are equivalent.*
(1) $\mathsf{CM}_R(M)$ *contains a nonempty open subset of* $\mathrm{V}(\mathfrak{p})$.
(2) $\mathsf{CM}(R/\mathfrak{p})$ *contains a nonempty open subset of* $\operatorname{Spec}(R/\mathfrak{p})$.
In addition, if $\mathfrak{p}$ belongs to $\mathsf{MCM}_R(M)$, then the following is also equivalent.
(3) $\mathsf{MCM}_R(M)$ *contains a nonempty open subset of* $\mathrm{V}(\mathfrak{p})$.

The result below is a Cohen–Macaulay version of Corollary 5.

Corollary 9. *Suppose that $\mathsf{CM}(R/\mathfrak{p})$ contains a nonempty open subset of $\operatorname{Spec}(R/\mathfrak{p})$ for any $\mathfrak{p} \in \operatorname{Supp}_R(M)$. Then $\mathsf{CM}_R(M)$ and $\mathsf{MCM}_R(M)$ are open.*

Remark 10. The same assertion as Corollary 5 holds for the Gorenstein property, the Cohen–Macaulay property, the maximal Cohen–Macaulay property, and Serre's conditions. In particular, the module version of the Nagata criterion holds for all of the aforementioned properties; see [3]. Those results recover theorems of Greco and Marinari [1] and of Massaza and Valabrega [5] about the Nagata criterion.

References

[1] S. Greco; M. G. Marinari, *Nagata's criterion and openness of loci for Gorenstein and complete intersection*, Math. Z. **160** (1978), no. 3, 207–216.

[2] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math.

[3] K. Kimura, *Openness of various loci over Noetherian rings*, preprint (2022), arXiv:2201.11955.

[4] G. J. Leuschke, *Gorenstein modules, finite index, and finite Cohen–Macaulay type*, Comm. Algebra **30** (2002), no. 4, 2023–2035.

[5] C. Massaza; P. Valabrega, *Sull'apertura di luoghi in uno schema localmente noetheriano*, Boll. Un. Mat. Ital. A (5) **14** (1977), no. 3, 564–574.

[6] M. Nagata, *On the closedness of singular loci*, Inst. Hautes Études Sci. Publ. Math. **2** (1959), 29–36.

[7] R. Y. Sharp, *Acceptable rings and homomorphic images of Gorenstein rings*, J. Algebra **44** (1977), no. 1, 246–261.

[8] R. Takahashi, *Openness of FID–loci*, Glasg. Math. J. **48** (2006), no. 3, 431–435.

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COMBINATORICS OF QUASI-HEREDITARY STRUCTURES

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Abstract. A quasi-hereditary algebra is a finite dimensional algebra together with a partial order on its set of isomorphism classes of simple modules which satisfies certain conditions. In this research, for a given algebra A, we study that how many partial orders make A to be quasi-hereditary. In particular, we classify such orders for path algebras of Dynkin type A. This proceeding is based on a paper [7].

1. Introduction

Quasi-hereditary algebras were defined in [11] as an algebraic axiomatization of the theory of rational representations of semi-simple algebraic groups. They were generalized to the concept of highest weight categories soon after in [2] as a tool to study highest weight theories which arise in the representation theories of semi-simple complex Lie algebras and reductive groups. There are many examples of such algebras, Schur algebras, algebras of global dimension at most two, incidence algebras and many more.

A quasi-hereditary algebra is a finite dimensional algebra together with a partial order on its set of isomorphism classes of simple modules which satisfies certain conditions. In the examples above, the partial order predated (and motivated) the theory, so the choice was clear (see [4]). However, there are instances of quasi-hereditary algebras having many possible choices of the partial order. So one may wonder about all the possible orders. In this research, we will study such all possible choices of orders.

Throughout this paper, let K be a field, A a finite dimensional K-algebra. We denote by $\mathsf{mod}A$ the category of finitely generated right A-modules and denote by $\{S(i) \mid i \in I\}$ a complete set of isomorphism classes of simple A-modules with an indexing set I. Let $P(i)$ and $I(i)$ be the projective cover and the injective envelop of $S(i)$, respectively. For an A-module M, we denote by $[M : S(i)]$ the Jordan-Hölder multiplicity of $S(i)$ in M.

Standard modules and costandard modules are fundamental concepts to define and study quasi-hereditary algebras.

Definition 1. Let $\lhd$ be a partial order on I. A *standard module* $\Delta(i)$ with weight $i \in I$ is the largest factor module of $P(i)$ such that each composition factor $S(j)$ satisfies $j \lhd i$. Dually, a *costandard module* $\nabla(i)$ with weight $i \in I$ is the largest submodule of $I(i)$ such that each composition factor $S(j)$ satisfies $j \lhd i$. We write $\Delta = \{\Delta(i) \mid i \in I\}$ and $\nabla = \{\nabla(i) \mid i \in I\}$.

Let Θ be a class of A-modules. We denote by $\mathcal{F}(\Theta)$ the subcategory of $\mathsf{mod}A$ consisting of all A-modules which have a Θ-filtration, that is, a module M with a chain of submodules $M_n \subset M_{n-1} \subset \cdots \subset M_1 \subset M_0 = M$ such that $M_i/M_{i-1} \in \Theta$.

The detailed version of this paper is [7].

Definition 2. [6] A partial order $\lhd$ on I is *adapted* to A if it satisfies that, for an A-module M with its top $S(i)$ and its socle $S(j)$, where i and j are incomparable by $\lhd$, there exists $k \in I$ such that $i \lhd k$ and $j \lhd k$ and $[M : S(k)] \neq 0$.

For example, any total order on I is adapted to A. In general, the standard, and the costandard modules will change when we refine the order. Dlab and Ringel [6] introduced adapted orders on I in order to consider refinements of partial orders. Namely, if $\lhd_2$ is a refinements of $\lhd_1$, then $\Delta_1(i) = \Delta_2(i)$ (also $\nabla_1(i) = \nabla_2(i)$) holds for any $i \in I$, where $\Delta_j(i)$ is a standard module with weight $i \in I$ associated to $\lhd_j$.

We define quasi-hereditary algebras.

Definition 3. [2, 6] Let $\lhd$ be a partial order on I. A pair $(A, \lhd)$ is *quasi-hereditary* if it satisfies the following statements.

(1) $\lhd$ is adapted to A.
(2) $[\Delta(i) : S(i)] = 1$ for any $i \in I$.
(3) $A \in \mathcal{F}(\Delta)$.

Quasi-hereditary algebras were introduced by Scott in [11] by using the existence of certain chain of ideals of A. In [2], Cline, Parshall and Scott gave a characterization of quasi-hereditary algebras by using highest weight categories and the existence of ∇-filtrations of injective modules. In their work, the order $\lhd$ on I was not assumed to be adapted, and the definition of quasi-hereditary algebras needs axioms which are different from the above.

For a partial order $\lhd$ on I, it is known by Conde [3] that if a pair $(A, \lhd)$ satisfies the axiom of quasi-hereditary algebras in [2], then the order $\lhd$ is adapted to A. Therefore assuming $\lhd$ to be adapted gives no restriction comparing with the definition in [2, 11].

For example, if A has global dimension at most two, then A is quasi-hereditary with some partial order. Any directed algebra is a quasi-hereditary algebra with some partial order. It is known that any quasi-hereditary algebra has finite global dimension.

We end this introduction to state the following characterization of hereditary algebras from the viewpoint of quasi-hereditary algebras.

Proposition 4. [5] *Let A be a finite dimensional K-algebra. Then $(A, \lhd)$ is quasi-hereditary for any adapted order $\lhd$ on I if and only if A is quasi-hereditary.*

2. Quasi-hereditary structures

To define quasi-hereditary structures on A, we need some notations. For an A-module T, let $T^{\perp}$ be a subcategory of $\mathsf{mod}A$ consisting of X such that $\mathrm{Ext}_A^i(T, X) = 0$ for all $i > 0$. Dually, we define ${}^{\perp}T$.

An A-module T is called a *tilting module* if T has finite projective dimension, $\mathrm{Ext}_A^i(T, T) = 0$ for all $i > 0$, and there exists an exact sequence with $T_i \in \mathsf{add}T$

$$0 \to A \to T_0 \to \cdots \to T_\ell \to 0.$$

We denote by $\mathsf{tilt}A$ the set of isomorphism classes of basic tilting A-modules. This set is a partially ordered set by $T_1 \leq_{\mathrm{tilt}} T_2$ if and only if $T_1^{\perp} \subseteq T_2^{\perp}$, see [8].

Theorem 5. [10] *Let $(A, \lhd)$ be a quasi-hereditary algebra. For each $i \in I$, there exists a unique indecomposable A-module $T(i)$ and short exact sequences*

$$0 \to \Delta(i) \to T(i) \to X(i) \to 0, \qquad 0 \to Y(i) \to T(i) \to \nabla(i) \to 0,$$

where $X(i)$ belongs to $\mathcal{F}(\Delta(j) \mid j \lhd i, j \neq i)$ and $Y(i)$ belongs to $\mathcal{F}(\nabla(j) \mid j \lhd i, j \neq i)$ such that

(1) *$T = \bigoplus_{i\in I} T(i)$ is a tilting A-module satisfying $\mathsf{add}T = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$.*
(2) *$\mathcal{F}(\Delta) = {}^{\perp}T$ and $\mathcal{F}(\nabla) = T^{\perp}$ hold.*

We say that the tilting module T in the above theorem the *characteristic tilting module* of $(A, \lhd)$. We have the following lemma.

Lemma 6. *Let $(A, \lhd_1)$ and $(A, \lhd_2)$ be quasi-hereditary algebras with basic characteristic tilting modules T_1, T_2, respectively. We denote by Δ_j the standard modules associated to $\lhd_j$ for $j = 1, 2$. The following statements are equivalent.*

(1) *$\Delta_1(i) = \Delta_2(i)$ for any $i \in I$.*
(2) *$\mathcal{F}(\Delta_1) = \mathcal{F}(\Delta_2)$.*
(3) *$T_1 \simeq T_2$.*

We are ready to define quasi-hereditary structures.

Definition 7. Let $(A, \lhd_1)$ and $(A, \lhd_2)$ be quasi-hereditary algebras with basic characteristic tilting modules T_1, T_2, respectively.

(1) We write $\lhd_1 \sim \lhd_2$ if $T_1 \simeq T_2$ holds.
(2) We denote by $\mathsf{qh.str}A$ the set of all equivalence classes of adapted orders to A defining A to be quasi-hereditary algebra modulo $\sim$ above, that is,

$$\mathsf{qh.str}A := \{\lhd \mid \lhd \text{ is an adapted order on } I,\ (A, \lhd) \text{ is quasi-hereditary}\}/\sim$$

We say that each element of $\mathsf{qh.str}A$ a *quasi-hereditary structure* of A. We denote by $[\lhd_1] \in \mathsf{qh.str}A$ the quasi-hereditary structure represented by $\lhd_1$.
(3) We write $[\lhd_1] \leq_{\mathrm{qh}} [\lhd_2]$ if $T_1 \leq_{\mathrm{tilt}} T_2$ holds. This gives a partial order on $\mathsf{qh.str}A$.

Note that for a quasi-hereditary structure $[\lhd]$, $\mathcal{F}(\Delta) = {}^{\perp}\mathcal{F}(\nabla)$ and $\mathcal{F}(\Delta)^{\perp} = \mathcal{F}(\nabla)$ hold [11]. Using this and Theorem 5, for quasi-hereditary structures $[\lhd_1], [\lhd_2] \in \mathsf{qh.str}A$, we have that $[\lhd_1] \leq_{\mathrm{qh}} [\lhd_2]$ if and only if $\mathcal{F}(\nabla_1) \subset \mathcal{F}(\nabla_2)$ if and only if $\mathcal{F}(\Delta_2) \subset \mathcal{F}(\Delta_1)$.

By Lemma 6, $(\mathsf{qh.str}A, \leq_{\mathrm{qh}})$ is a subposet of $(\mathsf{tilt}A, \leq_{\mathrm{tilt}})$. We study this partially ordered set. We first give some known results about $\mathsf{qh.str}A$.

Theorem 8. [4] *Let $(A, \lhd)$ be a quasi-hereditary algebra. Assume that there is a duality $F:\ \bmod A \to\ \bmod A$ such that $F(S(i)) \simeq S(i)$ for any $i \in I$ and $F^2 \simeq \mathrm{id}$. Then we have $|\mathsf{qh.str}(A)| = 1$.*

Since any refinement of an adapted order is also adapted, we have the following lemma.

Lemma 9. *We have $|\mathsf{qh.str}A| \leq |I|!$.*

Proof. Let $[\lhd] \in \mathsf{qh.str}A$ and $\lhd'$ a total order which is a refinement of $\lhd$. Then by the discussion [6, page 4] (see also [7, Lemma 2.3]), $\Delta(i) = \Delta'(i)$ holds for any $i \in I$. Namely, $\lhd \sim \lhd'$ holds. Therefore any quasi-hereditary structure is represented by a total order. We have the assertion. $\square$

There exists an algebra such that the above inequality is an equality.

Example 10. [7, Example 2.26] Let $\mathcal{C}_n$ be a quiver such that the set of vertices is $I = \{1, 2, \ldots, n\}$ and there is a unique arrow from i to j whenever $i > j$. In particular, the underlying graph of $\mathcal{C}_n$ is a complete graph. It is easy to see that any adapted order to $K\mathcal{C}_n$ is a total order on I, and two distinct total orders on I induce different quasi-hereditary structures on $K\mathcal{C}_n$. Therefore $|\mathsf{qh.str}(K\mathcal{C}_n)| = n!$ holds.

More precisely, one can show that $(\mathsf{qh.str}(K\mathcal{C}_n), \leq_{\mathrm{qh}})$ is isomorphic to the symmetric group S_n of rank n with the weak (Bruhat) order as partially ordered sets.

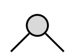
3. Quasi-hereditary structures of the path algebras of equioriented quivers of type A

Let $A_n = 1 \to 2 \to \cdots \to n-1 \to n$ be an equioriented A_n quiver. In this section, we see that $(\mathsf{qh.str}(KA_n), \leq_{\mathrm{qh}})$ is isomorphic to $(\mathsf{tilt}(KA_n), \leq_{\mathrm{tilt}})$ as partially ordered sets. By definition, taking characteristic tilting module is an injective morphism of posets from $(\mathsf{qh.str}(KA_n), \leq_{\mathrm{qh}})$ to $(\mathsf{tilt}(KA_n), \leq_{\mathrm{tilt}})$. To see that this is surjective, we use another description of tilting KA_n-modules via binary trees.

Binary trees can be defined inductively as follows. A *binary tree* T is either the empty set or a tuple $T = (r, L, R)$ where r is a singleton set, called the root of T, and L and R are two binary trees. The empty set has no vertex but has one leaf. The set of leaves of $T = (r, L, R)$ is the disjoint union of the set of leaves of L and R. The *size* of the tree is its number of vertices (equivalently the number of leaves minus 1).

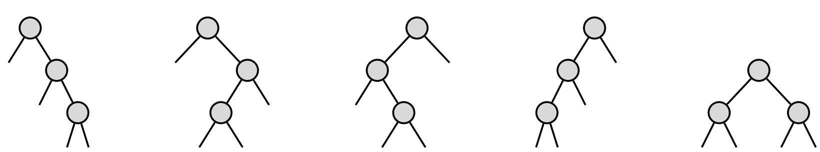
The followings are the binary trees of size 1, 2 and 3:

- binary tree of size 1:



- binary tree of size 2:

- binary tree of size 3:



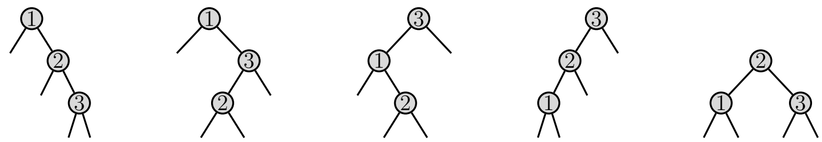
For each binary tree, there exists a unique labeling of its vertices, called a binary search tree, as follows.

Definition 11. A *binary search tree* of size n is a binary tree with n vertices labeled by $I = \{1, 2, \ldots, n\}$ with the following rule:

- if a vertex v is labeled by $k \in I$, then the vertices of the left subtree of v are labeled by integers less than k, and the vertices of the right subtree of v are labeled by integers superior to k.

This procedure is sometimes called the in-order traversal of the tree or simply as the in-order algorithm (recursively visit left subtree, root and right subtree).

Example 12. The binary search trees of size 3 are as follows.



We denote these binary search trees from left to right by $\mathbb{T}_i (i = 1, 2, \ldots, 5)$.

Since any binary tree admits a unique labeling of vertices making it to be a binary search tree, we always consider binary search trees.

Definition 13. For a binary search tree $\mathbb{T}$ of seize n, we define an order $\lhd_{\mathbb{T}}$ on $I = \{1, 2, \ldots, n\}$ by $i \lhd_{\mathbb{T}} j$ if and only if i labels a vertex of a subtree of a vertex labeled by j.

The following is one of main results of this study.

Theorem 14. [7] *Let $A_n = 1 \to 2 \to \cdots \to n-1 \to n$ be an equioriented A_n quiver and $I = \{1, 2, \ldots, n\}$. There exist bijections between the following three sets.*

(1) *The set of binary trees of size n.*
(2) *The set $\mathsf{tilt}(KA_n)$.*
(3) *The set $\mathsf{qh.str}(KA_n)$.*

In particular, we have that $|\mathsf{qh.str}(KA_n)|$ is equal to the Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$.

The bijection between (1) and (2) was shown by [1, Theorem 5.2], see also [9]. A bijection from (1) to (3) is given by $\mathbb{T} \mapsto \lhd_{\mathbb{T}}$. The point is that this map is well-defined and surjective. Therefore the natural map from $\mathsf{qh.str}(KA_n)$ to $\mathsf{tilt}(KA_n)$ is bijective. In particular, $(\mathsf{qh.str}(KA_n), \leq_{\mathrm{qh}})$ is isomorphic to $(\mathsf{tilt}(KA_n), \leq_{tilt})$ as partially ordered sets.

Note that each indecomposable KA_n-module is determined by its composition factors. For a given binary search tree $\mathbb{T}$, we have the composition factors of standard (resp. costandard) KA_n-modules $\Delta_{\mathbb{T}}(i)$ (resp. $\nabla_{\mathbb{T}}(i)$) associated to $\lhd_{\mathbb{T}}$ as follows.

Lemma 15. [7] *Let $\mathbb{T}$ be a binary search tree and $\Delta(i)$ (resp. $\nabla(i)$) the standard (resp. costandard) module associated to $\lhd_{\mathbb{T}}$. We denote by $T(i)$ the indecomposable direct summand of the characteristic tilting module of $(A, \lhd_{\mathbb{T}})$ as in Theorem 5.*

(1) *A simple module $S(j)$ is a composition factor of $\Delta_{\mathbb{T}}(i)$ (resp. $\nabla_{\mathbb{T}}(i)$) if and only if j labels a vertex in the right (resp. left) subtree of a vertex labeled by i.*
(2) *A simple module $S(j)$ is a composition factor of $T(i)$ if and only if j labels a vertex in either a left or a right subtree of a vertex labeled by i.*

Example 16. Let $\mathbb{T}_1, \ldots, \mathbb{T}_5$ be binary trees of size 3 as Example 12. For simplicity, we write $\lhd_{\mathbb{T}_i} = \lhd_i$. We denote by T_i the characteristic tilting modules of $(KA_n, \lhd_i)$. For each $\mathbb{T}_i$, we have

- $\lhd_1 = \{3 \lhd_1 2 \lhd_1 1\}, \quad T_1 = P(1) \oplus P(2) \oplus P(3).$
- $\lhd_2 = \{2 \lhd_2 3 \lhd_2 1\}, \quad T_2 = P(1) \oplus S(2) \oplus P(2).$
- $\lhd_3 = \{2 \lhd_3 1 \lhd_3 3\}, \quad T_3 = (P(1)/S(3)) \oplus S(2) \oplus P(1).$

- $\lhd_4 = \{1 \lhd_4 2 \lhd_4 3\}$, $\quad T_4 = S(1) \oplus (P(1)/S(3)) \oplus P(1)$.
- $\lhd_5 = \{1 \lhd_5 2, 3 \lhd_5 2\}$, $\quad T_5 = S(1) \oplus P(1) \oplus S(3)$.

We have cover relatons $T_4 \leq_{\text{tilt}} T_3 \leq_{\text{tilt}} T_2 \leq_{\text{tilt}} T_1$ and $T_4 \leq_{\text{tilt}} T_5 \leq_{\text{tilt}} T_1$.

4. Concatenations of quivers and quasi-hereditary structures

Definition 17. Let Q^1, Q^2 be quivers and $v_i \in Q_0^i$ a sink. A *concatenation* of Q^1 and Q^2 at v_1 and v_2 is a quiver Q such that

- $Q_0 = (Q_0^1 \setminus \{v_1\}) \sqcup (Q_0^2 \setminus \{v_2\}) \sqcup \{v\}$
- $Q_1 = Q_1^1 \sqcup Q_1^2$, where we identify $v = v_1 = v_2$.

If $v_i \in Q_0^i$ is a source, we similarly define a concatenation at v.

Let Q be a concatenation of Q^1 and Q^2 at v. For a partial order $\lhd$ on Q_0, we have partial orders $\lhd|_{Q_0^1}$ on Q_0^1 and $\lhd|_{Q_0^2}$ on Q_0^2. Let $\overline{1} = 2$ and $\overline{2} = 1$. Conversely, we construct a partial order on Q_0 from partial orders on Q_0^ℓ. Let $\lhd^\ell$ be partial orders on Q_0^ℓ for $\ell = 1, 2$. Then we have a partial order $\lhd = \lhd(\lhd^1, \lhd^2)$ on Q_0 as follows: for $i, j \in Q_0$, $i \lhd j$ if one of the following two statements holds:

(1) $i, j \in Q_0^\ell$ and $i \lhd^\ell j$ holds for some ℓ,
(2) $i \in Q_0^\ell$, $j \in Q_0^{\overline{\ell}}$, $i \lhd^\ell v$ and $v \lhd^{\overline{\ell}} j$ hold.

We have the following theorem.

Theorem 18. [7] *Let Q be a concatenation of Q^1 and Q^2. Let A be a factor algebra of KQ and $A^i := A/\langle e_u \mid u \in Q_0 \setminus Q_0^i\rangle$ for $i = 1, 2$. Then we have an isomorphism of posets*

$$\mathsf{qh.str}(A) \longrightarrow \mathsf{qh.str}(A^1) \times \mathsf{qh.str}(A^2),$$

given by $[\lhd] \mapsto \big([\lhd|_{Q_0^1}], [\lhd|_{Q_0^2}]\big)$. The converse map is given by $(\lhd_1, \lhd_2) \mapsto \lhd(\lhd_1, \lhd_2)$.

Example 19. Let Q be a quiver $1 \to 2 \leftarrow 3$. This Q is a concatenation of $Q^1 = 1 \to 2$ and $Q^2 = 2 \leftarrow 3$ at 2. We have $\mathsf{qh.str}(KQ^1) = \{[1 \lhd 2], [2 \lhd 1]\}$ and $\mathsf{qh.str}(KQ^2) = \{[2 \lhd 3], [3 \lhd 2]\}$. So $|\mathsf{qh.str}(KQ)| = 4$ and we have

$$\mathsf{qh.str}(KQ) = \{[1 \lhd 2 \lhd 3], [1 \lhd 2, 3 \lhd 2], [2 \lhd 1, 2 \lhd 3], [3 \lhd 2 \lhd 1]\}.$$

Clearly, each path algebra of quivers of type A_n can be obtained by iterated concatenations of equioriented A_n quivers. So we can classify quasi-hereditary structures of such algebras.

Corollary 20. *Let Q be a quiver of type A_n obtained by iterated concatenations of $Q^1, Q^2, \ldots, Q^\ell$ such that each Q^i is an equioriented quiver of type A_{n_i} for some $n_i \in \mathbb{Z}_{\geq 1}$. Then there is a bijection*

$$\mathsf{qh.str}(KQ) \longrightarrow \prod_{i=1}^{\ell} \mathsf{qh.str}(KA_{n_i})$$

given by $[\lhd] \mapsto \big([\lhd|_{Q_0^i}]\big)_{i=1}^{\ell}$.

The bijection in Theorem 18 enables us to calculate characteristic tilting modules.

Let Q be a concatenation of Q^1 and Q^2 at a sink v. Let A be a factor algebra of KQ and $A^\ell := A/\langle e_u \mid u \in Q_0 \setminus Q_0^\ell \rangle$ for $\ell = 1, 2$. Fix two quasi-hereditary structures $[\lhd^\ell] \in \mathsf{qh.str}(A^\ell)$ and denote by $T^\ell(i)$ an indecomposable direct summands of the characteristic tilting module T^ℓ of $(A^\ell, \lhd^\ell)$ as in Theorem 5 for $i \in Q_0^\ell$. We denote by $\lhd = \lhd(\lhd^1, \lhd^2)$ the partial order on Q_0 as in Theorem 18. Let $T(i)$ be an indecomposable direct summands of the characteristic tilting module T of $(A, \lhd)$. Since v is a sink of Q^2, there is an injective morphism $S(v) \to T^2(v)$ by Theorem 5. Since A^ℓ is a factor algebra of A, we regard an A^ℓ-module as an A-module by a natural way.

Theorem 21. *Under the notation as above, for $i \in Q_0^1$, let m be the length of an $S(v)$-socle of $T^1(i)$. Then the push-out $U(i)$ of $T^2(v)^{\oplus m} \leftarrow S(v)^{\oplus m} \to T^1(i)$ is isomorphic to $T(i)$.*

Example 22. Consider Example 19. Put $\lhd = (1 \lhd 2 \lhd 3)$ which is an image of $([1 \lhd 2], [2 \lhd 3])$ by the map in Theorem 18. Then $T^1(1) = S(1)$ and $T^1(2) = P^1(1)$ are indecomposable direct summands of the characteristic tilting module of $(KQ^1, [1 \lhd 2])$. Also, $T^2(2) = S(2)$ and $T^2(3) = P^2(3)$ are indecomposable direct summands of the characteristic tilting module of $(KQ^2, [2 \lhd 3])$. By the above theorem, we have $T(1) = S(1)$, $T(2) = P(1)$ and $T(3) = I(2)$.

References

[1] A. B. Buan, H. Krause, *Tilting and cotilting for quivers of type $\widetilde{A}_n$*, Journal of Pure and Applied Algebra 190 (1-3), 1–21.

[2] E. T. Cline, B. J. Parshall, and L. L. Scott, *Finite dimensional algebras and highest weight categories*, J. Reine Angew. Math. 391 (1988), pp. 85–99.

[3] T. Conde, *On certain strongly quasihereditary algebras*, PhD thesis. University of Oxford, 2016.

[4] K. Coulembier, *The classification of blocks in BGG category*, Math. Z. 295, 821–837 (2020).

[5] V. Dlab, C.M. Ringel, *Quasi-hereditary algebras*, Ill. J. Math. 33 (2) (1989) 280–291.

[6] V. Dlab, C. M. Ringel, *The module theoretical approach to quasi-hereditary algebras*, In Representations of algebras and related topics (Kyoto, 1990), volume 168 of London Math. Soc. Lecture Note Ser., pages 200–224. Cambridge Univ. Press, Cambridge, 1992.

[7] M. Flores, Y. Kimura and B. Rognerud, *Combinatorics of quasi-hereditary structures*, J. Comb. Theory, Ser. A, 187, (2022), 105559.

[8] D. Happel, L. Unger, *On a partial order of tilting modules*, Algebr. Represent. Theory 8 (2) (2005) 147–156.

[9] L. Hille, *On the volume of a tilting module*, Abh. Math. Semin. Univ. Hamb. 76 (2006) 261–277.

[10] C.M. Ringel, *The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences*, Math. Z. 208 (2) (1991) 209–223.

[11] L. L. Scott, *Simulating algebraic geometry with algebra. I. The algebraic theory of derived categories*, The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), 271–281, Proc. Sympos. Pure Math., 47, Part 2, Amer. Math. Soc., Providence, RI, 1987.

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TILTING COMPLEXES OVER BLOCKS COVERING CYCLIC BLOCKS

YUTA KOZAKAI

ABSTRACT. Let p be a prime number, k an algebraically field of characteristic p, $\tilde{G}$ a finite group, and G a normal subgroup of $\tilde{G}$ having a p-power index in $\tilde{G}$. Moreover let B be a block of kG and $\tilde{B}$ the unique block of $k\tilde{G}$ covering B. In this note, we show that the set of isomorphism classes of basic tilting complexes over B is isomorphic to that of $\tilde{B}$ as partially ordered sets under some kinds of assumptions. Moreover, as an application, we give the result that the block $\tilde{B}$ of $k\tilde{G}$ covering a cyclic block is tilting-discrete block.

1. Introduction

In representation theory of finite groups, there is a well-known and important conjecture called Broué's abelian defect group conjecture.

Conjecture 1. *Let k be an algebraically closed field of characteristic $p > 0$, G a finite group, B a block of the group algebra kG with defect group D, and b the Brauer correspondent of B in $kN_G(D)$. If D is abelian, then the block B is derived equivalent to b.*

There are many cases that Broué's abelian defect group conjecture holds. Also, it is known that Broué's abelian defect group conjecture does not hold generally without the assumption that the defect group D is abelian. However even if the defect group D is not abelian, it is thought that the similar statement holds in some situations and that how we may state the non-abelian version conjecture. The one situation we are interested in is as follows: $\tilde{G}$ is a finite group with a normal subgroup G of p-power index in $\tilde{G}$ and G has a cyclic Sylow p-subgroup P. In fact, it is expected that the principal block $B_0(k\tilde{G})$ of $k\tilde{G}$ is derived equivalent to that $B_0(N_{\tilde{G}}(P))$ of $kN_{\tilde{G}}(P)$ (for example see [5]). To solve this, it is essential to find a suitable tilting complex over $B_0(k\tilde{G})$, but it is not easy. On the other hand, the study on tilting complexes over the principal block $B_0(kG)$ is well known and they have some kinds of good properties because the block $B_0(kG)$ is a cyclic block, which implies that it is a Brauer tree algebra (for example, see [1]). Based on these, we try to compare tilting complexes over $B_0(k\tilde{G})$ and those over $B_0(kG)$, and to give a classification of that over $B_0(k\tilde{G})$.

2. Silting theory

Let Λ be a finite dimensional algebra over an algebraically closed field. In [2], the set of isomorphism classes of basic silting complexes over Λ has a partially ordered set structure.

The detailed version of this paper will be submitted for publication elsewhere.

Definition 2. [2] Let Λ be a finite dimensional algebra over an algebraically closed field. For silting complexes P and Q of $K^b(\operatorname{proj}\Lambda)$, we define a relation $\geq$ between P and Q as follows;

$$P \geq Q :\Leftrightarrow \operatorname{Hom}_{K^b(\operatorname{proj}\Lambda)}(P, Q[i]) = 0 \ ({}^\forall i > 0).$$

Then the relation $\geq$ gives a partial order on $\operatorname{silt}\Lambda$, where $\operatorname{silt}\Lambda$ means the set of isomorphism classes of basic silting complexes over Λ.

Here, we remark that any silting complex over B is a tilting complex over B for any block algebra B of a finite group since it is symmetric algebra (for example, see [2, Example 2.8]). Hence, for a block algebra B of a finite group, the set of isomorphism classes of basic tilting complexes over B has a partially ordered set structure too. We denote this partially ordered set by $\operatorname{tilt} B$.

We recall the definition of mutations for silting complexes of $K^b(\operatorname{proj}\Lambda)$ [2, Definition 2.30, Theorem 3.1].

Definition 3. Let P be a basic silting complex of $K^b(\operatorname{proj}\Lambda)$ and decompose it as $P = X \oplus M$. We take a triangle

$$X \xrightarrow{f} M' \to Y \to$$

with a minimal left $(\operatorname{add} M)$-approximation f of X. Then the complex $\mu_X^-(P) := Y \oplus M$ is a silting complex in $K^b(\operatorname{proj}\Lambda)$ again. We call the complex $\mu_X^-(P)$ a left mutation of P with respect to X. If X is indecomposable, then we say that the left mutation is irreducible. We define the (irreducible) right mutation $\mu_X^+(P)$ dually. Mutation will mean either left or right mutation.

Remark 4. If B is a block algebra of finite group, then, for any tilting complex $P = X \oplus M$ over Λ, the complex $\mu_X^\epsilon(P)$ is a tilting complex again where ϵ means $+$ or $-$.

The following result is very important to study of partially ordered structure of the sets of silting complexes.

Theorem 5 ([2, Theorem 2.35])**.** *For any silting complexes P and Q over Λ, the following conditions are equivalent:*
(1) Q is an irreducible left mutation of P;
(2) P is an irreducible right mutation of Q;
(3) $P > Q$ and there is no silting complex L satisfying $P > L > Q$.

We recall the definition of tilting-discrete algebras.

Definition 6. We say that an algebra (which is not necessarily a symmetric algebra) Λ is a tilting-discrete algebra if for all $\ell > 0$ and any tilting complex P over Λ, the set

$$\{T \in \operatorname{tilt}\Lambda \mid P \geq T \geq P[\ell]\}$$

is a finite set.

Tilting-discrete algebras have the following nice property.

Theorem 7 ([3, Theorem 3.5])**.** *If Λ is a tilting-discrete algebra, then Λ is a strongly tilting connected algebra, that is, for any tilting complexes T and U, the complex T can be obtained from U by either iterated irreducible left mutation or iterated irreducible right mutation.*

3. Block theory

3.1. **Block theory.** In this section, let k be an algebraically closed field of characteristic $p > 0$. We denote by G a finite group, and by k_G the trivial module of kG, that is, a one-dimensional vector space on which each element in G acts as the identity. We recall the definition of blocks of group algebras. The group algebra kG has a unique decomposition

$$kG = B_1 \times \cdots \times B_n$$

into a direct product of subalgebras B_i each of which is indecomposable as an algebra. Then each direct product component B_i is called a block of kG. For any indecomposble kG-module M, there exists a unique block B_i of kG such that $M = MB_i$ and $MB_j = 0$ for all $j \in \{1, \ldots, n\} - \{i\}$. Then we say that M lies in the block B_i or that M is a B_i-module. Also we denote by $B_0(G)$ the principal block of kG, that is, the unique block of kG which does not annihilate the trivial kG-module k_G.

First, we recall the definition of defect groups of blocks of finite groups and their properties.

Definition 8. Let B be a block of kG. A minimal subgroup D of G which satisfies the following condition is uniquely determined up to conjugacy in G: the B-bimodule epimorphism

$$B \otimes_{kD} B \to B \ \ (b_1 \otimes_{kD} b_2 \mapsto b_1 b_2)$$

is a split epimorphism. We call the subgroup a defect group of the block B.

The following results are well known (for example, see [1]).

Proposition 9. *For the principal block $B_0(G)$ of kG, its defect group is a Sylow p-subgroup of G.*

Blocks with cyclic defect groups are called cyclic blocks. The cyclic blocks have good properties.

Proposition 10. *For a block B of kG and a defect group D of B, the following are equivalent:*

(1) D is a non-trivial cyclic group;
(2) B is of finite representation type and is not semisimple;
(3) B is a Brauer tree algebra.

We introduce the induced modules, induced complexes and covering blocks.

Definition 11. Let H be a subgroup of G. For a kH-module U, we denote by $\mathrm{Ind}_H^G U := U \otimes_{kH} kG$ the induced module of U from H to G. Also, for a complex $X = (X^i, d^i)$, we denote by $\mathrm{Ind}_H^G X$ the complex $(X^i \otimes_{kH} kG, d^i \otimes_{kH} kG)$. This induces a functor from $K^b(\mathrm{proj}\, kH)$ to $K^b(\mathrm{proj}\, kG)$.

Definition 12. Let G be a normal subgroup of $\tilde{G}$ and $\tilde{B}$ a block of $k\tilde{G}$. We say a block B of kG is covered by $\tilde{B}$ or $\tilde{B}$ covers B if there exists a non-zero $\tilde{B}$-module $\tilde{U}$ such that $\tilde{U}$ has a non-zero summand lying in B as a kG-module.

Let G be a normal subgroup of a finite group $\tilde{G}$. In general, for indecomposable kG-module U, the induced module $\mathrm{Ind}_G^{\tilde{G}}U$ is not indecomposable. Moreover, for a block B of kG, there are several blocks of $k\tilde{G}$ covering the block B. However, in case where G is a normal subgroup of a finite group $\tilde{G}$ and has a p-power index, the following propositions hold.

Proposition 13 ([4, Green's indecomposability theorem])**.** *If G is a normal subgroup of a finite group $\tilde{G}$ of p-power index, then for any indecomposable kG-module V the induced $k\tilde{G}$-module $\mathrm{Ind}_G^{\tilde{G}}V$ is an indecomposable $k\tilde{G}$-module.*

Proposition 14 ([6, Corollary 5.5.6])**.** *Let G be a normal subgroup of $\tilde{G}$, and B a block of G. If the index of G in $\tilde{G}$ is a p-power, then there exists a unique block of $k\tilde{G}$ covering B.*

Remark 15. Let G be a normal subgroup of a finite group $\tilde{G}$ of a p-power index, B a block of kG, and $\tilde{B}$ the unique block of $k\tilde{G}$ covering B. Then by Propositions 13 and 14, for any indecomposable complex X of $K^b(\mathrm{proj}\, B)$, we can easily show that the induced complex $\mathrm{Ind}_G^{\tilde{G}}X$ is an indecomposable complex of $K^b(\mathrm{proj}\, \tilde{B})$.

4. Main Results

In this section, we give our main results. Let G be a normal subgroup of $\tilde{G}$ with index in $\tilde{G}$ a p-power. First we give the tilting-discreteness of $\tilde{B}$ and an isomorphism between $\mathrm{tilt}\, B$ and $\mathrm{tilt}\, \tilde{B}$ as partially ordered sets, where B is a block of kG with some properties and $\tilde{B}$ is a unique block covering B.

Theorem 16. *Let $\tilde{G}$ be a finite group, G a normal subgroup such that the index $|\tilde{G} : G|$ is a p-power, k an algebraically closed field of characteristic $p > 0$, B a block of kG, and $\tilde{B}$ the unique block of $k\tilde{G}$ covering B. Assume B satisfies the following conditions:*

(i) Any indecomposable B-module is $\tilde{G}$-invariant,
(ii) B is a tilting-discrete algebra,
(iii) Any algebra derived equivalent to B has a finite number of two-term tilting complexes.

Then $\tilde{B}$ is a tilting-discrete algebra. Moreover the induction functor $\mathrm{Ind}_G^{\tilde{G}} : K^b(\mathrm{proj}\, B) \to K^b(\mathrm{proj}\, \tilde{B})$ induces an isomorphism between $\mathrm{tilt}\, B$ and $\mathrm{tilt}\, \tilde{B}$ as partially ordered sets, here $\mathrm{tilt}\, B$ and $\mathrm{tilt}\, \tilde{B}$ mean the set of all tilting complexes over B and $\tilde{B}$ respectively.

As an application of above theorem, we can apply it to the case where we state in the introduction, that is, the case $\tilde{G}$ has a normal subgroup G with a p-power index in $\tilde{G}$ and with a cyclic Sylow p-subgroup. In fact, in this setting, the assumptions in Theorem 16 are satisfied automatically. Hence we get the following theorem.

Theorem 17. *Let $\tilde{G}$ be a finite group having G as a normal subgroup with index in $\tilde{G}$ a p-power. Let B be a block of the finite group G with cyclic defect group and $\tilde{B}$ the unique block of $k\tilde{G}$ covering B. Then the following hold.*

(1) $\tilde{B}$ is a tilting-discrete algebra.
(2) The induction functor $\mathrm{Ind}_G^{\tilde{G}} : \mathrm{tilt}\, B \to \mathrm{tilt}\, \tilde{B}$ induces an isomorphism of partially ordered sets.

References

[1] J. L. Alperin, *Local representation theory*, Cambridge University Press, 1986.

[2] T. Aihara, O. Iyama, *Silting mutation in triangulated categories.* J. Lond. Math. Soc. (2) **85** (2012), no. 3, 633–668.

[3] T. Aihara, *Tilting-connected symmetric algebras.* Algebr. Represent. Theory **16** (2013), no. 3, 873–894.

[4] J. A. Green, *On the indecomposable representations of a finite group.* Math. Z. 70 (1958/59), 430–445.

[5] M. Holloway, S. Koshitani, N. Kunugi, *Blocks with nonabelian defect groups which have cyclic subgroups of index p.* Arch. Math. (Basel) **94** (2010), no. 2, 101–116.

[6] H. Nagao, Y. Tsushima, *Representations of finite groups,* Academic Press, Inc., Boston, MA, 1989, Translated from the Japanese.

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RELATIVE STABLE EQUIVALENCES OF MORITA TYPE FOR THE PRINCIPAL BLOCKS OF FINITE GROUPS

NAOKO KUNUGI AND KYOICHI SUZUKI

Abstract. Wang and Zhang introduced the notion of stable equivalences of Morita type relative to pairs of modules for blocks of finite groups. First, in this paper, we give a method of constructing, in certain situations, relative stable equivalences of Morita type for the principal blocks of finite groups. Second, we introduce the notion of relative Brauer indecomposability, and give an equivalent condition for certain modules to be relatively Brauer indecomposable.

1. Introduction

Let G be a finite group, and k a filed of characteristic $p > 0$. We can decompose kG as a direct product of indecomposable k-algebras:

$$kG = B_1 \times \cdots \times B_n.$$

Each B_i is called a *block* of G. For any indecomposable kG-module U, there exists a unique block B_i such that $UB_i = U$. We write k_G for the *trivial kG-module*, that is, a one-dimensional k-vector space on which every element of G acts trivially. The group algebra kG has a unique block B such that $k_G B = k_G$, which is called the *principal block* of G and denoted by $B_0(G)$. We are interested in constructing Morita equivalences for the principal blocks of finite groups.

Broué [1] introduced the notion of stable equivalences of Morita type, and developed a method of constructing them for the principal blocks. This method has been used as one of the useful tools for constructing Morita equivalences for the principal blocks. However, we cannot use the method for finite groups having a common nontrivial central p-subgroup. On the other hand, Wang and Zhang [10] introduced the notion of relative stable equivalences of Morita type for blocks of finite groups, which is a generalization of stable equivalences of Morita type. In this paper, we state, as our first main theorem, a method for constructing relative stable equivalences of Morita type for the principal blocks.

In [5], the notion of Brauer indecomposability was introduced. The Brauer indecomposability of modules called Scott modules plays an important role in the method of Broué. Ishioka and the first author [4] gave an equivalent condition for Scott modules to be Brauer indecomposable. Although Brauer indecomposability of Scott modules is also useful for our first main theorem, somewhat more general condition is more appropriate. Therefore, in this paper, we introduce the notion of relative Brauer indecomposability, and state, as our second main theorem, an equivalent condition for Scott modules to be relatively Brauer indecomposable.

The detailed version of this paper will be submitted for publication elsewhere.

2. Preliminaries

In this section, we recall basic notation and definitions on modular representation theory and fusion systems.

Throughout this paper, we assume that k is an algebraically closed field of characteristic p, G is a finite group, and, unless other wise stated, modules are finitely generated right modules. We write $Z(G)$ for the center of G. Let H be a subgroup of G. We write $[H\backslash G]$ for a set of representatives of the right cosets of H in G. For a kG-module M, we write $M{\downarrow}_H^G$ for the restriction of M to H, and for a kH-module N, we write $N\uparrow_H^G = N \otimes_{kH} kG$ for the induced kG-module of N. For a kG-module M, we write $M^* = \mathrm{Hom}_k(M, k)$ for the k-dual of M, considered as a left kG-module.

For a p-subgroup Q of G, there is a unique indecomposable summand of $k_Q{\uparrow}^G$ such that it has k_G as a direct summand of the top. This indecomposable summand is called the *Scott kG-module* with vertex Q, and denoted by $S(G, Q)$.

Let M be a kG-module. For a subgroup H of G, we write M^H for the set of fixed points of H in M. For a p-subgroup Q of G, the *Brauer construction* of M with respect to Q is the $kN_G(Q)$-module $M(Q)$ defined as follows:

$$M(Q) = M^Q / \sum_R \mathrm{tr}_R^Q(M^R),$$

where R runs over the set of proper subgroups of Q, and $\mathrm{tr}_R^Q : M^R \to M^Q$, $\mathrm{tr}_R^Q(m) = \sum_{t\in[R\backslash Q]} mt$.

For subgroups H and K of G, we write

$$\mathrm{Hom}_G(H, K) = \{\varphi \in \mathrm{Hom}(H, K) \mid \varphi = c_g \text{ for some } g \in G \text{ such that } H^g \leq K\},$$

where c_g is a conjugation map. Let P be a p-subgroup of G. The *fusion system* of G over P is the category $\mathcal{F}_P(G)$ whose objects are the subgroups of P and morphisms are given by $\mathrm{Hom}_{\mathcal{F}_P(G)}(Q, R) = \mathrm{Hom}_G(Q, R)$. For subgroups Q and R of P, we say that Q and R are *$\mathcal{F}_P(G)$-conjugate* if Q and R are isomorphic in $\mathcal{F}_P(G)$. Let Q be a subgroup of P. We say that Q is *fully automized* in $\mathcal{F}_P(G)$ if $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}_P(G)}(Q)$. We say that Q is *receptive* in $\mathcal{F}_P(G)$ if for any subgroup R of P and any $\varphi \in \mathrm{Iso}_{\mathcal{F}_P(G)}(R, Q)$, there is an element $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}_P(G)}(N_\varphi, P)$ such that $\bar{\varphi}|_Q = \varphi$, where $N_\varphi = \{g \in N_P(R) \mid {c_g}^{\varphi^{-1}} \in \mathrm{Aut}_P(Q)\}$. We say that Q is *fully normalized* in $\mathcal{F}_P(G)$ if $|N_P(Q)| \geq |N_P(R)|$ for any subgroup R of P that is $\mathcal{F}_P(G)$-conjugate to Q. The fusion system $\mathcal{F}_P(G)$ is *saturated* if any subgroup of P is $\mathcal{F}_P(G)$-conjugate to a subgroup that is fully automized and receptive.

3. Relative stable equivalences of Morita type

In this section, we first introduce results of Broué [1] and Linckelmann [7]. Next, we define the notion of relative stable equivalences of Morita type that was introduced by Wang and Zhang [10]. Finally, we state the first main theorem of this paper.

Broué [1] gave a method of constructing stable equivalences of Morita type:

Theorem 1. *(see [1, Theorem 6.3]) Let G and G' be finite groups with a common Sylow p-subgroup P such that $\mathcal{F}_P(G) = \mathcal{F}_P(G')$, and $M = S(G \times G', \Delta P)$. If $(M(\Delta Q), M(\Delta Q)^*)$*

induces a Morita equivalence between $B_0(C_G(Q))$ and $B_0(C_{G'}(Q))$ for any nontrivial subgroup Q of P, then (M, M^*) induces a stable equivalence of Morita type between $B_0(G)$ and $B_0(G')$.

Linckelmann showed the following:

Theorem 2. *(see [7, Theorem 2.1]) Let B and B' be blocks of G and G', and M a B-B'-bimodule that is projective as a left module and a right module. Assume that $- \otimes_B M$ induces $\underline{mod}(B) \cong \underline{mod}(B')$. If for any simple B-module S, the B'-module $S \otimes_B M$ is simple, then $- \otimes_B M$ induces an equivalence between mod(B) and mod(B').*

Theorem 1 has been used as one of the useful tools for constructing Morita equivalences for principal blocks. In fact, we may construct a stable equivalences of Morita type by using Theorem 1, and lift it to a Morita equivalence by using Theorem 2. In this way, Morita equivalences has been confirmed in some cases, for example see [9] and [6]. However, we cannot use Theorem 1 if the common Sylow p-subgroup has a nontrivial subgroup Z that is a subgroup of $Z(G)$ and $Z(G')$. We see that $C_G(Z) = G$ and $C_{G'}(Z) = G'$. Hence we need to show that $B_0(G)$ and $B_0(G')$ are Morita equivalent in order to apply Theorem 1.

Okuyama [8] introduced the notion of projectivity relative to modules:

Definition 3. (see [8] and also [2, Section 8]) Let W be a kG-module. A kG-module M is *relatively W-projective* if M is a direct summand of $V \otimes W$ for some kG-module V.

Then we can define a relative stable category that is an analogue of the stable category. Let W be a kG-module. The *relative W-stable category* $\underline{\mathrm{mod}}^W(kG)$ of mod(kG) is the category whose objects are the finitely generated kG-modules, and whose morphisms are given by

$$\underline{\mathrm{Hom}}_{kG}^W(M, N) = \mathrm{Hom}_{kG}(M, N)/\mathrm{Hom}_{kG}^W(M, N),$$

where $\mathrm{Hom}_{kG}^W(M, N)$ is the subspace of $\mathrm{Hom}_{kG}(M, N)$ consisting of all homomorphisms that factor through a W-projective kG-module. Let B be a block of G. We write $\underline{\mathrm{mod}}^W(B)$ for the full subcategory of $\underline{\mathrm{mod}}^W(kG)$ whose objects are all B-modules. It follows that $\underline{\mathrm{mod}}^W(kG)$ has a structure of triangulated category (see [3, Theorem 6.2]), and $\underline{\mathrm{mod}}^W(B)$ is a triangulated subcategory of $\underline{\mathrm{mod}}^W(kG)$ (see [10, Proposition 3.1]).

Wang and Zhang [10] introduced the notion of relative stable equivalences of Morita type by using the notion of projectivity relative to modules:

Definition 4. (see [10, Definition 5.1]) Let G and G' be finite groups and B and B' blocks of G and G', respectively. For a kG-module W, a kG'-module W', a B-B'-bimodule M, and a B'-B-bimodule N, we say that the pair (M, N) induces a *relative (W, W')-stable equivalence of Morita type* between B and B' if M and N are finitely generated projective as left modules and right modules with the property that there are isomorphisms of bimodules

$$M \otimes_{B'} N \cong B \oplus X \quad \text{and} \quad N \otimes_B M \cong B' \oplus Y,$$

where X is $W^* \otimes W$-projective as a $k[G \times G]$-module and Y is $W'^* \otimes W'$-projective as a $k[G' \times G']$-module.

If $W = kG$, then it follows that X is projective as B-B-bimodule. Hence the notion of relative stable equivalences is a generalization of the notion of stable equivalences.

Finally, we state the first main theorem of this paper:

Theorem 5. *Let G and G' be finite groups with a common Sylow p-subgroup P such that $\mathcal{F}_P(G) = \mathcal{F}_P(G')$, and $M = S(G\times G', \Delta P)$. Assume that Z is a subgroup of P that is central in G and G'. If $(M(\Delta Q), M(\Delta Q)^*)$ induces a Morita equivalence between $B_0(C_G(Q))$ and $B_0(C_{G'}(Q))$ for any subgroup Q of P properly containing Z, then (M, M^*) induces a relative $(k_Z{\uparrow}^G, k_Z{\uparrow}^{G'})$-stable equivalence of Morita type between $B_0(G)$ and $B_0(G')$.*

Note that Theorem 5 with $Z = 1$ implies Theorem 1 as $k_1{\uparrow}^G \cong kG$.

4. Relative Brauer indecomposability

In this section, we first recall from [5] and [4] the definition and some results of the Brauer indecomposability of kG-modules. Next, we introduce the notion of relative Brauer indecomposability and then state the second main theorem of this paper.

In [5], the notion of Brauer indecomposability was introduced:

Definition 6. (see [5]) A kG-module M is *Brauer indecomposable* if $M(Q)$ is indecomposable as $kQC_G(Q)$-module or zero.

In order to apply Theorem 1, the Scott module M must be Brauer indecomposable. In fact, for any nontrivial subgroup Q of P, we have to confirm that $(M(\Delta Q), M(\Delta Q)^*)$ induces a Morita equivalence between $B_0(C_G(Q))$ and $B_0(C_{G'}(Q))$, that is, the following bimodule isomorphisms hold:

$M(\Delta Q)\otimes_{B_0(C_{G'}(Q))}M(\Delta Q)^* \cong B_0(C_G(Q))$ and $M(\Delta Q)^*\otimes_{B_0(C_G(Q))}M(\Delta Q) \cong B_0(C_{G'}(Q))$.

Since $B_0(C_G(Q))$ and $B_0(C_{G'}(Q))$ are indecomposable, the Brauer construction $M(\Delta Q)$ must be indecomposable as a $B_0(C_G(Q))$-$B_0(C_{G'}(Q))$-bimodule, or equivalently, as a $kC_{G\times G'}(\Delta Q)$-module. This means that M must be Brauer indecomposable.

Ishioka and the first author [4] gave conditions for Scott modules to be Brauer indecomposable:

Theorem 7. *(see* [4, Theorem 1.3]*) Let P be a p-subgroup of G, and $M = S(G, P)$. Suppose that the fusion system $\mathcal{F}_P(G)$ is saturated. Then the following are equivalent:*

(i) The module M is Brauer indecomposable.

(ii) The module $S(N_G(Q), N_P(Q)){\downarrow}^{N_G(Q)}_{QC_G(Q)}$ is indecomposable for each fully normalized subgroup Q of P.

Moreover, if these conditions hold, then $M(Q) \cong S(N_G(Q), N_P(Q))$ for any fully normalized subgroup Q of P.

Theorem 8. *(see* [4, Theorem 1.4]*) Let P be a p-subgroup of G, and Q a fully normalized subgroup of P. Assume that $\mathcal{F}_P(G)$ is saturated. If there exists a subgroup H_Q of $N_G(Q)$ satisfying the following conditions:*

(a) $N_P(Q)$ is a Sylow p-subgroup of H_Q,

(b) $|N_G(Q) : H_Q| = p^a, a \geq 0$,

then $S(N_G(Q), N_P(Q)){\downarrow}_{QC_G(Q)}$ is indecomposable.

Theorem 7 enables us to confirm the Brauer indecomposability of Scott modules by using the group-theoretic conditions in Theorem 8.

In Theorem 5, it suffices to show that $M(\Delta Q)$ is indecomposable as a $kC_{G\times G'}(\Delta Q)$-module for any subgroup Q of P properly containing Z while in Theorem 1, M must be Brauer indecomposable. Therefore we introduce the notion of relative Brauer indecomposability:

Definition 9. Let M be an indecomposable kG-module with vertex P, and R a subgroup of P. We say that M is *relatively R-Brauer indecomposable* if for any p-subgroup Q of G containing R, the Brauer construction $M(Q)$ is indecomposable (or zero) as a $kQC_G(Q)$-module.

Finally, we state the second main theorem of this paper:

Theorem 10. *Let P be a p-subgroup of G, and $M = S(G,P)$. Suppose that the fusion system $\mathcal{F}_P(G)$ is saturated, and Z is a subgroup of $Z(G)\cap P$. Then the following are equivalent.*

(i) The module M is relatively Z-Brauer indecomposable.

(ii) The module $S(N_G(Q), N_P(Q))\downarrow^{N_G(Q)}_{QC_G(Q)}$ is indecomposable for each fully normalized subgroup Q of P containing Z.

Moreover, if these conditions hold, then $M(Q) \cong S(N_G(Q), N_P(Q))$ for any fully normalized subgroup Q of P containing Z.

Note that Theorem 10 with $Z = 1$ implies Theorem 7. Theorem 10 enables us to confirm the relative Brauer indecomposability of Scott modules by using the group-theoretic conditions in Theorem 8.

References

[1] M. Broué, *Equivalences of blocks of group algebras*, in Finite-dimensional algebras and related topics (Ottawa 1992), eds. V. Dlab and L.L. Scott, Kluwer Acad. Pub., Dordrecht, 1994, pp. 1–26.

[2] J.F. Carlson, *Modules and group algebras*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1996.

[3] J.F. Carlson, C. Peng and W.W. Wheeler, *Transfer maps and virtual projectivity*, J. Algebra. **204(1)** (1998), 286–311.

[4] H. Ishioka and N. Kunugi, *Brauer indecomposability of Scott modules*, J. Algebra. **470** (2017), 441–449.

[5] R. Kessar, N. Kunugi and N. Mitsuhashi, *On saturated fusion systems and Brauer indecomposability of Scott modules*, J. Algebra. **340** (2011), 90–103.

[6] S. Koshitani and C. Lassueur, *Splendid Morita equivalences for principal 2-blocks with dihedral defect groups*, Math. Z. **294(1-2)** (2020), 639–666.

[7] M. Linckelmann, *Stable equivalences of Morita type for self-injective algebras and p-groups*, Math. Z. **223(1)** (1996), 87–100.

[8] T. Okuyama, *A generalization of projective covers of modules over group algebras*, preprint.

[9] ——, *Some examples of derived equivalent blocks of finite groups*, preprint (1997).

[10] L. Wang and J. Zhang, *Relatively stable equivalences of Morita type for blocks*, J. Pure Appl. Algebra, **222(9)** (2018), 2703–2717.

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CATEGORICAL ENTROPY OF THE FROBENIUS PUSHFORWARD FUNCTOR

HIROKI MATSUI

Abstract. For a triangulated category $\mathcal{T}$ with a split generator G and an exact endofunctor $\Phi : \mathcal{T} \to \mathcal{T}$, Dimitrov-Haiden-Katzarkov-Kontsevich introduced the invariant $h_t^{\mathcal{T}}(\Phi)$ which is called the categorical entropy. In this article, we will determine the categorical entropy of the Frobenius pushforward functor.

1. Introduction

This report is based on joint work with Ryo Takahashi [5].

For a triangulated category $\mathcal{T}$ and an exact endofunctor $\Phi : \mathcal{T} \to \mathcal{T}$, Dimitrov, Haiden, Katzarkov, and Kontsevich [1] introduced the invariant $h_t^{\mathcal{T}}(\Phi)$ which is called the *categorical entropy* of Φ as a categorical analog of the topological entropy. The categorical entropy $h_t^{\mathcal{T}}(\Phi)$ is a function in one real variable t with values in $\mathbb{R} \cup \{-\infty\}$ and measures the complexity of the exact endofunctor Φ.

For a commutative noetherian local ring with prime characteristic p, the ring endomorphism $F : R \to R$, which is called the *Frobenius endomorphism*, is defined by $F(a) = a^p$. Assume further that $F : R \to R$ is module finite. The Frobenius endomorphism F induces two exact endofunctors: the *Frobenius pushforward*

$$\mathbb{R}F_* : \mathrm{D^b}(R) \to \mathrm{D^b}(R)$$

on the bounded derived category $\mathrm{D^b}(R)$ of finitely generated R-modules and the *Frobenius pullback*

$$\mathbb{L}F^* : \mathrm{K^b}(R) \to \mathrm{K^b}(R)$$

on the bounded homotopy category $\mathrm{K^b}(R)$ of finitely generated projective R-modules. Both these functors are the main tools to study singularities with positive characteristics.

For the Frobenius pullback functor $\mathbb{L}F^*$, Majidi-Zolbanin and Miasnikov [3] considered the full subcategory $\mathrm{K^b_{fl}}(R)$ of $\mathrm{K^b}(R)$ consisting of perfect complexes with finite length cohomologies and computed the categorical entropy $h_t^{\mathrm{K^b_{fl}}(R)}(\mathbb{L}F^*)$.

In this report, we study the Frobenius pushforward functor F_* and compute its categorical entropy $h_t^{\mathrm{D^b}(R)}(\mathbb{R}F_*)$. We will also discuss the relation between the categorical entropy $h_t^{\mathrm{D^b}(R)}(\mathbb{R}\phi_*)$ of the pushforward functor along a local ring endomorphism $\phi : R \to R$ and the *local entropy* $h_{loc}(\phi)$ of ϕ which has been introduced in [4].

The detailed version of this paper will be submitted for publication elsewhere.

2. Categorical Entropy

Let $\mathcal{T}$ be a triangulated category. We begin with fixing notations.

Notation 1. (1) For an object $X \in \mathcal{T}$, denote by $\mathrm{thick}(X)$ the smallest thick subcategory containing $\mathcal{T}$.

(2) For objects $X_1, X_2, \ldots, X_r \in \mathcal{T}$, we write $X_1 * X_2 * \cdots * X_r$ the subcategory of $\mathcal{T}$ consisting of objects $Y \in \mathcal{T}$ such that there are exact triangles

$$X_i \to Y_i \to Y_{i+1} \to X_i[1] \quad (i = 1, 2, \ldots, r-1)$$

with $Y_1 = Y$ and $Y_r = X_r$.

The following fact is basic:

Lemma 2. *The following conditions are equivalent for $X, Y \in \mathcal{T}$:*

(1) $Y \in \mathrm{thick}(X)$.

(2) *There are $Y' \in \mathcal{T}$, $n_1, n_2, \ldots, n_r \in \mathbb{Z}$ such that $Y \oplus Y' \in X[n_1] * X[n_2] * \cdots * X[n_r]$.*

Now, let us state the definitions of complexities and categorical entropies introduced in [1], which play central roles in this report.

Definition 3. (Dimitrov-Haiden-Katzarkov-Kontsevich)

(1) For $X, Y \in \mathcal{T}$ and $t \in \mathbb{R}$, define the *complexity* of Y relative to X by

$$\delta_t(X, Y) := \inf \left\{ \sum_{i=1}^{r} e^{n_i t} \;\middle|\; \begin{array}{c} \exists Y' \in \mathcal{T}, \exists n_1, n_2, \ldots, n_r \in \mathbb{Z} \text{ s.t.} \\ Y \oplus Y' \in X[n_1] * X[n_2] * \cdots * X[n_r] \end{array} \right\} \in [0, \infty].$$

By Lemma 2, $\delta_t(X, Y) < \infty$ if and only if $Y \in \mathrm{thick}(X)$.

(2) Assume that $\mathcal{T}$ has a split generator G (i.e., $\mathcal{T} = \mathrm{thick}(G)$). For an exact endofunctor $\Phi : \mathcal{T} \to \mathcal{T}$, define the *categorical entropy* of $(\mathcal{T}, \Phi)$ by

$$h_t^{\mathcal{T}}(\Phi) := \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, \Phi^n(G)).$$

This limit exists in $[-\infty, \infty)$ and is independent of the choice of G by [1, Lemma 2.6].

Here, we list basic properties of $\delta_t(X, Y)$.

Lemma 4. *Let $\mathcal{T}$ be a triangulated category.*

(1) *For $X, Y, Z \in \mathcal{T}$ with $Z \in \mathrm{thick}Y \subseteq \mathrm{thick}X$, one has $\delta_t(X, Z) \le \delta_t(X, Y)\delta_t(Y, Z)$.*

(2) *For $X, Y, Z \in \mathcal{T}$, one has $\delta_t(X, Y) \le \delta_t(X, Y \oplus Z) \le \delta_t(X, Y) + \delta_t(X, Z)$.*

(3) *For $X, Y, Z \in \mathcal{T}$, one has $\delta_t(X \oplus Y, Z) \le \delta_t(X, Z)$.*

(4) *For $X, Y \in \mathcal{T}$, one has $\delta_t(X, Y[n]) = \delta_t(X, Y)e^{nt}$.*

(5) *For $X, Y, Y_1, \ldots, Y_r \in \mathcal{T}$ with $Y \in Y_1 * \cdots * Y_r$, one has $\delta_t(X, Y) \le \sum_{i=1}^{r} \delta_t(X, Y_i)$.*

(6) *For an exact functor $\Phi : \mathcal{T} \to \mathcal{T}'$ and $X, Y \in \mathcal{T}$, one has $\delta_t(\Phi(X), \Phi(Y)) \le \delta_t(X, Y)$.*

3. Local and Categorical Entropies

Let $(R, \mathfrak{m}, k)$ be a d-dimensional commutative noetherian local ring. Let $\phi : R \to R$ be a finite local ring homomorphism.

Majidi-Zolbanin, Miasnikov, and Szpiro [4] defined the *local entropy* which measures the complexity of ϕ:

Definition 5. (Majidi Zolbanin-Miasnikov-Szpiro) Define the *local entropy* by

$$h_{loc}(\phi) := \lim_{n\to\infty} \frac{1}{n} \log(\mathrm{length}_R(R/\phi^n(\mathfrak{m})R)).$$

This limit exists and non-negative by [4, Theorem 1].

They determined the local entropy for the Frobenius homomorphism.

Proposition 6. ([4, Theorem 1]) *If R has prime characteristic p, then the equality*

$$h_{loc}(F) = d \log p$$

holds.

The aim of this report is to compare the local entropy of ϕ and the categorical entropy of exact endofunctors associated with ϕ. Let us recall two basic exact endofunctors. The pushforward functor

$$\phi_* : \mathrm{Mod}\, R \to \mathrm{Mod}\, R$$

along ϕ is defined as follows: for an R-module M, $\phi_*(M) := M$ is an abelian group together with the R-module structure via ϕ. This functor is exact by definition. The pullback functor

$$\phi^* : \mathrm{Mod}\, R \to \mathrm{Mod}\, R$$

along ϕ is defined by $\phi^*(M) := M \otimes_R \phi_*(R)$. Deriving these functors, we obtain exact functors

$$\begin{array}{rccc} \mathbb{R}\phi_* : & \mathrm{D^b}(R) & \longrightarrow & \mathrm{D^b}(R) \\ & \cup\!\mid & & \cup\!\mid \\ & \mathrm{D^b_{fl}}(R) & \longrightarrow & \mathrm{D^b_{fl}}(R), \end{array}$$

and

$$\begin{array}{rccc} \mathbb{L}\phi^* : & \mathrm{K^b}(R) & \longrightarrow & \mathrm{K^b}(R) \\ & \cup\!\mid & & \cup\!\mid \\ & \mathrm{K^b_{fl}}(R) & \longrightarrow & \mathrm{K^b_{fl}}(R). \end{array}$$

Here, $\mathrm{D^b_{fl}}(R)$ and $\mathrm{K^b_{fl}}(R)$ stand for the subcategories of $\mathrm{D^b}(R)$ and $\mathrm{K^b}(R)$ consisting of complexes with finite length cohomologies, respectively.

For the pullback functor $\mathbb{L}\phi^*$, Majidi-Zolbanin and Miasnikov compared the categorical entropy of $\mathbb{L}\phi^*$ and the local entropy of ϕ. Moreover, they determined the categorical entropy for the Frobenius pullback functor:

Theorem 7. *Let R be a d-dimensional commutative noetherian local ring and $\phi : R \to R$ a finite local ring homomorphism.*

(1) *For any $t \in \mathbb{R}$, one has the inequality*

$$h_t^{\mathrm{K^b_{fl}}(R)}(\mathbb{L}\phi^*) \geq h_{loc}(\phi).$$

(2) *Assume further that R is a complete noetherian local ring with prime characteristic p. For any $t \in \mathbb{R}$, the equality*

$$h_t^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{pf}}(R)}(\mathbb{L}F^*) = h_{loc}(F) = d \log p$$

holds.

On the other hand, we can also compute $h_t^{\mathrm{K}^{\mathrm{b}}(R)}(\mathbb{L}F^*)$ and $h_t^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{b}}(R)}(\mathbb{R}F^*)$.

Proposition 8. *Assume that R has prime characteristic and the Frobenius homomorphism $F : R \to R$ is finite. For any $t \in \mathbb{R}$, the following equalities hold:*

(1) $h_t^{\mathrm{K}^{\mathrm{b}}(R)}(\mathbb{L}F^*) = 0$.
(2) $h_t^{\mathrm{D}_{\mathrm{fl}}^{\mathrm{b}}(R)}(\mathbb{R}F^*) = \log[F_*(k) : k]$.

Remark 9. The triangulated categories $\mathrm{K}_{\mathrm{fl}}^{\mathrm{b}}(R), \mathrm{K}^{\mathrm{b}}(R), \mathrm{D}_{\mathrm{fl}}^{\mathrm{b}}(R)$ have generators $\mathrm{K}(\underline{x})$ (the Koszul complex of a system of generators $\underline{x}$ of $\mathfrak{m}$), R, k, respectively. Therefore the categorical entropies that appeared in the preceding results are defined.

From the above two results, the remained problem is to compute $h_t^{\mathrm{D}^{\mathrm{b}}(R)}(\mathbb{R}F_*)$, which we will consider in the next section.

4. Main Theorem

First note that if R is excellent, then the derived category $\mathrm{D}^{\mathrm{b}}(R)$ has a split generator and hence we can consider the categorical entropy $h_t^{\mathrm{D}^{\mathrm{b}}(R)}(\mathbb{R}\phi_*)$.

Theorem 10. *Let R be a d-dimensional excellent noetherian local ring.*

(1) *Let $\phi : R \to R$ be a finite local ring homomorphism. For any $t \in \mathbb{R}$, the equality*

$$h_t^{\mathrm{D}^{\mathrm{b}}(R)}(\mathbb{R}\phi_*) \geq h_{loc}(\phi) + \log[\phi_*(k) : k]$$

holds.

(2) *Assume further that R has prime characteristic p and the Frobenius homomorphism $F : R \to R$ is finite. For any $t \in \mathbb{R}$, the equality*

$$h_t^{\mathrm{D}^{\mathrm{b}}(R)}(\mathbb{R}F_*) = h_{loc}(F) + \log[F_*(k) : k] = d \log p + \log[F_*(k) : k]$$

holds.

Using Theorem 10(1), we can globalize the inequality for the Frobenius pushforward functor:

Corollary 11. *Let X be a connected noetherian scheme with prime characteristic p. Assume that the Frobenius homomorphism $F : X \to X$ is finite. For any $t \in \mathbb{R}$ and $x \in X$, the inequality*

$$h_t^{\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)}(\mathbb{R}F_*) \geq \dim \mathcal{O}_{X,x} \cdot \log p + \log[F_*(k(x)) : k(x)]$$

holds.

Remark 12. Since X is connected the number $\dim \mathcal{O}_{X,x} \cdot \log p + \log[F_*(k(x)) : k(x)]$ is independent of $x \in X$.

For the rest of this report, let us give a sketch of the proof of Theorem 10.

The proof of Theorem 10(1) needs the following lemma which generalizes [3, Lemma 2.1].

Lemma 13. *Let* $(R, \mathfrak{m}, k)$ *be a commutative noetherian local ring. Let* $0 \neq G \in \mathrm{D^b}(R), 0 \neq P \in \mathrm{K^b_{fl}}(R)$ *and take an integer* N *with* $\mathrm{H}^i(G \otimes P) = 0$ *for all* $|i| > N$. *Set* $B := \max\{\mathrm{length}_R(\mathrm{H}^i(G \otimes P)) \mid -N \leq i \leq N\}$. *Then for any* $E \in \mathrm{D^b}(R)$, $m \in \mathbb{Z}$, *and* $t \in \mathbb{R}$, *the inequality*

$$\delta_t(G, E) \geq B^{-1} e^{-mt} e^{-N|t|} \cdot \mathrm{length}_R(\mathrm{H}^m(E \otimes_R P))$$

holds.

(Proof of Theorem 10(1)). We note that for a finitely generated R-module M with finite length, one has

$$\mathrm{length}_R((\phi^*)^n(M)) = [\phi^n(k) : k] \cdot \mathrm{length}_R(M) = [\phi(k) : k]^n \cdot \mathrm{length}_R(M).$$

In particular, $\mathrm{length}_R((\phi^*)^n(R)/\mathfrak{m}(\phi^*)^n(R))) = [\phi(k) : k]^n \cdot \mathrm{length}_R(R/\phi^n(\mathfrak{m})R)$.

Take a split generator $G \in \mathrm{D^b}(R)$ such that $\mathrm{H}^i(G) = 0$ for $i < 0$ and that R is a direct summand of $\mathrm{H}^0(G)$. Let $\underline{x}$ be a system of generators of $\mathfrak{m}$ and set $P = K(\underline{x})$ the Koszul complex of $\underline{x}$. Then it follows from Lemma 13 that

$$\begin{aligned}
\delta_t(G, (\mathbb{R}\phi_*)^n(G)) &\geq B^{-1} e^{-N|t|} \cdot \mathrm{length}_R(\mathrm{H}^0((\mathbb{R}\phi_*)^n(G) \otimes_R P)) \\
&= B^{-1} e^{-N|t|} \cdot \mathrm{length}_R(\mathrm{H}^0((\mathbb{R}\phi_*)^n(G)) \otimes_R \mathrm{H}^0(P)) \\
&= B^{-1} e^{-N|t|} \cdot \mathrm{length}_R((\phi_*)^n(\mathrm{H}^0(G)) \cdot \otimes_R R/\mathfrak{m}) \\
&\geq B^{-1} e^{-N|t|} \cdot \mathrm{length}_R((\phi_*)^n(R) \otimes_R R/\mathfrak{m}) \\
&= B^{-1} e^{-N|t|} \cdot \mathrm{length}_R((\phi_*)^n(R/\phi^n(\mathfrak{m})R)) \\
&= B^{-1} e^{-N|t|} \cdot [\phi(k) : k]^n \cdot \mathrm{length}_R(R/\phi^n(\mathfrak{m})R)
\end{aligned}$$

Taking $\lim_{n\to\infty} \frac{1}{n} \log(-)$, we obtain $h_t(\mathbb{R}\phi_*) \geq h_{loc}(\phi) + \log[\phi(k) : k]$. □

Since the proof of Theorem 10(2), i.e., the proof of

$$h_t^{\mathrm{D^b}(R)}(\mathbb{R}F_*) \leq h_{loc}(F) + \log[\phi_*(k) : k]$$

is more difficult and complicated than the converse inequality, we shall give quite rough sketch of the proof. This difficulty comes from the fact that no explicit descriptions of a split generator of $\mathrm{D^b}(R)$ is known unlike $\mathrm{K^b_{fl}}(R), \mathrm{K^b}(R), \mathrm{D^b_{fl}}(R)$. Therefore, we use induction to reduce the case of $d = 0$ so that $\mathrm{D^b}(R) = \mathrm{D^b_{fl}}(R)$. The proof is done as follows:

- The case of $d = 0$ follows from Proposition 8(2).
- For the case of $d > 0$, first we reduce to domain case and then take a regular element x with $x\mathrm{Ext}_R^{2d+1}(-, -) = 0$. We can take such a regular element by [2, Theorem 5.3]. Then for a split generator G' of $\mathrm{D^b}(R/xR)$, $G := G' \oplus R$ is a split generator of $\mathrm{D^b}(R)$.

- Using Lemma 4, reduce to the computations of $\delta_t(G', (\mathbb{R}F_*)^n(G'))$, $\delta_t(G, (F_*)^n(R))$. $\delta_t(G', (\mathbb{R}F_*)^n(G'))$ is known by induction hypothesis. To compute $\delta_t(G, (F_*)^n(R))$, we use an exact sequence

$$0 \to \Omega^{2d}((F_*)^n(R)) \to R^{\oplus \beta_{2d-1}((F_*)^n(R))} \to \cdots \to R^{\oplus \beta_1((F_*)^n(R))} \to R^{\oplus \beta_0((F_*)^n(R))} \to (F_*)^n(R) \to 0$$

and the equality

$$\lim_{n\to\infty} \frac{1}{n} \log \beta_i((F_*)^n(R)) = d \log p + \log[F_*(k) : k];$$

see [6].

References

[1] G. Dimitrov, F. Haiden, L. Katzarkov, and M. Kontsevich, *Dynamical systems and categories*, Contemp. Math. **621** (2014), 133–170.

[2] S. B. Iyengar and R. Takahashi, *Annihilation of cohomology and strong generation of module categories*, Int. Math. Res. Not. IMRN **2** (2016), 499–535.

[3] M. Majidi-Zolbanin and N. Miasnikov, *Entropy in the category of perfect complexes with cohomology of finite length*, JPAA **223** (2019), no. 6, 2585–2597.

[4] M. Majidi-Zolbanin and N. Miasnikov, and L. Szpiro, *Entropy and flatness in local algebraic dynamics*, Publ. Mat. **57** (2013), 509–544.

[5] H. Matsui and R. Takahashi, *On the categorical entropy of the Frobenius pushforward functor*, `arXiv:2207.13774`.

[6] G. Seibert, *Complexes with homology of finite length and Frobenius functors*, J. Algebra **125** (1989), 278–287.

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CHARACTERIZATION OF 4-DIMENSIONAL NON-THICK IRREDUCIBLE REPRESENTATIONS

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ABSTRACT. We say that a group representation $\rho : G \to \mathrm{GL}(V)$ is *thick* if it has "enough transitivity" of the group action on the set of subspaces of V. If $\dim V \leq 3$, then ρ is thick if and only if it is irreducible. In the case $\dim V \geq 4$, if ρ is thick, then it is irreducible, but the converse is not true. In this paper, we give a characterization of 4-dimensional non-thick irreducible representations of an arbitrary group G.

Key Words: Thick representation, Dense representation, Characterization, 4-dimensional non-thick irreducible representation.

2020 *Mathematics Subject Classification*: Primary 20C99; Secondary 14D22, 20E05.

1. Introduction

In [1], we described the moduli of 4-dimensional non-thick irreducible representations for the free group of rank 2. In this paper, we deal with characterization of 4-dimensional non-thick irreducible representations of an arbitrary group. Throughout this paper, V denotes a finite-dimensional vector space over a field k.

Definition 1 (*cf.* [2] and [3])**.** Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a group G. We say that ρ is *m-thick* if for any subspaces V_1, V_2 of V with $\dim_k V_1 = m$ and $\dim_k V_2 = \dim_k V - m$ there exists $g \in G$ such that $(\rho(g)V_1) \cap V_2 = 0$. If ρ is m-thick for any $0 < m < \dim_k V$, then we say that ρ is *thick*.

Roughly speaking, m-thick representations $\rho : G \to \mathrm{GL}(V)$ have enough transitivity of the group action of G on the set of m-dimensional vector subspaces of V.

Definition 2 (*cf.* [2] and [3])**.** Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. We say that ρ is *m-dense* if the exterior representation

$$\wedge^m \rho : G \to \mathrm{GL}(\Lambda^m V)$$

is irreducible. Here, we define

$$(\wedge^m \rho)(g)(v_1 \wedge v_2 \wedge \cdots \wedge v_m) = \rho(g)v_1 \wedge \rho(g)v_2 \wedge \cdots \wedge \rho(g)v_m$$

for $g \in G, v_1, v_2, \ldots, v_m \in V$. If ρ is m-dense for any $0 < m < \dim_k V$, then we say that ρ is *dense*.

In [2], we obtained several results on thick representations and dense representations.

The detailed version of this paper will be submitted for publication elsewhere.

The first author was partially supported by JSPS KAKENHI Grant Number JP20K03509.

Proposition 3 ([2, Proposition 2.6])**.** For an n-dimensional representation $\rho : G \to \mathrm{GL}(V)$,

$$m\text{-thick} \Longleftrightarrow (n-m)\text{-thick}.$$

Proposition 4 ([2, Proposition 2.6])**.** For an n-dimensional representation $\rho : G \to \mathrm{GL}(V)$,

$$m\text{-dense} \Longleftrightarrow (n-m)\text{-dense}$$

Proposition 5 ([2, Proposition 2.7])**.** Let $0 < m < n$. For an n-dimensional representation $\rho : G \to \mathrm{GL}(V)$,

$$\begin{array}{ccccc} m\text{-dense} & \Longrightarrow & m\text{-thick} & & \\ & & \Downarrow & & \\ 1\text{-dense} & \Longleftrightarrow & 1\text{-thick} & \Longleftrightarrow & \text{irreducible.} \end{array}$$

Corollary 6 ([2, Corollary 2.8])**.** For a finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$,

$$\textit{dense} \Rightarrow \textit{thick} \Rightarrow \text{irreducible}$$

Corollary 7 ([2, Corollary 2.9])**.** Let $n \leq 3$. For an n-dimensional representation $\rho : G \to \mathrm{GL}(V)$,

$$\textit{dense} \Leftrightarrow \textit{thick} \Leftrightarrow \text{irreducile.}$$

Not all 4-dimensional irreducible representations are thick. In this paper, we describe a characterization of 4-dimensional non-thick irreducible representations for an arbitrary group G.

2. Motivation

In this section, we explain our motivation for considering thick reprsentations and dense representations.

Let $\mathrm{Rep}_n(G)$ be the *representation variety* of degree n for G. For simplicity, let us consider $\mathrm{Rep}_n(G)$ over an algebraically closed field Ω. We can regard

$$\mathrm{Rep}_n(G) = \{\rho \mid \rho : G \to \mathrm{GL}_n(\Omega)\}$$

as an affine algebraic scheme (variety) over Ω if G is a finitely generated group. Even if G is not finitely generated, $\mathrm{Rep}_n(G)$ can be defined, although it is not necessarily of finite type over Ω. The group scheme $\mathrm{PGL}_n(\Omega)$ acts on $\mathrm{Rep}_n(G)$ by

$$\rho \mapsto P^{-1}\rho P$$

for $\rho \in \mathrm{Rep}_n(G)$ and $P \in \mathrm{PGL}_n(\Omega)$. We define the $\mathrm{PGL}_n(\Omega)$-invariant open subscheme $\mathrm{Rep}_n(G)_{air}$ of $\mathrm{Rep}_n(G)$ by

$$\mathrm{Rep}_n(G)_{air} = \{\rho \in \mathrm{Rep}_n(G) \mid \rho \text{ is (absolutely) irreducible }\}.$$

We also define the character variety $\mathrm{Ch}_n(G)_{air}$ of n-dimensional irreducible representations for G by

$$\begin{aligned} \mathrm{Ch}_n(G)_{air} &= \mathrm{Rep}_n(G)_{air}/\mathrm{PGL}_n(\Omega) \\ &= \{[\rho] \mid \text{ eq. classes of } n\text{-dim. irreducible representations }\}. \end{aligned}$$

Theorem 8 ([2, Theorem 3.9 and Proposition 3.11]). *The representation variety* $\mathrm{Rep}_n(G)_{air}$ *has open subschemes*

$$\begin{array}{rcl} \mathrm{Rep}_n(G)_{thick} & = & \{\rho \mid \rho \textit{ is (absolutely) thick }\} \\ \mathrm{Rep}_n(G)_{dense} & = & \{\rho \mid \rho \textit{ is (absolutely) dense }\}. \end{array}$$

Let us define

$$\begin{array}{rcl} \mathrm{Ch}_n(G)_{thick} & = & \mathrm{Rep}_n(G)_{thick}/\mathrm{PGL}_n(\Omega) \\ \mathrm{Ch}_n(G)_{dense} & = & \mathrm{Rep}_n(G)_{dense}/\mathrm{PGL}_n(\Omega). \end{array}$$

Then we have the following diagram:

$$\begin{array}{ccccc} \mathrm{Rep}_n(G)_{dense} & \subseteq & \mathrm{Rep}_n(G)_{thick} & \subseteq & \mathrm{Rep}_n(G)_{air} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ch}_n(G)_{dense} & \subseteq & \mathrm{Ch}_n(G)_{thick} & \subseteq & \mathrm{Ch}_n(G)_{air}. \end{array}$$

Let us consider the morphism

$$\begin{array}{rcccl} \wedge^m & : & \mathrm{Rep}_n(G)_{dense} & \to & \mathrm{Rep}_{\binom{n}{m}}(G)_{air} \\ & & \rho & \mapsto & \wedge^m \rho. \end{array}$$

Theorem 9 ([4]). *Let* $2 \leq m \leq n-2$. *Then*

$$\mathrm{Im}\wedge^m \subseteq \mathrm{Rep}_{\binom{n}{m}}(G)_{non-thick} := \mathrm{Rep}_{\binom{n}{m}}(G)_{air} \setminus \mathrm{Rep}_{\binom{n}{m}}(G)_{thick}.$$

Roughly speaking, any exterior representations can not become thick representations.

Any exterior representations are contained in $\mathrm{Rep}_n(G)_{non-thick}$, while $\mathrm{Rep}_n(G)_{thick}$ is open in $\mathrm{Rep}_n(G)_{air}$. Then how can we describe $\mathrm{Rep}_n(G)_{non-thick}$? Not only the exterior representations but also the tensor products of two representations can not become thick representations.

Theorem 10 ([4]). Let $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ be finite-dimensional representations of G over k. If $\dim_k V \geq 2$ and $\dim_k W \geq 2$, then $\rho\otimes\tau : G \to \mathrm{GL}(V\otimes_k W)$ is not 2-thick. In particular, $\rho \otimes \tau$ is not thick.

Moreover, we obtain:

Theorem 11 ([4]). Let $\rho : G \to \mathrm{GL}(V)$ be an n-dimensional representation over k. If $2 \leq m \leq n-2$, then $\wedge^m \rho : G \to \mathrm{GL}(\Lambda^m V)$ is not 3-thick. In particular, $\wedge^m \rho$ is not thick.

Theorem 12 ([4]). Let $\rho : G \to \mathrm{GL}(V)$ be an n-dimensional representation over k. If $n \geq 3$ and $m \geq 2$, then the m-th symmetric tensor $S^m(\rho) : G \to \mathrm{GL}(S^m(V))$ is not 3-thick. In particular, $S^m(\rho)$ is not thick.

By these theorems, we can construct many non-thick representations. It is difficult to investigate which representations are thick for finite groups and discrete groups. However, we have already classified (finite-dimensional) thick representations over $\mathbb{C}$ of connected complex simple Lie groups.

Let G be a connected semi-simple Lie group over $\mathbb{C}$. Let $\mathfrak{g}, \mathfrak{h}, \Delta^+(\subset \mathfrak{h}^*)$ be the Lie algebra of G, a Cartan subalgebra, the set of positive roots, respectively. For a finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$ over $\mathbb{C}$, denote by $W(V)$ the set of weights of V. We can regard $W(V)$ as a partially ordered set by using Δ^+. We say that $\rho : G \to \mathrm{GL}(V)$ is *weight multiplicity-free* if the dimension of the φ-eigenspace is 1 for any $\varphi \in W(V)$.

Theorem 13 ([3])**.** For a connected semi-simple Lie group G over $\mathbb{C}$, a finite-dimensional irreducible representation $\rho : G \to \mathrm{GL}(V)$ over $\mathbb{C}$ is thick if and only if ρ is weight multiplicity-free and $W(V)$ is a totally ordered set.

By the classification of weight multiplicity-free irreducible representations by Howe and Panyushev, we have the following theorem:

Theorem 14 ([3, Theorem 3.6])**.** *The classification of thick representations of connected simple Lie groups:*

(1) *the trivial* 1*-dimensional representation for any groups*
(2) A_n $(n \geq 1)$
  - *the standard representation V for A_n $(n \geq 1)$ with highest weight ω_1*
  - *the dual representation V^* of V for A_n $(n \geq 1)$ with highest weight ω_n*
  - *the symmetric tensor $S^m(V)$ $(m \geq 2)$ of V for A_1 with highest weight $m\omega_1$*

(3) B_n $(n \geq 2)$
  - *the standard representation V for B_n $(n \geq 2)$ with highest weight ω_1*
  - *the spin representation for B_2 with highest weight ω_2*

(4) C_n $(n \geq 3)$
  - *the standard representation V for C_n $(n \geq 3)$ with highest weight ω_1*

(5) G_2
  - *the* 7*-dimensional representation V for G_2 with highest weight ω_1.*

For constructing non-thick representations, the following lemma is useful.

Lemma 15. *Let $\rho : G \to \mathrm{GL}(V)$ be a non-thick representation. For a group homomorphism $\phi : G' \to G$, $\rho \circ \phi : G' \to \mathrm{GL}(V)$ is a non-thick representation.*

By this lemma, we can construct a non-thick representation $\rho \circ \phi : G' \xrightarrow{\phi} G \xrightarrow{\rho} \mathrm{GL}(V)$, where $\rho : G \to \mathrm{GL}(V)$ is a non-thick representation of a connected complex simple Lie group G which is not listed in Theorem 14.

Anyway, what is $\mathrm{Rep}_n(G)_{non-thick}$? In this paper, we would like to investigate a characterization of 4-dimensional non-thick irreducible representations to describe $\mathrm{Rep}_4(G)_{non-thick}$ and $\mathrm{Ch}_4(G)_{non-thick}$.

3. Four-dimensional non-thick irreducible representations

Let us give an example of a non-thick irreducible representation. We denote by S_n the symmetric group of degree n. We regard $\mathrm{GL}(V)^n$ and S_n as subgroups of $\mathrm{GL}(V^{\oplus n})$ by $(A_1, \ldots, A_n) \cdot (v_1, \ldots, v_n) = (A_1 v_1, \ldots, A_n v_n)$ and $\sigma \cdot (v_1, \ldots, v_n) = (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)})$ for $(A_1, \ldots, A_n) \in \mathrm{GL}(V)^n, \sigma \in S_n$ and $(v_1, \ldots, v_n) \in V^{\oplus n}$, respectively. Then the semidirect product $\mathrm{GL}(V)^n \rtimes S_n$ is defined as a subgroup of $\mathrm{GL}(V^{\oplus n})$. The inclusion $\rho_{V,n} : \mathrm{GL}(V)^n \rtimes S_n \to \mathrm{GL}(V^{\oplus n})$ gives a representation of $\mathrm{GL}(V)^n \rtimes S_n$.

Theorem 16 ([4])**.** Let $n \geq 2$ and $\dim_k V \geq 2$. The representation $\rho_{V,n} : \mathrm{GL}(V)^n \rtimes S_n \to \mathrm{GL}(V^{\oplus n})$ is a non-thick irreducible representation. More precisely, $\rho_{V,n}$ is neither n-thick nor $\dim_k V$-thick.

Let us consider 4-dimensional non-thick irreducible representations. Our main theorem is the following:

Theorem 17 ([4])**.** Let $\rho : G \to \mathrm{GL}(V)$ be a 4-dimensional non-thick irreducible representation of a group G over a field k. Then ρ is equivalent to one of the following two cases:

(1) the composition $G \xrightarrow{\phi} \mathrm{GL}(W)^2 \rtimes S_2 \xrightarrow{\rho_{W,2}} \mathrm{GL}(W \oplus W)$ with $\dim_k W = 2$, where $\phi : G \to \mathrm{GL}(W)^2 \rtimes S_2$ is a group homomorphism

(2) a representation $\rho' : G \to \mathrm{GL}(V_1 \otimes_k V_2)$ with $\dim_k V_1 = \dim_k V_2 = 2$ which is equivalent to $\tau_1 \otimes \tau_2$ as projective representations, where $\tau_i : G \to \mathrm{PGL}(V_i)$ is a projective representation for $i = 1, 2$.

Remark 18. In Theorem 17, the case (2) can not be replaced with

(2)' a representation $\tau_1 \otimes \tau_2 : G \to \mathrm{GL}(V_1 \otimes_k V_2)$ with $\dim_k V_1 = \dim_k V_2 = 2$, where $\tau_i : G \to \mathrm{GL}(V_i)$ is a linear representation for $i = 1, 2$.

Indeed, there exists an example of (2) which does not satisfy (2)'. Let us give such an example. Suppose that $\dim_k V_1 = \dim_k V_2 = 2$. Let us consider the fiber product G of ψ and π:

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{GL}(V_1 \otimes V_2) \\ \downarrow_{(\rho_1, \rho_2)} & & \downarrow_{\pi} \\ \mathrm{PGL}(V_1) \times \mathrm{PGL}(V_2) & \xrightarrow{\psi} & \mathrm{PGL}(V_1 \otimes V_2) \\ (g_1, g_2) & \mapsto & g_1 \otimes g_2. \end{array}$$

In other words,

$$G = \{(g_1, g_2, A) \in \mathrm{PGL}(V_1) \times \mathrm{PGL}(V_2) \times \mathrm{GL}(V_1 \otimes V_2) \mid \pi(A) = g_1 \otimes g_2\}.$$

If $\mathrm{ch}\, k \neq 2$, then $\rho : G \to \mathrm{GL}(V_1 \otimes V_2)$ is a 4-dimensional non-thick irreducible representation such that the projective representation $\rho_i : G \to \mathrm{PGL}(V_i)$ can not be lifted as a linear representation ($i = 1, 2$).

Remark 19. In [1, Propositions 18 and 21], we claimed that any 4-dimensional non-thick irreducible representation is equivalent to (1) or (2)' in Theorem 17 instead of (2). However, this is not true, as seen in Remark 18. On the other hand, any projective representation can be lifted as a linear representation for free groups.

Before giving an outline of the proof of Theorem 17, we need some definitions.

Definition 20 (perfect pairing)**.** Let $\rho : G \to \mathrm{GL}(V)$ be an n-dimensional representation. For $0 < m < n$, the perfect pairing $\Lambda^m V \otimes_k \Lambda^{n-m} V \xrightarrow{\wedge} \Lambda^n V$ is defined by

$$(v_1 \wedge v_2 \wedge \cdots \wedge v_m) \otimes (v_{m+1} \wedge v_{m+2} \wedge \cdots \wedge v_n) \\ \mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_m \wedge v_{m+1} \wedge v_{m+2} \wedge \cdots \wedge v_n.$$

The perfect pairing $\wedge$ is G-equivariant. For a subspace $W \subseteq \Lambda^m V$, we put

$$W^\perp := \{w' \in \Lambda^{n-m} V \mid w \wedge w' = 0 \text{ for any } w \in W\} \subseteq \Lambda^{n-m} V$$

Note that $(W^\perp)^\perp = W$.

Definition 21 ([2])**.** We say that $W \subseteq \Lambda^m V$ is *realizable* if there exist linearly independent vectors $v_1, v_2, \ldots, v_m \in V$ such that $0 \neq v_1 \wedge v_2 \wedge \cdots \wedge v_m \in W$.

The following proposition gives a characterization of m-thickness and m-denseness.

Proposition 22 ([2])**.** Let $\rho : G \to \mathrm{GL}(V)$ be an n-dimensional representation. Then

(1) ρ is <u>not m-dense</u> if and only if there exist non-trivial G-invariant subspaces $W_1 \subseteq \Lambda^m V$ and $W_2 \subseteq \Lambda^{n-m} V$ such that $W_1^\perp = W_2$.
(2) ρ is <u>not m-thick</u> if and only if there exist non-trivial G-invariant <u>realizable</u> subspaces $W_1 \subseteq \Lambda^m V$ and $W_2 \subseteq \Lambda^{n-m} V$ such that $W_1^\perp = W_2$.

The following proposition is trivial, but useful.

Proposition 23. *For a* 4*-dimensional irreducible representation* $\rho : G \to \mathrm{GL}(V)$*, the following statements are equivalent:*

(1) ρ *is thick.*
(2) ρ *is* 2*-thick.*

Proof. Since ρ is irreducible, it is 1-thick by Proposition 5. It is also 3-thick by Proposition 3. Hence, ρ is thick if and only if it is 2-thick by the definition. ☐

Thereby, any 4-dimensional non-thick irreducible representation $\rho : G \to \mathrm{GL}(V)$ is not 2-thick. By Proposition 22, the 6-dimensional vector space $\Lambda^2 V$ has non-trivial G-invariant realizable subspaces W_1 and W_2 such that $W_1^\perp = W_2$. Using [2, Corollary 4.5], we obtain $\dim_k W_1 \geq 2$ and $\dim_k W_2 \geq 2$. Since $\dim_k W_1 + \dim_k W_2 = 6$, there are only two types:

(1) $\Lambda^2 V$ has a 2-dimensional G-invariant realizable subspace.
(2) $\Lambda^2 V$ has a 3-dimensional G-invariant realizable subspace.

Let us discuss Case (1):

Proposition 24 ([4])**.** Assume that $\Lambda^2 V$ has a 2-dimensional G-invariant realizable subspace. Then there exists a basis $e_1, e_2, e_3, e_4 \in V$ such that

$$W := \langle e_1 \wedge e_2, e_3 \wedge e_4 \rangle \subset \Lambda^2 V$$

is G-invariant. Furthermore, with respect to this basis, we can write

$$\rho(g) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}$$

for any $g \in G$, where $A_1, A_2 \in \mathrm{GL}_2(k)$.

In this case, we can define a group homomorphism $\phi : G \to \mathrm{GL}(W)^2 \rtimes S_2$ such that ρ is equivalent to $\rho_{W,2} \circ \phi$.

Let us discuss Case (2):

Proposition 25 ([4])**.** Assume that $\Lambda^2 V$ has a 3-dimensional G-invariant realizable subspace and that $\Lambda^2 V$ has no 2-dimensional G-invariant realizable subspace. Then there exists a basis $e_1, e_2, e_3, e_4 \in V$ such that

$$W := \langle e_1 \wedge e_2, e_3 \wedge e_4, (e_1 + e_3) \wedge (e_2 + e_4) \rangle \subset \Lambda^2 V$$

is G-invariant. Furthermore, with respect to this basis, we can write

$$\rho(g) = \begin{pmatrix} aA' & bA' \\ cA' & dA' \end{pmatrix}$$

for any $g \in G$, where $a, b, c, d \in k$ with $ad - bc \neq 0$ and $A' \in \mathrm{GL}_2(k)$.

In this case, ρ can be decomposed as the tensor product of two projective representations.

Summarizing the two cases, we can prove Theorem 17.

4. Appendix

In this appendix, we deal with $\mathrm{Rep}_4(F_2)_{thick}$ for the free group F_2 of rank 2 over an algebraically closed field Ω.

Example 26. Let F_m be a free group of rank m with $m \geq 2$. The representation variety $\mathrm{Rep}_n(F_m)$ is an irreducible smooth variety of dimension mn^2. The character variety $\mathrm{Ch}_n(F_m)_{air}$ is an irreducible smooth variety of dimension $(m-1)n^2 + 1$.

For $n = 4$ and $m = 2$, $\dim \mathrm{Rep}_4(F_2) = \dim \mathrm{Rep}_4(F_2)_{air} = 32$.
Let $F_2 = \langle \alpha, \beta \rangle$. In Case (1), by conjugating ρ by $\mathrm{PGL}_4(\Omega)$ we obtain

$$\rho(\alpha) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}$$

$$\rho(\beta) = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}.$$

Roughly speaking,

$$\dim \mathrm{Rep}_4(F_2)_{Case(1)} \leq 4 + 4 + 4 + 4 + \dim \mathrm{PGL}_4(\Omega) = 31 < \dim \mathrm{Rep}_4(F_2)_{air}.$$

In Case (2), note that any projective representation can be lifted as a linear representation for the free group F_2. Set

$$(\mathrm{Rep}_2(F_2) \times \mathrm{Rep}_2(F_2))^o = \{(\rho_1, \rho_2) \in \mathrm{Rep}_2(F_2) \times \mathrm{Rep}_2(F_2) \mid \rho_1 \otimes \rho_2 \text{ is irreducible } \}.$$

Then we obtain a surjective morphism

$$\begin{array}{rcl} (\mathrm{Rep}_2(F_2) \times \mathrm{Rep}_2(F_2))^o \times \mathrm{PGL}_4(\Omega) & \to & \mathrm{Rep}_4(F_2)_{Case(2)} \\ (\rho_1, \rho_2, P) & \mapsto & P^{-1}(\rho_1 \otimes \rho_2)P. \end{array}$$

Roughly speaking,

$$\dim \mathrm{Rep}_4(F_2)_{Case(2)} \leq 2 \dim \mathrm{Rep}_2(F_2) + \dim \mathrm{PGL}_2(\Omega) = 2 \times 8 + 15 = 31.$$

Since $\dim \mathrm{Rep}_4(F_2)_{air} = 32 > \dim \mathrm{Rep}_4(F_2)_{non-thick}$, we obtain $\mathrm{Rep}_4(F_2)_{thick} \neq \emptyset$.

References

[1] K. Nakamoto and Y. Omoda, The moduli spaces of non-thick irreducible representations for the free group of rank 2. (Japanese) Proceedings of the 41st Symposium on Ring Theory and Representation Theory, 55–62, Symp. Ring Theory Represent. Theory Organ. Comm., Tsukuba, 2009.

[2] ———, *Thick representations and dense representations I*, Kodai Math. J. **42** (2019), 274–307.

[3] ———, *The classification of thick representations of simple Lie groups*, Kodai Math. J. **45** (2022), 259–269.

[4] ———, *Thick representations and dense representations II*, in preparation.

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LOCALIZATION OF TRIANGULATED CATEGORIES WITH RESPECT TO EXTENSION-CLOSED SUBCATEGORIES

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Abstract. The aim of this article is to develop a framework of the localization theory of triangulated categories via extriangulated categories. Actually, given the pair of a triangulated category $\mathcal{C}$ and an extension-closed subcategory $\mathcal{N}$, we establish an exact sequence $\mathcal{N} \to \mathcal{C} \to \mathcal{C}/\mathcal{N}$ of extriangulated categories. Such a construction unifies the Verdier quotient and the heart of a t-structure.

1. Introduction

The abelian categories and triangulated categories, introduced by A. Grothendieck and J.-L. Verdier [5, 7], serve a foundation of the homological algebra. In the many blanches of mathematics, we often encounter interplays between abelian categories and triangulated categories. To name just a few important instances in the representation theory of algebra:

- for a given t-structure of a triangulated category $\mathcal{C}$, there exists a cohomological functor from $\mathcal{C}$ to the abelian heart [2];
- for a 2-cluster tilting subcategory $\mathcal{U}$ of a triagnulated category $\mathcal{C}$, the ideal quotient $\mathcal{C}/[\mathcal{U}]$ is abelian [9];
- for a Frobenius exact category $\mathcal{C}$ and the subcategory $\mathcal{N}$ of projective-injective objects, the ideal quotient $\mathcal{C}/[\mathcal{N}]$ is triangulated [6];
- the derived category $\mathsf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is defined to be the localization of the abelian category $\mathsf{C}(\mathcal{A})$ of complexes in $\mathcal{A}$ with respect to the subcategory of acyclic complexes.

To capture such phenomena in a more conceptual framework, the notion of *extriangulated category* was introduced by Nakaoka and Palu [10] as a simultaneous generalization of exact and triangulated categories. As a benefit of revealing an extriangulated structure, it is closed under basic categorical operations; taking extension-closed subcategories, ideal quotients by projective-injective objects and the relative theory [8]. Recently, it was shown that the extriangulated structure is still closed by certain localizations which were introduced in [11] as a unification of the Serre/Verdier quotients. Our aim is to formulate a new framework of localization of triangulated categories as an application of [11], namely, we establish an exact sequence

$$\mathcal{N} \to \mathcal{C} \to \mathcal{C}/\mathcal{N} \tag{1.1}$$

of extriangulated categories arising from the pair of a triangulated category $\mathcal{C}$ and an extension-closed subcategory $\mathcal{N}$. Precisely, the main theorem summarized as follows.

The detailed version of this paper will be submitted for publication elsewhere.

Theorem 1. [13, Thm. 2.20, 3.2, 4.2] *Let $\mathcal{C}$ be a triangulated category and regard it as a natural extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Assume that a full subcategory $\mathcal{N}$ of $\mathcal{C}$ is closed under direct summands and isomorphisms.*

(0) *If $\mathcal{N}$ is extension-closed in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, then $\mathcal{N}$ naturally defines a relative extriangulated structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$.*

(1) *If $\mathcal{N}$ is extension-closed in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, then $\mathcal{N}$ is thick with respect to the relative structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$. Moreover, we have an extriangulated localization $(Q, \mu) : (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}) \to (\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$.*

(2) *Suppose that $\mathcal{N}$ is extension-closed. Then, $\mathcal{N}$ is thick in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ if and only if $\mathcal{N}$ is biresolving with respect to the relative structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$ if and only if the resulting category $(\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ is triangulated. In this case, the localization (Q, μ) is nothing but the Verdier quotient.*

(3) *Suppose that $\mathcal{N}$ is extension-closed and functorially finite. Then, $\mathcal{N}$ satisfies* $\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$ *in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ if and only if $\mathcal{N}$ is Serre with respect to the relative structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$ if and only if the resulting category $\mathcal{C}/\mathcal{N}$ is abelian. Furthermore, the functor $Q : (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to \mathcal{C}/\mathcal{N}$ from the original triangulated category is cohomological.*

$\mathcal{N}$	*extension-closed*	*thick*	$\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$	*in* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$
	thick	*biresolving*	*Serre*	*in* $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$
$\mathcal{C}/\mathcal{N}$	*extriangulated*	*triangulated*	*abelian*	

The assertion (1) in the above theorem shows that the Verdier quotient is a typical example of the exact sequence (1.1). The assertion (2) contains some types of cohomological functors such as the heart of a t-structures, see Examples 10, 11 and 12.

Notation and convention. All categories and functors in this article are always assumed to be additive. All subcategory $\mathcal{U} \subseteq \mathcal{C}$ is always assumed to be full, additive and closed under isomorphisms. For $X \in \mathcal{C}$, if $\mathcal{C}(U, X) = 0$ for any $U \in \mathcal{U}$, we write abbreviately $\mathcal{C}(\mathcal{U}, X) = 0$. Similar notations will be used in obvious meanings.

2. Localization with respect to extension-closed subcategories

In the reset, we fix a triangulated category $\mathcal{C}$ with a suspension [1] and an extension-closed subcategory $\mathcal{N}$ of $\mathcal{C}$ and regard $\mathcal{C}$ as a natural extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Note that $\mathbb{E}(C, A) = \mathcal{C}(C, A[1])$ for any objects $A, C \in \mathcal{C}$. First, we show that $\mathcal{N}$ naturally determines an extriangulated structure on $\mathcal{C}$ relative to the triangulated structure.

Proposition 2. *For any objects $A, C \in \mathcal{C}$, we define subsets of $\mathbb{E}(C, A)$ as follows.*

(1) *A subset $\mathbb{E}^L_{\mathcal{N}}(C, A)$ is defined to be the set of morphisms $h : C \to A[1]$ satisfying the condition that, for any morphism $N \xrightarrow{x} C$ with $N \in \mathcal{N}$, $h \circ x$ factors through an object in $\mathcal{N}[1]$.*

(2) *A subset $\mathbb{E}^R_{\mathcal{N}}(C, A)$ is defined to be the set of morphisms $h : C \to A[1]$ satisfying the condition that, for any morphism $A \xrightarrow{y} N$ with $N \in \mathcal{N}$, $y \circ h[-1]$ factors through an object in $\mathcal{N}[-1]$.*

Then, both $\mathbb{E}^L_{\mathcal{N}}$ and $\mathbb{E}^R_{\mathcal{N}}$ give rise to closed subfunctors of $\mathbb{E}$ in the sense of [8, Prop. 3.16]. *In particular, putting $\mathbb{E}_{\mathcal{N}} := \mathbb{E}^L_{\mathcal{N}} \cap \mathbb{E}^R_{\mathcal{N}}$, we have three extriangulated structures*

$$(\mathcal{C}, \mathbb{E}^L_{\mathcal{N}}, \mathfrak{s}^L_{\mathcal{N}}), \quad (\mathcal{C}, \mathbb{E}^R_{\mathcal{N}}, \mathfrak{s}^R_{\mathcal{N}}), \quad (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$$

which are relative to $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Here $\mathfrak{s}_{\mathcal{N}}$ is a restriction of $\mathfrak{s}$ to $\mathbb{E}_{\mathcal{N}}$ and other symbols are used in similar meanings.

To understand the above relative extriangulated structures, we observe the following two extremal cases.

Example 3. (1) Suppose that the subcategory $\mathcal{N}$ is *thick* in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, namely it is closed under taking cones and cocones. Then, since $\mathcal{N} = \mathcal{N}[1] = \mathcal{N}[-1]$, we have equalities $\mathbb{E}^L_{\mathcal{N}} = \mathbb{E}^R_{\mathcal{N}} = \mathbb{E}$. In particular, the relative structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$ coincides with the original triangulated structure.

(2) Suppose that the subcategory $\mathcal{N}$ is *rigid*, namely $\mathbb{E}(\mathcal{N}, \mathcal{N}) = 0$. Then, $\mathcal{N}$ forms a subcategory of projective-injective objects in $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$. Moreover, the structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$ is maximal with respect to the above property. In this case, due to [10, Prop. 3.30], the ideal quotient $\overline{\mathcal{C}} := \mathcal{C}/[\mathcal{N}]$ admits a natural extriangulated structure $(\overline{\mathcal{C}}, \overline{\mathbb{E}}_{\mathcal{N}}, \overline{\mathfrak{s}}_{\mathcal{N}})$.

Recall that a subcategory $\mathcal{N}$ of an arbitrary extriangulated category is said to be *thick* if it satisfies the 2-out-of-3 property for $\mathfrak{s}$-conflations, namely, for any $\mathfrak{s}$-conflation $A \longrightarrow B \longrightarrow C$, if two of $\{A, B, C\}$ belong to $\mathcal{N}$, so does the third[1]. It is easily checked that any extension-closed subcategory $\mathcal{N}$ of $\mathcal{C}$ becomes a thick subcategory of $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$. Keeping in mind the case of the Verdier localization, we define the class $\mathcal{S}_{\mathcal{N}}$ of morphisms which one would like to consider to be isomorphisms in the quotient category $\mathcal{C}/\mathcal{N}$.

Definition 4. For a thick subcategory $\mathcal{N}$ of an arbitrary extriangulated category, we associate the following classes of morphisms.

(1) $\mathcal{L} = \{f \in \operatorname{Mor} \mathcal{C} \mid f \text{ is an } \mathfrak{s}\text{-inflation with } \mathsf{Cone}(f) \in \mathcal{N}\}$.
(2) $\mathcal{R} = \{g \in \operatorname{Mor} \mathcal{C} \mid g \text{ is an } \mathfrak{s}\text{-deflation with } \mathsf{CoCone}(g) \in \mathcal{N}\}$.

Define $\mathcal{S}_{\mathcal{N}}$ to be the smallest subclass closed by compositions containing both $\mathcal{L}$ and $\mathcal{R}$.

For the pair of triangulated category $\mathcal{C}$ and an extension-closed subcategory $\mathcal{N}$, the above class $\mathcal{S}_{\mathcal{N}}$ possesses nice properties.

Lemma 5. *We consider the class $\overline{\mathcal{S}_{\mathcal{N}}}$ of morphisms $\overline{s}$ with $s \in \mathcal{S}_{\mathcal{N}}$.*

(1) *Let us denote by $\overline{\mathcal{S}_{\mathcal{N}}}^{*}$ the closure of $\overline{\mathcal{S}_{\mathcal{N}}}$ with respect to compositions with isomorphisms in $\overline{\mathcal{C}}$. Then, we have $\overline{\mathcal{S}_{\mathcal{N}}} = \overline{\mathcal{S}_{\mathcal{N}}}^{*}$.*
(2) *The class $\mathcal{S}_{\mathcal{N}}$ forms a* multiplicative system *in the ideal quotient $\overline{\mathcal{C}}$. In particular, we have the additive localization $Q : \mathcal{C} \to \mathcal{C}/\mathcal{N} := \mathcal{C}[\mathcal{S}_{\mathcal{N}}^{-1}]$ as follows:*

(2.1)
$$\begin{tikzcd} \mathcal{C} \arrow[rr, "Q"] \arrow[dr, "\text{ideal quot.}"'] & & \mathcal{C}/\mathcal{N} \\ & \overline{\mathcal{C}} \arrow[ur, "\text{Localization}"'] & \end{tikzcd}$$

[1]Note that this definition is a generalization of thick subcategories of triangulated categories.

(3) *The class $\overline{\mathcal{S}_{\mathcal{N}}}$ is saturated in the sense that, for any morphism $f \in \operatorname{Mor}\mathcal{C}$, if $Q(f)$ is an isomorphism, then $f \in \overline{\mathcal{S}_{\mathcal{N}}}$.*

Theorem 1 (2) says that the multiplicative system $\overline{\mathcal{S}_{\mathcal{N}}}$ satisfies the needed *compatibility with extriangulation* (see the conditions (MR1), …, (MR4) in [11, Thm. 3.5]). In particular, the relative structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$ determines a natural extriangulated structures $(\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ on the localization $\mathcal{C}/\mathcal{N}$ which makes the natural quotient functor $Q : \mathcal{C} \to \mathcal{C}/\mathcal{N}$ exact. The construction so far is depicted below.

$$
\begin{tikzcd}
\underset{\text{extension-closed sub.}}{(\mathcal{N}, \mathbb{E}|_{\mathcal{N}}, \mathfrak{s}|_{\mathcal{N}})} \arrow[r, "\mathsf{inc}"] & \underset{\text{triangulated cat.}}{(\mathcal{C}, \mathbb{E}, \mathfrak{s})} & \\
\underset{\text{thick sub.}}{(\mathcal{N}, \mathbb{E}_{\mathcal{N}}|_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}|_{\mathcal{N}})} \arrow[u, "\mathsf{id}"] \arrow[r, "\mathsf{inc}"] & \underset{\text{extriangulated cat.}}{(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})} \arrow[u, "\mathsf{id}"] \arrow[r, "Q"] & \underset{\text{extriangulated cat.}}{(\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})}
\end{tikzcd}
$$

Note that the all appearing functors are *exact* in the sense in [11, Def. 2.11].

We push further an observation on what the above diagram means in Example 3.

Example 6. (1) Suppose that the subcategory $\mathcal{N}$ is thick in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Then, we get $\mathbb{E} = \mathbb{E}_{\mathcal{N}}$ and the quotient functor Q is the usual Verdier quotient.

(2) Suppose that the subcategory $\mathcal{N}$ is *rigid*, namely $\mathbb{E}(\mathcal{N}, \mathcal{N}) = 0$. Then, $\overline{\mathcal{S}_{\mathcal{N}}}$ becomes the set of isomorphisms and the quotient functor Q is nothing other than the ideal quotient $\mathcal{C} \to \mathcal{C}/\mathcal{N} = \overline{\mathcal{C}}$. The extriangulated structure $(\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ coincides with the natural one in $\overline{\mathcal{C}}$.

3. The triangulated case

As we have already seen in Example 6, if a given subcategory $\mathcal{N} \subseteq \mathcal{C}$ is thick, our extriangulated category $\mathcal{C}/\mathcal{N}$ corresponds to a triangulated category. Conversely, if the quotient $\mathcal{C}/\mathcal{N}$ is triangulated, then $\mathcal{N}$ must be thick. To sharpen this assertion, we recall that a thick subcategory $\mathcal{N}$ is said to be *biresolving* if, for any object $C \in \mathcal{C}$, there exist an $\mathfrak{s}$-inflation $C \to N$ and an $\mathfrak{s}$-deflation $N' \to C$ with $N, N' \in \mathcal{N}$

Corollary 7. *We consider the extriangulated category $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$ and the localization $Q : (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}) \to (\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ with respect to the subcategory $\mathcal{N}$. Then the following three conditions are equivalent.*

(i) *The extriangulated category $(\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ corresponds to a triangulated category.*
(ii) *$\mathcal{N}$ is a thick subcategory of the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$.*
(iii) *$\mathcal{N}$ is a biresolving subcategory of the extriangulated category $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$.*

Under the above equivalent conditions, the localization $Q : (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}) \to (\widetilde{\mathcal{C}}_{\mathcal{N}}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ coincides with the usual Verdier quotient.

4. The exact case

It is natural to ask when the extriangulated category $\mathcal{C}/\mathcal{N}$ corresponds to an exact category. We denote by $\mathsf{Cone}(\mathcal{N}, \mathcal{N})$ the subcategory of $\mathcal{C}$ consisting of objects X appearing in a triangle $N' \longrightarrow N \longrightarrow X \longrightarrow N'[1]$ with $N, N' \in \mathcal{N}$. The following is an exact version of Corollary 7.

Corollary 8. *Let us consider the following conditions.*

(i) *The extriangulated category $(\widetilde{\mathcal{C}}_{\mathcal{N}}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ corresponds to an exact category.*
(ii) *$\mathcal{N}$ satisfies the condition $\mathsf{Cone}(\mathcal{N},\mathcal{N}) = \mathcal{C}$ in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$.*
(iii) *$\mathcal{N}$ is a Serre subcategory of the extriangulated category $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$.*

The condition (ii) *always implies* (i) *and* (iii). *Suppose that $\mathcal{N}$ is functorially finite in $\mathcal{C}$. Then, the all conditions are equivalent.*

We do not know whether the functorial finiteness on $\mathcal{N}$ are really needed for the above corollary.

The following shows that $\mathcal{C}/\mathcal{N}$ is actually an abelian category under the assumption $\mathsf{Cone}(\mathcal{N},\mathcal{N}) = \mathcal{C}$ and provide a new construction of cohomological functors.

Corollary 9. *Assume that $\mathsf{Cone}(\mathcal{N},\mathcal{N}) = \mathcal{C}$ holds in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Then, the following assertions hold.*

(1) *The resulting extriangulated category $(\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ corresponds to an abelian exact category.*
(2) *The exact functor $Q : (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}) \to (\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ induces a cohomological functor $Q : (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to \mathcal{C}/\mathcal{N}$ from the original triangulated category.*
(3) *The exact functor Q induces a right exact functor $Q : (\mathcal{C}, \mathbb{E}^R_{\mathcal{N}}, \mathfrak{s}^R_{\mathcal{N}}) \to \mathcal{C}/\mathcal{N}$ and a left exact functor $Q : (\mathcal{C}, \mathbb{E}^L_{\mathcal{N}}, \mathfrak{s}^L_{\mathcal{N}}) \to \mathcal{C}/\mathcal{N}$ in the sense of* [12, Def. 2.7].

As mentioned so far, we have half/left/right exact functors $Q : \mathcal{C} \to \mathcal{C}/\mathcal{N}$ from an extension closed subcategory $\mathcal{N}$ with $\mathsf{Cone}(\mathcal{N},\mathcal{N}) = \mathcal{C}$ as depicted in the following commutative diagram.

$$
\begin{tikzcd}
 & (\mathcal{C}, \mathbb{E}^R_{\mathcal{N}}, \mathfrak{s}^R_{\mathcal{N}}) \arrow[ddr] \arrow[drr, bend left, "\text{right exact}"] & & \\
(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}) \arrow[ur] \arrow[rrr, "\text{exact}"] \arrow[ddr] & & & \widetilde{\mathcal{C}}_{\mathcal{N}} \\
 & & (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \arrow[ur, "\text{cohomological}"] & \\
 & (\mathcal{C}, \mathbb{E}^L_{\mathcal{N}}, \mathfrak{s}^L_{\mathcal{N}}) \arrow[ur] \arrow[uurr, bend right, "\text{left exact}"'] & &
\end{tikzcd}
$$

Unlike the triangulated case, the above contains many important examples.

Example 10. [9] Let $\mathcal{C}$ be a triangulated category and assume that $\mathcal{U}$ is a 2-cluster tilting subcategory of $\mathcal{C}$, equivalently, $(\mathcal{U}, \mathcal{U})$ forms a cotorsion pair. Then, the ideal quotient $\mathcal{C}/[\mathcal{U}]$ is abelian and the natural functor $\pi : \mathcal{C} \to \mathcal{C}/[\mathcal{U}]$ is cohomological.

Sketch. Due to [1, Thm. 5.7], the pair $(\mathcal{U}, \mathcal{U})$ forms a cotorsion pair and we get its abelian heart $\mathcal{C}/[\mathcal{U}]$. We put $\mathcal{N} := \mathcal{U}$ and consider the relative structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$. Since $\mathcal{N}$ is rigid and $\mathsf{Cone}(\mathcal{N},\mathcal{N}) = \mathcal{C}$, Example 6 guarantees that our quotient functor $Q : \mathcal{C} \to \mathcal{C}/\mathcal{N}$ is nothing but the ideal quotient $\mathcal{C} \to \mathcal{C}/[\mathcal{N}]$. Corollary 9(2) shows Q is cohomological. $\square$

Example 11. [2] Let $\mathcal{C}$ be a triangulated category and $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ be a t-structure of $\mathcal{C}$. Then, the subcategory $\mathcal{H} := \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$ is abelian and there exists a natural cohomological functor $H : \mathcal{C} \to \mathcal{H}$.

Sketch. Due to [1, Thm. 5.7], the pair $(\mathcal{U}, \mathcal{V}) := (\mathcal{C}^{\leq -1}, \mathcal{C}^{\geq 1})$ forms a cotorsion pair and we get its heart $\mathcal{H}$. We put $\mathcal{N} := \mathsf{add}(\mathcal{U} * \mathcal{V})$ and consider the relative structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$. Then, since $\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$ holds, by Corollary 9, we have the cohomolgical functor $Q : \mathcal{C} \to \mathcal{C}/\mathcal{N}$. By the universality, we can easily check an equivalence $\mathcal{C}/\mathcal{N} \simeq \mathcal{H}$. □

Note that the general heart construction due to Abe-Nakaoka unifies the heart of a t-structure and Koenig-Zhu's abelian quotient $\mathcal{C}/[\mathcal{N}]$ as mentioned above. Abe-Nakaoka's construction can be still understood through Corollary 9. However, we skip the details. The following example can not be explained by Abe-Nakaoka's construction.

Example 12. [3, 4] Let $\mathcal{C}$ be a triangulated category and $\mathcal{U}$ a contravariantly finite rigid subcategory of $\mathcal{C}$. Then, we have a cohomological functor $H := \mathcal{C}(\mathcal{U}, -) : \mathcal{C} \to \mathsf{mod}\,\mathcal{U}$ which is factored as follows:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "H"] \arrow[dr, "\pi"'] & & \mathsf{mod}\,\mathcal{U} \\ & \mathcal{C}/[\mathcal{U}^{\perp}] \arrow[ur, "\mathsf{Loc}"'] & \end{tikzcd} \tag{4.1}$$

where $\mathcal{U}^{\perp}$ denotes the subcategory of objects X in $\mathcal{C}$ with $\mathcal{C}(\mathcal{U}, X) = 0$ and Loc is a Gabriel-Zisman localization which admits left and right fractions.

Sketch. We clarify how the diagram (4.1) relates to our localization. Firstly, we put $\mathcal{N} := \mathcal{U}^{\perp}$ and note that $\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$ holds. Thus, the localization $Q : \mathcal{C} \to \mathcal{C}/\mathcal{N}$ is, by definition, factored as the ideal quotient $\pi : \mathcal{C} \to \overline{\mathcal{C}}$ followed by the localization of $\overline{\mathcal{C}}$ with respect to the multiplicative system $\overline{\mathcal{S}_{\mathcal{N}}}$ which is same as Loc in (4.1). As a bit more advantage of our results, Corollary 9 explains how the abelian exact structure on $\mathsf{mod}\,\mathcal{U} \simeq \mathcal{C}/\mathcal{N}$ inherits from the relative extriangulated structure on the triangulated category $\mathcal{C}$. Thus, their diagram (4.1) is nothing but our construction (2.1) of the quotient functor Q. □

References

[1] N. Abe, H. Nakaoka, *General heart construction on a triangulated category* (II)*: Associated homological functor*, Appl. Categ. Structures **20** (2012), no. 2, 161–174.

[2] A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux Pervers (Perverse sheaves)*, Analysis and Topology on Singular Spaces, I, Luminy, 1981, Asterisque **100** (1982) 5–171 (in French).

[3] A. Beligiannis, *Rigid objects, triangulated subfactors and abelian localizations*, Math. Z. **274** (2013), no. 3-4, 841–883.

[4] A. Buan, R. Marsh, *From triangulated categories to module categories via localization* II*: calculus of fractions*, J. Lond. Math. Soc. (2) **86** (2012), no. 1, 152–170.

[5] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tohoku Math. J. (2) **9** (1957), 119–221.

[6] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, London Mathematical Society Lecture Note Series, 119. Cambridge University Press, Cambridge, 1988.

[7] R. Hartshorne, *Algebraic geometry*, Springer, 1977.

[8] M. Herschend, Y. Liu, H. Nakaoka, *n-exangulated categories* (I)*: Definitions and fundamental properties*, J. Algebra **570** (2021), 531—586.

[9] S. Koenig, B. Zhu, *From triangulated categories to abelian categories: cluster tilting in a general framework*, Math. Z. **258** (2008), no. 1, 143–160.

[10] H. Nakaoka, Y. Palu, *Extriangulated categories, Hovey twin cotorsion pairs and model structures*, Cah. Topol. Géom. Différ. Catég. **60** (2019), no. 2, 117–193.

[11] H. Nakaoka, Y. Ogawa, A. Sakai, *Localization of extriangulated categories*, J. Algebra **611** (2022), 341–398.

[12] ———, *Auslander's defects over extriangulated categories: an application for the general heart construction*, J. Math. Soc. Japan **73** (2021), no. 4, 1063–1089.

[13] Y. Ogawa, *Localization of triangulated categories with respect to extension-closed subcategories*, arXiv:2205.12116v2.

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A NEW FRAMEWORK OF PARTIALLY ADDITIVE ALGEBRAIC GEOMETRY

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ABSTRACT. In this article, we develop an elementary theory of partially additive rings as a foundation of $\mathbb{F}_1$-geometry. Our approach is so concrete that an analog of classical algebraic geometry is established very straightforwardly.

As applications, we construct (1) a kind of affine group scheme $\mathbb{GL}_n$ whose value at a commutative ring R is the group of $n \times n$ invertible matrices over R and at $\mathbb{F}_1$ is the n-th symmetric group, and (2) a projective space $\mathbb{P}^n$ as a kind of scheme and count the number of points of $\mathbb{P}^n(\mathbb{F}_q)$ for $q = 1$ or $q = p^n$ a power of a rational prime.

Key Words: Partially additive ring, Field with one element, Affine group scheme.

2020 *Mathematics Subject Classification*: Primary 14A23; Secondary 14L17.

1. INTRODUCTION

In his 1957 paper [5], J.Tits observed that the correspondence of geometries over a field k and the Chevalley groups over k developed in that paper specializes, when k is the hypothetical "field of characteristic one", to the correspondence of finite complexes and the Weyl groups of those Chevalley groups. In the early 1990s, Manin, based on the 'beautiful ideas of Deninger and Kurokawa', proposed to use this hypothetical field of characteristic one to solve the Riemann hypothesis. Since then, there have been many attempts to establish a foundation for these ideas, but it seems that the project has not been settled down yet.

In this paper, we develop an elementary theory of partially additive rings and based on it, we also develop a kind of scheme theory. As such, Deitmar's $\mathbb{F}_1$-schemes [2] and Lorscheid's blue schemes [3] are relevant to our theory. Deitmar's theory of $\mathbb{F}_1$-schemes is a direct analog of the classical scheme theory, where a commutative (multiplicative) monoid is used in place of a commutative ring. Lorscheid's theory of blue schemes is based on the theory of blueprints. A blueprint is a generalization of a commutative ring with identity in which addition is replaced by a congruence on a semiring $\mathbb{N}[A]$, the monoid-semiring of a commutative monoid A. A Partially additive ring, defined in this paper (and in [4]), is a special case of a blueprint developed in [3]. It is a part of blueprint which is the direct partially-additive analog of the non-additive setting of Deitmar and it can be thought of as an interpolation between commutative rings and commutative monoids. It is so concrete that an analog of classical algebraic geometry is established very straightforwardly. As applications, we construct a kind of affine group scheme $\mathbb{GL}_n$ whose value at a commutative ring R is the group of $n \times n$ invertible matrices over R and at $\mathbb{F}_1$ is the n-th symmetric group, and we construct a projective space $\mathbb{P}^n$ as a kind

The detailed version of this paper has been submitted to mathematics arXiv [4].

of scheme and count the number of points of $\mathbb{P}^n(\mathbb{F}_q)$ for $q = 1$ or $q = p^n$ a power of a rational prime.

2. Partially additive algebra

In this section we summarize definitions of the notions in partially additive algebra.

The definition of a partial monoid given below is due to G.Segal.

Definition 1. A **partial monoid** is a set A, a distinguished element $0 \in A$, a subset A_2 of $A \times A$ and a map $+\colon A_2 \to A$ such that

1. $A \times \{0\} \cup \{0\} \subseteq A_2$ and $a + 0 = a = 0 + a$, for any $a \in A$.
2. $(a, b) \in A_2$ if $(b, a) \in A_2$ and $a + b = b + a$, for any $a, b \in A$.
3. $(a, b) \in A_2$ and $(a+b, c) \in A_2$ if and only if $(b, c) \in A_2$ and $(a, b+c) \in A_2$, for any $a, b, c \in A$.

For example, any based set is considered as a partial monoid by giving it a trivial structure — only the base point 0 can be added to other elements. For another example, a commutative monoid (thus a commutative group) is a partial monoid where A_2 is taken to be the whole set $A \times A$.

Definition 2. Let A, B be partial monoids. A map $f\colon A \to B$ is a **homomorphism** of partial monoids if

1. $f(0) = 0$ and
2. for all $a, b \in A$, $(a, b) \in A_2$ implies $(f(a), f(b)) \in B_2$ and $f(a + b) = f(a) + f(b)$.

Definition 3. A **partial ring** is a partial monoid with a bilinear, associative and commutative product $\cdot\colon A \times A \to A$ and a multiplicative identity $1 \in A$. More explicitly, a partial ring is a partial monoid with a product $\cdot\colon A \times A \to A$ such that, for all $a, b, c \in A$,

1. $0 \cdot a = 0$,
2. if $(a, b) \in A_2$ then $(a \cdot c, b \cdot c) \in A_2$ and $(a + b) \cdot c = a \cdot b + a \cdot c$,
3. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
4. $a \cdot b = b \cdot a$, and
5. $1 \cdot a = a$.

If every element except for 0 of a partial ring is invertible, we call it a **partial field**.

For example, any commutative monoid with an absorbing element 0 is a partial ring in which only 0 can be added to other elements, and any commutative semiring (thus a commutative ring) with identity is a partial ring. A commutative group with an absorbing element 0 adjoined and a field are examples of a partial field.

Definition 4. Let A and B be partial rings. A **homomorphism** of $A \to B$ is a homomorphism of partial monoids $f\colon A \to B$ which is compatible with the multiplications in A and B.

For example, if A and B are commutative monoids with an absorbing element, considered as partial rings, then a homomorphism $A \to B$ as partial rings is nothing but a homomorphism $A \to B$ as commutative monoids. For another example, if A and B are commutative rings with identity considered as partial rings, then a homomorphism

$A \to B$ as partial rings is nothing but a homomorphism $A \to B$ as commutative rings. Thus we have full embeddings of categories $\mathcal{CM}on_0 \to \mathcal{PR}ing$ and $\mathcal{CR}ing \to \mathcal{PR}ing$, where $\mathcal{CM}on_0, \mathcal{PR}ing$ and $\mathcal{CR}ing$ denote the category of commutative monoids with absorbing element, of partial rings and of commutative rings with identity. In the rest of this paper, a commutative monoid with absorbing element and a commutative ring with identity are considered as a partial ring via this embedding.

In the rest of this paper, A is a partial ring.

Definition 5. An A**-module** is a partial monoid M with an action of A which is bilinear, associative and unital. More explicitly, an A-module is a partial monoid M and a map $\cdot\colon A \times M \to M$ such that

(1) for any $a \in A$, the map $M \to M$ given by $m \mapsto a \cdot m$ is a homomorphism of partial monoids,
(2) for any $m \in M$, the map $A \to M$ given by $a \mapsto a \cdot m$ is a homomorphism of partial monoids,
(3) $(a \cdot b) \cdot m = a \cdot (b \cdot m)$ for any $a, b \in A$ and $m \in M$,
(4) $1 \cdot m = m$ for any $m \in M$.

For example, a based set M with an action of a commutative monoid A on it is an A-module. For another example, an A-module in the usual sense, where A is a commutative ring, is an A-module in our sense.

Definition 6. Let M and N be A-modules. A **homomorphism** of A-modules is a homomorphism of partial monoids $f\colon M \to N$ which is compatible with the actions of A on M and N.

Let M be an A-module and $S \subseteq A$ be a multiplicative subset. As usual, let $\frac{m}{s}$ denote the equivalence class of $(s, m) \in S \times M$ under the equivalence relation $(s, m) \sim (t, n) \iff \exists u \in S$ s.t. $usn = utm$. If we put

$$S^{-1}M = \left\{ \frac{m}{s} \,\middle|\, m \in M, s \in S \right\}, \text{ and}$$
$$(S^{-1}M)_2 = \left\{ \left(\frac{m}{s}, \frac{n}{s} \right) \,\middle|\, (m, n) \in M_2 \right\},$$

then $S^{-1}M$ is an A-module in a natural manner. (The definition of $(S^{-1}M)_2$ may seem too easy at first look, but it can do, since we can make a common denominator.)

Definition 7. An A-submodule of A is called an **ideal** of A. An ideal I is a **prime ideal** if $A \setminus I$ is multiplicatively closed.

3. Examples

In this section, partial rings and A-modules of major interests are listed.

Partial rings.

- Partial rings of order 2
  If $A = \{0, 1\}$ is a partial ring, operations other than $1 + 1$ are determined by the axioms. There are three possibilities of $1+1$, namely, $1+1 = 0, 1$ and 'undefined'. If $1 + 1 = 0, A = \mathbb{F}_2$, the field of two elements. If $1 + 1 = 1, A = \mathbb{B}$, a "Boolean"

semiring. If $1+1$ is undefined, we denote this partial ring by $\mathbb{F}_1$. All these three are partial fields.

- Let $\langle x_1, \dots, x_n \rangle$ be the commutative monoid generated freely by n indeterminates $x_1, \dots, x_n$ with an absorbing element 0 adjoined.
- Let S be a set of elements of $\mathbb{N}[x_1, \dots, x_n]$. We denote by $\mathbb{F}_1\langle x_1, \dots, x_n | S\rangle$ the smallest partial subring of $\mathbb{N}[x_1, \dots, x_n]$ which contains $\langle x_1, \dots, x_n\rangle$ and any subsum of an element of S can be calculated in it.
- As a special case of the previous example, we consider the case where S consists of a single element $x_1 + \cdots + x_n$. In this case, we can show that

$$\mathbb{F}_1\langle x_1, \dots, x_n | x_1 + \cdots + x_n\rangle = \{ \text{ subsum of } (x_1 + \cdots + x_n)^r \mid r \in \mathbb{N}\}$$

and any two elements $\alpha, \beta \in \mathbb{F}_1\langle x_1, \dots, x_n | x_1 + \cdots + x_n\rangle$ are summable in this partial ring if the sum $\alpha+\beta$ taken in $\mathbb{N}[x_1, \dots, x_n]$ is contained in $\mathbb{F}_1\langle x_1, \dots, x_n | x_1 + \cdots + x_n\rangle$. This partial ring represents the summable n-tupples, as there exists an isomorphism of A-modules

$$\mathrm{Hom}_{\mathcal{PR}ing}(F_1\langle x_1, \dots, x_n | x_1 + \cdots + x_n\rangle, A) \cong A_n$$

for any partial ring A.of partial rings.

<u>A-modules</u>.

- Direct product

$$A^n = \{(a_1, \dots, a_n) \mid a_1, \dots, a_n \in A\},$$
$$(A^n)_2 = \{(a_1, \dots, a_n; b_1, \dots, b_n) \mid (a_i, b_i) \in A_2, \forall i\}.$$

- Summable n-tuples

$$A_n = \{(a_1, \dots, a_n) \mid a_1 + \cdots + a_n \text{ can be calculated in } A\},$$
$$(A_n)_2 = \left\{ (a_1, \dots, a_n; b_1, \dots, b_n) \;\middle|\; \sum_{i=1}^{n} (a_i + b_i) \text{ can be calculated in } A \right\}.$$

- "Hyper summable" n-tuples

$$A_{(m)} = \{(a_1, \dots, a_m) \in A^m \mid (c_1 a_1, \dots, c_m a_m) \in A_m \, \forall c_i \in A\},$$
$$(A_{(m)})_2 = \{(a_1, \dots, a_m; b_1, \dots, b_n) \mid (a_1 + b_1, \dots, a_n + b_n) \in A_{(m)}\}.$$

4. Congruences

In this section, we summarize some facts about congruences on a partial ring. Congruence is one of the main points where the analogy between partial algebras and classical commutative algebras does not go smoothly. This inconsistency is explained simply by stating the fact that congruences on a partial ring does not correspond bijectively with ideals in that partial ring. But since the congruence is one of the main tools to construct a new partial algebra from another, we need to establish the theory of congruences on a partial ring.

Recall from [1] that an **equivalence relation** on an object X of a category $\mathcal{C}$ with small limits is a subobject R of $X \times X$ such that the injection $\mathrm{Hom}_{\mathcal{C}}(C, R) \to \mathrm{Hom}_{\mathcal{C}}(C, X) \times \mathrm{Hom}_{\mathcal{C}}(C, X)$ induced by the monomorphism $R \to X \times X$ gives rise to an equivalence relation on the set $\mathrm{Hom}_{\mathcal{C}}(C, X)$ for all object C of $\mathcal{C}$. An equivalence relation R is called **effective** if it is a kernel pair of a morphism in $\mathcal{C}$. In this paper the word **congruence** will be used as a synonym of effective equivalence relation.

If $\mathcal{C} = \mathcal{CR}ing$, the following are true:

(1) Every equivalence relation is a congruence.
(2) A congruence R on a ring A gives rise to an ideal $I(R) = \{a - b \mid (a, b) \in C\}$ of A and, conversely, an ideal J of A gives rise to a congruence $C(J) = \{(a, b) \mid a - b \in J\}$ on A. This establishes a bijective correspondence between congruences on A and ideals of A.

On the other hand, if $\mathcal{C} = \mathcal{PR}ing$, the following are true:

(1) An equivalence relation R is a congruence if and only if $R_2 = (R \times R) \cap (A \times A)_2$.
(2) For a congruence R on a ring A, let $I(R)$ be the ideal of A defined by

$$I(R) = \{a \in A \mid (a, 0) \in R\}.$$

Conversely, for an ideal J of A, let $C(J)$ be the smallest congruence on A which contains $J \times J$. This establishes a two way correspondence

$$C\colon (\text{ideals of } A) \rightleftarrows (\text{congruences on } A) : I$$

We have $CI(R) \subseteq R$ and $J \subseteq IC(J)$ for any congruence R on A and for any ideal J of A. Thus we have a bijective correspondence between the congruences of the form $C(J)$ and the ideals of the form $I(R)$.

5. Partial schemes

Let X be the set of the prime ideals of A. For any $a \in A$, let $D(a)$ denote the set of prime ideals of A which does not contain a. Since $D(a) \cap D(b) = D(ab)$ for any $a, b \in A$, $D(a)$'s for all $a \in A$ constitute a base for a topology on X_A, with which we make X_A a topological space.

For any open set $U \subseteq X_A$, we put $S_U = \{s \in A | s \notin P, \forall P \in U\}$ and $\mathcal{O}'_A(U) = S_U^{-1}A$. Then $\mathcal{O}'_A(U)$ is a presheaf of partial rings on X_A. Let $\mathcal{O}_A$ denote the sheafification of $\mathcal{O}'_A$. We put $\mathrm{Spec}A = (X_A, \mathcal{O}_A)$.

Definition 8. An **affine partial scheme** is a partial-ringed space $(X, \mathcal{O}_X)$ which is isomorphic to $\mathrm{Spec}A$ for some partial ring A. A **partial scheme** is a partial-ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine partial scheme.

Proposition 9. *Let Spec* $A = (X, \mathcal{O}_X)$.

(1) X *is quasi-compact.*
(2) *We have a natural monomorphism* $A \to \mathcal{O}_X(X)$.
(3) *If* A *is a partial field,* $A \cong \mathcal{O}_X(X)$.

For a proof, see [4].

6. Projective space $\mathbb{P}^n$ as a partial scheme

Let B be a partial ring given by $B = \mathbb{F}_1\langle y_0, \dots, y_n \mid y_0 + \cdots + y_n\rangle$. Let A_i be the 0-th part of the localization of B by the multiplicative set $\{x_i^r \mid r \in \mathbb{N}\}$. It is readily seen that A_i consists of subsums of $\left(\frac{y_1}{y_i} + \cdots + \frac{y_n}{y_i}\right)^r_.$ If $A_{i,j}$ denotes the localization of A_i by the multiplicative set $\left\{\left(\frac{x_j}{x_i}\right)^r \mid r \in \mathbb{N}\right\}_,$ we have that $A_{i,j} = A_{j,i}$. This allows us to patch affine pieces $X_i = \mathrm{Spec} A_i$ together to get a partial scheme, which we denote by $\mathbb{P}^n_B$. If $\{i_1, \dots, i_k\} \subseteq \{0, 1, \dots, n\}$ is a k-element set, let $A_{i_1,\dots,i_k}$ denote the 0-th part of the localization of B by the multiplicative set generated by $x_{i_1}, \dots, x_{i_k}$. Then we have an isomorphism $X_{i_1} \cap \cdots \cap X_{i_k} \cong \mathrm{Spec} A_{i_1,\dots,i_k}$. Let F be a partial field. Assume that there exists a finite subset E of F such that for any $a_1, \dots, a_r \in E, (1, a_1, \dots, a_r) \in F_{r+1}$ and elements in $F \setminus E$ is not summable with 1. If the cardinal of E is $\kappa(F)$, we have that the number of F-valued points of $X_{i_1} \cap \cdots \cap X_{i_k}$, *i.e.* that of homomorphisms from $A_{i_1,\dots,i_k}$ to F is $(\kappa(F) - 1)^k \kappa(F)^{n-k}$. So we can calculate the number of F-valued points of $\mathbb{P}^n_B$ as

$$\begin{aligned} \#\mathbb{P}^n_B(F) &= \sum_{k=0}^{n} (-1)^k \binom{n+1}{k} (\kappa(F) - 1)^k \kappa(F)^{n-k} \\ &= \begin{cases} \kappa(F)^n + \cdots + \kappa(F) + 1 & \text{if } \kappa(F) \neq 1 \\ (n+1)\kappa(F)^n = n + 1 & \text{if } \kappa(F) = 1. \end{cases} \end{aligned}$$

Since $\kappa(\mathbb{F}_q) = q$ including the case $q = 1$, we have that

$$\#\mathbb{P}^n_B(\mathbb{F}_q) = \begin{cases} q^n + \cdots + q + 1 & \text{if } q = p^d \text{ is a power of a prime } p \\ n + 1 & \text{if } q = 1. \end{cases}$$

Since $n + 1$ is the number of vertices of an n-simplex, this result can be thought of as a supportive evidence for a part of the conjecture of Tits [5].

7. Linear algebra of A-modules

Let $\varphi \colon A^m \to A^n$ be a homomorphism of A-modules. If e_j $(1 \leq j \leq m)$ and f_i $(1 \leq i \leq n)$ denote the elementary vectors of M and N respectively, then φ determines an $n \times m$ matrix $\alpha = (a_{ij})$, where $\varphi(e_j) = \sum_{i=1}^n a_{ij} f_i$. For any $(c_1, \dots, c_m) \in A^m$, we have $(c_1 e_1, \dots, c_m e_m) \in (A^m)_m$. Since φ is a homomorphism, this implies that $(c_1\varphi(e_1), \dots, c_m\varphi(e_m)) \in (A^n)_m$. This implies that $(c_1 a_{i1}, \dots c_m a_{im}) \in A_2$ for all $i = 1, \dots, n$, so that for a matrix $\alpha = (a_{ij})$ determined by a homomorphism $A^m \to A^n$, we have $(a_{i1}, \dots, a_{im}) \in A_{(m)}$ for all $i = 1, \dots, n$. Conversely, a matrix α with this property determines a homomorphism $A^m \to A^n$ in an obvious way.

Proposition 10. *There exists a bijection between the set of homomorphisms $A^m \to A^n$ and the set of $n \times m$ matrices whose rows are in $A_{(m)}$. $A_{(m)}$ is naturally isomorphic to the A-module dual of A^m.*

Let $M_{n,m}(A)$ denote the set of $n \times m$ matrices whose rows are in $A_{(m)}$. Unfortunately, $M_{n,m}(A)$ is not a functor of A, since the correspondence $A \mapsto A_{(n)}$ is not functorial. We remedy this defect by considering A_n as a functorial approximation to $A_{(n)}$ and use A_n in place of $A_{(n)}$. A supporting evidence that we can use A_n as an approximation to

$A_{(n)}$ is that these two A-modules coincide for an important class of partial rings such as commutative monoids with absorbing element and commutative rings with identity. This observation leads to the following definition:

In this paper we say that a partial ring A **is good** if $A_n = A_{(n)}$.

Now, let $M'_{n,m}(A)$ denote the set of $n \times m$ matrices whose rows are in A_m. Since an element of $M'_{n,m}(A)$ is not "genuine" matrices, *i.e.* does not correspond to an A-module homomorphism, $M'_n(A) = M'_{n,n}(A)$ is only a non-commutative partial magma, while $M_n(A) = M_{n,n}(A)$ is a (genuine) monoid by the usual matrix product. Similarly, if we put $GL'_n(A) = GL_n(A) \cap M'_n(A)$, where $GL_n(A)$ is the group of invertible matrices in $M_n(A)$, then $GL'_n(A)$ is only a non-commutative partial group. A definition of partial group is given in [4].

8. Main result

Let $\mathcal{PG}rp$ and $\mathcal{G}rp$ denote the category of partial groups and groups, respectively. We will give a definition of a good partial ring in the talk. Commutative monoids with absorbing element 0 and commutative rings with identity are examples of good partial rings.

Theorem 11. *There exists a representable functor* $\mathbb{GL}'_n \colon \mathcal{PR}ing \to \mathcal{PG}rp$ *which enjoys the following properties:*

(1) *its restriction to the category of good partial rings factors as* $\iota \circ \mathbb{GL}_n$*, where* ι *is the canonical inclusion* $\mathcal{G}rp \to \mathcal{PG}rp$*.*
(2) *If* A *is a commutative rings with identity, then* $\mathbb{GL}'_n(A) = \mathbb{GL}_n(A)$ *is the group of* n*-th general linear group with entries in* A*, and*
(3) $\mathbb{GL}'_n(\mathbb{F}_1) = \mathbb{GL}_n(\mathbb{F}_1) = \mathfrak{S}_n$ *is the* n*-th symmetric group.*

Proof. Let $\mathbb{N}[x_{ij}, y_{ij}\,(1 \le i,j \le n)]$ be the semiring of polynomials of $2n^2$ indeterminates $x_{ij}, y_{ij}\,(1 \le i,j \le n)$. Consider $n \times n$ matrices $X = (x_{ij}), Y = (y_{ij}), Z = XY = (z_{ij})$ and $W = YX = (w_{ij})$. Let K be the subset of $\mathbb{N}[x_{ij}, y_{ij}\,(1 \le i,j \le n)]$ consisting of $4n$ elements

$$\begin{aligned} &x_i = x_{i1} + \cdots + x_{in}\,(1 \le i \le n),\\ &y_i = y_{i1} + \cdots + y_{in}\,(1 \le i \le n),\\ &z_i = z_{i1} + \cdots + z_{in}\,(1 \le i \le n) \text{ and}\\ &w_i = w_{i1} + \cdots + w_{in}\,(1 \le i \le n). \end{aligned}$$

We put $G' = \mathbb{F}_1\langle x_{ij}, y_{ij}\,(1 \le i,j \le n)|K\rangle$. Let Q be the smallest congruence on G' which contains $2n^2$ pairs (z_{ij}, δ_{ij}) and $(w_{ij}, \delta_{ij})\,(1 \le i,j \le n)$. Then we put $G = G'/Q$.

Next, let $N = \mathbb{N}[x_{ij}, y_{ij}, x'_{ij}, y'_{ij}\,(1 \le i,j \le n)]$ be the semiring of polynomials of $4n^2$ indeterminates $x_{ij}, y_{ij}, x'_{ij}, y'_{ij}\,(1 \le i,j \le n)$. Consider $n \times n$ matrices

$$\begin{aligned} &X = (x_{ij}), Y = (y_{ij}), Z = XY = (z_{ij}), W = YX = (w_{ij}),\\ &X' = (x'_{ij}), Y' = (y'_{ij}), Z' = X'Y' = (z'_{ij}), W' = Y'X' = (w'_{ij}),\\ &S = XX' = (s_{ij}), T = Y'Y = (t_{ij}), U = ST = (u_{ij}), V = TS = (v_{ij}). \end{aligned}$$

We put

$$L = \{x_i, y_i, z_i, w_i, x'_i, y'_i, z'_i, w'_i, s_i, t_i, u_i, v_i | 1 \le i \le n\},$$

where $*_i$ denotes the sum of i-th row of a matrix indicated by the capital of the same letter $*$. We put $H' = \mathbb{F}_1\langle x_{ij}, y_{ij}, x'_{ij}, y'_{ij}\,(1 \leq i, j \leq n)|L\rangle$. Let R be the smallest congruence on H' which contains $6n^2$ pairs (z_{ij}, δ_{ij}), (w_{ij}, δ_{ij}), (z'_{ij}, δ_{ij}), (w'_{ij}, δ_{ij}), (u_{ij}, δ_{ij}) and $(v_{ij}, \delta_{ij})\,(1 \leq i, j \leq n)$. Then we put $H = H'/R$. A partial cogroup structure on G is given by a series of partial ring homomorphisms, for the details of which we refer the reader to [4]. □

Remark 12. Above theorem suggests that analogies between the symmetric group and the general linear group can be unified to a single statement about a single object $\mathbb{GL}_n$. It also shows that if we have an intermediate good partial ring A between $\mathbb{F}_1$ and $\mathbb{F}_q$, where $q = p^d$ is a power of a prime p, there exists an intermidiate group $\mathbb{GL}_n(A)$. It is proved that commutative submonoids with absorbing element and subfields of $\mathbb{F}_q$ exhausts the intermediate partial rings between $\mathbb{F}_1$ and $\mathbb{F}_q$. Anyway, a more detailed investigation is needed in this direction.

References

[1] F. Borceux, *Handbook of categorical algebra. 2*, Encyclopedia of Mathematics and its Applications, 51, Cambridge University Press, Cambridge, (1994).

[2] A. Deitmar, Anton, *Schemes over* $\mathbb{F}_1$, Number fields and function fields—two parallel worlds, Progr. Math., 239 (2005), 87–100.

[3] O. Lorscheid, *The geometry of blueprints: Part I: Algebraic background and scheme theory*, Advances in Mathematics (2012), 1804–1846.

[4] S. Okuyama, *Partially additive rings and group schemes over* $\mathbb{F}_1$, arXiv e-prints (2022), arXiv:2206.06084.

[5] J. Tits, *Sur les analogues algébriques des groupes semi-simples complexes*, Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956 (1957), 261–289.

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HIGHER VERSIONS OF MORPHISMS REPRESENTED BY MONOMORPHISMS

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Abstract. In this article, we introduce and study a new class of morphisms which includes morphisms represented by monomorphisms in the sense of Auslander and Bridger. As an application, we give a common generalization of several results due to Auslander and Bridger that describe relationships between torsionfreeness and the grades of Ext modules.

Key Words: morphism represented by monomorphisms, n-torsionfree module, syzygy, (Auslander) transpose, grade.

2000 *Mathematics Subject Classification*: 13D02, 16E05, 16D90.

1. Introduction

Throughout this article, let R be a two-sided noetherian ring. We assume that all modules are finitely generated right ones. It is a natural and classical question to ask when a given homomorphism of R-modules is stably equivalent to another homomorphism satisfying certain good properties. A well-studied one is about stable equivalence to a monomorphism: A homomorphism $f : X \to Y$ of R-modules is said to be *represented by monomorphisms* if there is an R-homomorphism $t : X \to P$ with P projective such that $\binom{f}{t} : X \to Y \oplus P$ is a monomorphism. This notion has been introduced by Auslander and Bridger [1], and later studied in detail by Kato [3]. Among other things, Kato gave the following characterization; we denote by $\operatorname{Tr}(-)$ the (Auslander) transpose.

Theorem 1 (Kato). *Let $f : X \to Y$ be an R-homomorphism. Then the following are equivalent.*

(1) *The morphism f is represented by monomorphisms.*
(2) *The map $\operatorname{Ext}^1_{R^{\mathrm{op}}}(\operatorname{Tr} f, R) : \operatorname{Ext}^1_{R^{\mathrm{op}}}(\operatorname{Tr} X, R) \to \operatorname{Ext}^1_{R^{\mathrm{op}}}(\operatorname{Tr} Y, R)$ is injective.*

Motivated by the above theorem, we define a condition which we call (T_n) for each integer $n \geq 0$ so that (T_1) is equivalent to being represented by monomorphisms, and find out several properties.

The notion of n-torsionfree modules was also introduced by Auslander and Bridger [1]: An R-module M is called *n-torsionfree* if $\operatorname{Ext}^i_{R^{\mathrm{op}}}(\operatorname{Tr} M, R) = 0$ for all $1 \leq i \leq n$. Auslander and Bridger found various important properties related to n-torsionfree modules. For example, for an R-module M, Auslander and Bridger figured out the relationship between the grade of the Ext module $\operatorname{Ext}^i_R(M, R)$ and the torsionfreeness of the syzygy $\Omega^i M$. This result has been playing an important role in studies on n-torsionfree modules. As an application of a result stated in Section 2, we give a higher version of Auslander and

The detailed version [5] of this article has been submitted for publication elsewhere.

Bridger's theorem in Section 3. It gives a common generalization of [1, Propositions 2.26, 2.28 and Corollary 2.32].

2. The condition (T_n)

We begin with introducing the new condition (T_n) for R-homomorphisms. It is a natural extension of the condition (2) in Theorem 1.

Definition 2. Let $n \geq 1$ be an integer. We say that a homomorphism $f : X \to Y$ of R-modules satisfies (T_n) if the map $\mathrm{Ext}^i_{R^{\mathrm{op}}}(\mathrm{Tr}\, f, R)$ is bijective for all $1 \leq i \leq n-1$ and $\mathrm{Ext}^n_{R^{\mathrm{op}}}(\mathrm{Tr}\, f, R)$ is injective. In addition, we provide that every R-homomorphism satisfies (T_0).

In order to describe several properties related to the condition (T_n), we use the following terminology.

Definition 3. [3, Definition and Lemma 2.11] Let $f : X \to Y$ be a homomorphism of R-modules. Let $s : P \to Y$ be an epimorphism with P projective. The module $\underline{\mathrm{Ker}} f$ is defined as $\mathrm{Ker}\,((f, s) : X \oplus P \to Y)$. The module $\underline{\mathrm{Ker}} f$ is uniquely determined by f up to projective summands.

For an R-homomorphism f, we denote by $\mathrm{Cok}\, f$ the cokernel of f. The kernel and the cokernel of the homomorphim f are related to the module $\underline{\mathrm{Ker}} f$ as follows.

Lemma 4. [3, Lemma 2.17, Theorem 4.12] *Let $f : X \to Y$ be a homomorphism of R-modules. Let $t : Q \to Y$ be an epimorphism with Q projective. Then there exists an exact sequence*

$$0 \to \mathrm{Ker}\, f \to \underline{\mathrm{Ker}} f \to Q \to \mathrm{Cok}\, f \to 0.$$

Let M be an R-module. The *grade* of M, which is denoted by $\mathrm{grade}_R\, M$, is defined to be the infimum of integers i such that $\mathrm{Ext}^i_R(M, R) = 0$. The relationship between the grades of Ext modules and the torsionfreeness of modules has been actively studied; the works of Auslander and Bridger [1] and Auslander and Reiten [2] are among the most celebrated studies. The following theorem is the first main theorem of this article, which interprets the condition (T_n) in terms of grades and torsionfreeness.

Theorem 5. *Let $n \geq 1$ be an integer. Consider the following conditions for an R-homomorphism $f : X \to Y$.*

- (a_n) *The homomorphism f satisfies the condition (T_n).*
- (b_n) *The R-module $\underline{\mathrm{Ker}} f$ is n-torsionfree.*
- (c_n) *There is an inequality $\mathrm{grade}_{R^{\mathrm{op}}}\, \mathrm{Ker}\, \mathrm{Ext}^1_R(f, R) \geq n$.*

Then the following implications hold.

$$(a_n) \wedge (b_{n+1}) \Longrightarrow (c_n), \qquad (b_n) \wedge (c_n) \Longrightarrow (a_n), \qquad (a_n) \wedge (c_{n-1}) \Longrightarrow (b_n).$$

Let us consider an application of the above theorem. The following corollary is none other than [3, Theorem 4.2], which gives a simple characterization of the morphisms represented by monomorphisms when R is commutative and generically Gorenstein (e.g., when R is an integral domain). We can deduce it from Theorem 5.

Corollary 6 (Kato)**.** *Suppose that R is commutative and the total ring $\mathrm{Q}(R)$ of fractions of R is Gorenstein. Let $f : X \to Y$ be a homomorphism of R-modules. Then f is represented by monomorphisms if and only if $\operatorname{Ker} f$ is torsionless.*

Proof. Since $\mathrm{Q}(R)$ is Gorenstein, the torsionless property is closed under extensions; see [4, Theorem 2.3] for instance. Hence, by Lemma 4, $\operatorname{Ker} f$ is torsionless if and only if so is $\underline{\operatorname{Ker}} f$. Suppose that $\underline{\operatorname{Ker}} f$ is torsionless. By [1, Proposition 4.21], we have $\operatorname{grade} \operatorname{Ker} \operatorname{Ext}^1(f, R) \geq 1$. It follows from Theorem 5 that f is represented by monomorphisms. □

3. Grade inequalities of Ext modules

In this section, as an application of Theorem 5, we consider the grades of Ext modules. Let M be an R-module and $n \geq 1$ an integer. Auslander and Bridger [1] state and prove a criterion for $\Omega^i M$ to be i-torsionfree for $1 \leq i \leq n$. By using Theorem 5, we can recover [1, Proposition 2.26], which is the most fundamental theorem in studies on n-torsionfree modules.

Corollary 7 (Auslander–Bridger)**.** *Let $n \geq 1$ be an integer and M an R-module. The following are equivalent.*

(1) *The inequality* $\operatorname{grade}_{R^{\mathrm{op}}} \operatorname{Ext}^i_R(M, R) \geq i - 1$ *holds for all* $1 \leq i \leq n$.
(2) *The syzygy $\Omega^i M$ is i-torsionfree for all $1 \leq i \leq n$.*

Proof. We use induction on n. The assertion is trivial for $n = 1$. Let $n > 1$. Assume that (1) or (2) holds. Then $\Omega^i M$ is i-torsionfree for all $1 \leq i \leq n-1$ by the induction hypothesis. Let $f : P \to \Omega^{n-1} M$ be an epimorphism with P projective. Then f satisfies (T_n). It follows from Theorem 5 that $\operatorname{grade} \operatorname{Ker} \operatorname{Ext}^1(f, R) \geq n - 1$ if and only if $\underline{\operatorname{Ker}} f$ is n-torsionfree. As $\operatorname{Ker} \operatorname{Ext}^1(f, R) \cong \operatorname{Ext}^1(\Omega^{n-1} M, R) \cong \operatorname{Ext}^n(M, R)$ and $\underline{\operatorname{Ker}} f \cong \Omega^n M$, we have the desired result. □

The results [1, Proposition 2.26 and Corollary 2.32] describe the relationship between the grades of Ext modules and the torsionfreeness of syzygy modules, and the relationship of them with the natural map $\psi^i_M : \operatorname{Tr} \Omega^i \operatorname{Tr} \Omega^i M \to M$ being represented by monomorphisms. The main result of this section is the following theorem, which gives a common generalization of [1, Propositions 2.26, 2.28 and Corollary 2.32].

Theorem 8. *Let $n \geq 1$ and $j \geq 0$ be integers and M an R-module. The following are equivalent.*

(1) *The inequality* $\operatorname{grade}_{R^{\mathrm{op}}} \operatorname{Ext}^i_R(M, R) \geq i + j - 1$ *holds for all* $1 \leq i \leq n$.
(2) *The syzygy $\Omega^i M$ is i-torsionfree and the natural map $\psi^i_M : \operatorname{Tr} \Omega^i \operatorname{Tr} \Omega^i M \to M$ satisfies the condition (T_j) for all $1 \leq i \leq n$.*

References

[1] M. Auslander; M. Bridger, Stable module theory, Memoirs of the American Mathematical Society **94**, *American Mathematical Society, Providence, R.I.*, 1969.
[2] M. Auslander; I. Reiten, Syzygy modules for Noetherian rings, *J. Algebra* **183** (1996), no. 1, 167–185.
[3] K. Kato, Morphisms represented by monomorphisms, *J. Pure Appl. Algebra* **208** (2007), no. 1, 261–283.

[4] H. Matsui; R. Takahashi; Y. Tsuchiya, When are n-syzygy modules n-torsionfree?, *Arch. Math. (Basel)* **108** (2017), no. 4, 351–355.
[5] Y. Otake, Morphisms represented by monomorphisms with n-torsionfree cokernel, preprint (2022), arXiv:2203.04436.

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THE REDUCTION NUMBER OF STRETCHED IDEALS

KAZUHO OZEKI

Abstract. The homological property of the associated graded ring of an ideal is an important problem in commutative algebra. In this report we explore the structure of the associated graded ring of stretched $\mathfrak{m}$-primary ideals in the case where the reduction number attains almost minimal value in a Cohen-Macaulay local ring $(A, \mathfrak{m})$. As an application, we present complete descriptions of the associated graded ring of stretched $\mathfrak{m}$-primary ideals with small reduction number.

Key Words: commutative ring, stretched local ring, stretched ideal, Cohen-Macaulay local ring, associated graded ring, Hilbert function, Hilbert coefficient.

2020 *Mathematics Subject Classification*: Primary 13H10; Secondary 13D40.

1. Introduction

Throughout this report, let A be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. For simplicity, we may assume the residue class field $A/\mathfrak{m}$ is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and let

$$R = \mathrm{R}(I) := A[It] \quad \subseteq A[t] \text{ and } \quad R' = \mathrm{R}'(I) := A[It, t^{-1}] \quad \subseteq A[t, t^{-1}]$$

denote, respectively, the Rees algebra and the extended Rees algebra of I. Let

$$G = \mathrm{G}(I) := R'/t^{-1}R' \cong \bigoplus_{n \geq 0} I^n/I^{n+1}$$

denotes the associated graded ring of I. Let $\ell_A(N)$ denotes, for an A-module N, the length of N.

Let $Q = (a_1, a_2, \cdots, a_d) \subseteq I$ be a parameter ideal in A which forms a reduction of I. We set

$$n_I = n_Q(I) := \min\{n \geq 0 \mid I^{n+1} \subseteq Q\} \quad \text{and} \quad r_I = r_Q(I) := \min\{n \geq 0 \mid I^{n+1} = QI^n\},$$

respectively, denote the index of nilpotency and the reduction number of I with respect to Q. Then it is easy to see that the inequality $r_I \geq n_I$ always holds true.

The notion of *stretched* Cohen-Macaulay local rings was introduced by J. Sally. We say that the ring A is *stretched* if $\ell_A(Q + \mathfrak{m}^2/Q + \mathfrak{m}^3) = 1$ holds true, i.e. the ideal $(\mathfrak{m}/Q)^2$ is principal, for some parameter ideal Q in A which forms a reduction of $\mathfrak{m}$ ([9]). We note here that this condition depends on the choice of a reduction Q (see [8, Example 2.3]). She showed that the equality $r_Q(\mathfrak{m}) = n_Q(\mathfrak{m})$ holds true if and only if the associated graded ring $\mathrm{G}(\mathfrak{m})$ of $\mathfrak{m}$ is Cohen-Macaulay in the case where the base local ring A is stretched.

The detailed version of this paper has been submitted for publication elsewhere.

In 2001, Rossi and Valla [8] gave the notion of stretched $\mathfrak{m}$-primary ideals. We say that the $\mathfrak{m}$-primary ideal I is stretched if the following two conditions

(1) $Q \cap I^2 = QI$ and
(2) $\ell_A(Q + I^2/Q + I^3) = 1$

hold true for some parameter ideal Q in A which forms a reduction of I. We notice that the first condition is naturally satisfied if $I = \mathfrak{m}$ so that this extends the classical definition of stretched local rings given in [9].

Throughout this report, a stretched $\mathfrak{m}$-primary ideal I will come always equipped with a parameter ideal Q in A which forms a reduction of I such that $Q \cap I^2 = QI$ and $\ell_A(I^2 + Q/I^3 + Q) = 1$. Rossi and Valla [8] showed that the equality $r_I = n_I$ holds true if and only if the associated graded ring G is Cohen-Macaulay in the case where I is stretched. Thus stretched $\mathfrak{m}$-primary ideals whose reduction number attends to minimal value enjoy nice properties.

In this report we will also study the Hilbert coefficients of stretched $\mathfrak{m}$-primary ideals. As is well known, for a given $\mathfrak{m}$-primary ideal I, there exist integers $\{\mathrm{e}_k(I)\}_{0 \le k \le d}$ such that the equality

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d\mathrm{e}_d(I)$$

holds true for all integers $n \gg 0$. For each $0 \le k \le d$, $\mathrm{e}_k(I)$ is called the k-th *Hilbert coefficient* of I. We set the power series

$$HS_I(z) = \sum_{n=0}^{\infty} \ell_A(I^n/I^{n+1})z^n$$

and call it the Hilbert series of I. It is also well known that this series is rational and that there exists a polynomial $h_I(z)$ with integer coefficients such that $h_I(1) \neq 0$ and

$$HS_I(z) = \frac{h_I(z)}{(1-z)^d}.$$

The purpose of this report is to explore the structure of associated graded ring of stretched $\mathfrak{m}$-primary ideal I in the case where the reduction number attains almost minimal value.

The main result of this report is the following.

Theorem 1. *Suppose that I is stretched and assume that the equality $r_I = n_I + 1$ is satisfied. Then the following assertions hold true where $s = \min\{n \ge 1 \mid Q \cap I^{n+1} \neq QI^n\}$.*

(1) $\operatorname{depth} G = d - 1$,
(2) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \binom{n_I+1}{2} - s + 1$,
(3) $\mathrm{e}_k(I) = \binom{n_I+2}{k+1} - \binom{s}{k}$ *for all* $2 \le k \le d$,
(4) $\ell_A(A/I^{n+1}) = \sum_{k=0}^{d}(-1)^k \mathrm{e}_k(I)\binom{n+d-k}{d-k}$ *for all* $n \ge \max\{0, n_I - d + 1\}$, *and*
(5) *the Hilbert series $HS_I(z)$ of I is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \{\mathrm{e}_0(I) - \ell_A(A/I) - n_I + 1\}z + \sum_{2 \le i \le n_I+1, i \neq s} z^i}{(1-z)^d}.$$

Corollary 2. *Suppose that I is stretched and assume that $I^{n_I+2} = QI^{n_I+1}$ (i.e. $r_I \leq n_I + 1$), then* $\operatorname{depth} G \geq d-1$.

As an application, we give the depth of the associated graded ring of stretched $\mathfrak{m}$-primary ideals with reduction number at most four as follows.

Corollary 3 (Corollary 19)**.** *Suppose that I is stretched and assume that $I^5 = QI^4$ (i.e. $r_I \leq 4$), then* $\operatorname{depth} G \geq d-1$.

2. Preliminary Steps

The purpose of this section is to summarize some results on the structure of the stretched $\mathfrak{m}$-primary ideals, which we need throughout this report.

We set $\alpha_n = \ell_A(I^{n+1}/QI^n)$ for $n \geq 1$. We then have the following lemma which was given by Rossi and Valla.

Lemma 4. ([7, Lemma 2.4]) *Suppose that I is stretched. Then we have the following.*

(1) *There exists $x, y \in I\backslash Q$ such that $I^{n+1} = QI^n + (x^n y)$ holds true for all $n \geq 1$.*
(2) *The map*
$$I^{n+1}/QI^n \xrightarrow{\widehat{x}} I^{n+2}/QI^{n+1}$$
is surjective for all $n \geq 1$. Therefore $\alpha_n \geq \alpha_{n+1}$ for all $n \geq 1$.
(3) $\ell_A(I^{n+1}/QI^n + I^{n+2}) \leq 1$ *for all $n \geq 1$.*

We set
$$\Lambda := \Lambda_I = \Lambda_Q(I) = \{n \geq 1 \mid QI^{n-1} \cap I^{n+1}/QI^n \neq (0)\}$$
and $|\Lambda|$ denotes the cardinality of the set Λ. Then the following proposition is satisfied.

Proposition 5. *Suppose that I is stretched. Then we have the following.*

(1) $\alpha_1 = \ell_A(I^2 + Q/Q) = n_I - 1$.
(2) $\alpha_n = \alpha_{n-1} - 1$ *if $n \notin \Lambda$, and $\alpha_n = \alpha_{n-1}$ if $n \in \Lambda$ for all $2 \leq n \leq r_I - 1$.*
(3) $|\Lambda| = r_I - n_I$.

We notice here that $\alpha_1 = \ell_A(I^2/QI) = \mathrm{e}_0(I) + (d-1)\ell_A(A/I) - \ell_A(I/I^2)$ holds true, so that $n_I = \alpha_1 + 1$ doesn't depend on a minimal reduction Q of I for stretched $\mathfrak{m}$-primary ideals I.

3. The structure of Sally modules

In this report we need the notion of Sally modules for computing to the Hilbert coefficients of ideals. The purpose of this section is to summarize some results and techniques on the Sally modules which we need throughout this report. Remark that in this section $\mathfrak{m}$-primary ideals I are not necessarily stretched.

Let $T = \mathrm{R}(Q) = A[Qt] \subseteq A[t]$ denotes the Rees algebra of Q. Following Vasconcelos [10], we consider
$$S = \mathrm{S}_Q(I) = IR/IT \cong \bigoplus_{n\geq 1} I^{n+1}/Q^nI$$
the Sally module of I with respect to Q.

We give one remark about Sally modules. See [3, 10] for further information.

Remark 6 ([3, 10]). We notice that S is a finitely generated graded T-module and $\mathfrak{m}^n S = (0)$ for all $n \gg 0$. We have $\mathrm{Ass}_T\, S \subseteq \{\mathfrak{m}T\}$ so that $\dim_T S = d$ if $S \neq (0)$.

From now on, let us introduce some techniques, being inspired by [1, 2], which plays a crucial role throughout this report. See [6, Section 3] (also [5, Section 2] for the case where $I = \mathfrak{m}$) for the detailed proofs.

We denote by $E(m)$, for a graded module E and each $m \in \mathbb{Z}$, the graded module whose grading is given by $[E(m)]_n = E_{m+n}$ for all $n \in \mathbb{Z}$.

We have an exact sequence

$$0 \to K^{(-1)} \to F \overset{\varphi_{-1}}{\to} G \to R/IR + T \to 0 \qquad (\dagger_{-1})$$

of graded T-modules induced by tensoring the canonical exact sequence

$$0 \to T \overset{i}{\hookrightarrow} R \to R/T \to 0$$

of graded T-modules with A/I where $\varphi_{-1} = A/I \otimes i$, $K^{(-1)} = \mathrm{Ker}\, \varphi_{-1}$, and $F = T/IT \cong (A/I)[X_1, X_2, \cdots, X_d]$ is a polynomial ring with d indeterminates over the residue class ring A/I.

Lemma 7. ([5]) *There exists an exact sequence*

$$0 \to K^{(0)}(-1) \to ([R/IR + T]_1 \otimes F)(-1) \overset{\varphi_0}{\to} R/IR + T \to S/IS(-1) \to 0 \qquad (\dagger_0)$$

of graded T-modules where $K^{(0)} = \mathrm{Ker}\, \varphi_0$.

Notice that $\mathrm{Ass}_T\, K^{(m)} \subseteq \{\mathfrak{m}T\}$ for all $m = -1, 0$, because $F \cong (A/I)[X_1, X_2, \cdots, X_d]$ is a polynomial ring over the residue ring A/I and $[R/IR+T]_1 \otimes F$ is a maximal Cohen-Macaulay module over F.

We then have the following proposition by the exact sequences $(\dagger_{-1})$ and $(\dagger_0)$.

Proposition 8. ([6, Lemma 3.3]) *We have*

$$\begin{aligned} \ell_A(I^n/I^{n+1}) &= \ell_A(A/[I^2+Q])\binom{n+d-1}{d-1} - \ell_A(I/[I^2+Q])\binom{n+d-2}{d-2} \\ &+ \ell_A([S/IS]_{n-1}) - \ell_A(K_n^{(-1)}) - \ell_A(K_{n-1}^{(0)}) \end{aligned}$$

for all $n \geq 0$.

We also need the notion of *filtration of the Sally module* which was introduced by M. Vaz Pinto [11] as follows.

Definition 9. ([11]) We set, for each $m \geq 1$,

$$S^{(m)} = I^m t^{m-1} R / I^m t^{m-1} T (\cong I^m R / I^m T(-m+1)).$$

We notice that $S^{(1)} = S$, and $S^{(m)}$ are finitely generated graded T-modules for all $m \geq 1$, since R is a module-finite extension of the graded ring T.

The following lemma follows by the definition of the graded module $S^{(m)}$.

Lemma 10. *Let $m \geq 1$ be an integer. Then the following assertions hold true.*

(1) $\mathfrak{m}^n S^{(m)} = (0)$ *for integers $n \gg 0$; hence* $\dim_T S^{(m)} \leq d$.

(2) *The homogeneous components* $\{S_n^{(m)}\}_{n\in\mathbb{Z}}$ *of the graded* T*-module* $S^{(m)}$ *are given by*

$$S_n^{(m)} \cong \begin{cases} (0) & \text{if } n \leq m-1, \\ I^{n+1}/Q^{n-m+1}I^m & \text{if } n \geq m. \end{cases}$$

Let $L^{(m)} = TS_m^{(m)}$ be a graded T-submodule of $S^{(m)}$ generated by $S_m^{(m)}$ and

$$\begin{aligned} D^{(m)} &= (I^{m+1}/QI^m) \otimes (A/\operatorname{Ann}_A(I^{m+1}/QI^m))[X_1, X_2, \cdots, X_d] \\ &\cong (I^{m+1}/QI^m)[X_1, X_2, \cdots, X_d] \end{aligned}$$

for $m \geq 1$ (c.f. [11, Section 2]).

We then have the following lemma.

Lemma 11. ([11, Section 2]) *The following assertions hold true for* $m \geq 1$.

(1) $S^{(m)}/L^{(m)} \cong S^{(m+1)}$ *so that the sequence*

$$0 \to L^{(m)} \to S^{(m)} \to S^{(m+1)} \to 0$$

is exact as graded T*-modules.*

(2) *There is a surjective homomorphism* $\theta_m : D^{(m)}(-m) \to L^{(m)}$ *graded* T*-modules.*

For each $m \geq 1$, tensoring the exact sequence

$$0 \to L^{(m)} \to S^{(m)} \to S^{(m+1)} \to 0$$

and the surjective homomorphism $\theta_m : D^{(m)}(-m) \to L^{(m)}$ of graded T-modules with A/I, we get the exact sequence

$$0 \to K^{(m)}(-m) \to D^{(m)}/ID^{(m)}(-m) \xrightarrow{\varphi_m} S^{(m)}/IS^{(m)} \to S^{(m+1)}/IS^{(m+1)} \to 0 \qquad (\dagger_m)$$

of graded F-modules where $K^{(m)} = \operatorname{Ker}\varphi_m$.

Notice here that, for all $m \geq 1$, we have $\operatorname{Ass}_T K^{(m)} \subseteq \{\mathfrak{m}T\}$ because $D^{(m)}/ID^{(m)} \cong (I^{m+1}/QI^m + I^{m+2})[X_1, X_2, \cdots, X_d]$ is a maximal Cohen-Macaulay module over F.

We then have the following result by Proposition 8 and exact sequences $(\dagger_m)$ for $m \geq 1$.

Proposition 12. *The following assertions hold true:*

(1) *We have*

$$\begin{aligned} \ell_A(I^n/I^{n+1}) &= \{\ell_A(A/I^2+Q) + \sum_{m=1}^{r_I-1} \ell_A(I^{m+1}/QI^m + I^{m+2})\}\binom{n+d-1}{d-1} \\ &+ \sum_{k=1}^{r_I}(-1)^k \left\{\sum_{m=k-1}^{r_I-1}\binom{m+1}{k}\ell_A(I^{m+1}/QI^m + I^{m+2})\right\}\binom{n+d-k-1}{d-k-1} \\ &- \sum_{m=-1}^{r_I-1} \ell_A(K_{n-m-1}^{(m)}) \end{aligned}$$

for all $n \geq \max\{0, r_I - d + 1\}$.

(2) $\mathrm{e}_0(I) = \ell_A(A/I^2+Q) + \sum_{m=1}^{r_I-1} \ell_A(I^{m+1}/QI^m + I^{m+2}) - \sum_{m=-1}^{r_I-1} \ell_{T_\mathcal{P}}(K_\mathcal{P}^{(m)})$ *where* $\mathcal{P} = \mathfrak{m}T$.

4. Proof of Main Theorem

Let $B = T/\mathfrak{m}T \cong (A/\mathfrak{m})[X_1, X_2, \cdots, X_d]$ which is a polynomial ring with d indeterminates over the field $A/\mathfrak{m}$. We notice that $D^{(m)}/ID^{(m)} \cong B$ for $m \geq 1$ if I is stretched. Then, thanks to Lemma 4 and Proposition 12, we can get the following proposition.

Proposition 13. *Suppose that I is stretched. Then the following assertions hold true:*

(1) *We have*

$$\begin{aligned}\ell_A(A/I^{n+1}) &= \{\mathrm{e}_0(I) + r_I - n_I\}\binom{n+d}{d} \\ &- \{\mathrm{e}_0(I) - \ell_A(A/I) + \binom{r_I}{2} + r_I - n_I\}\binom{n+d-1}{d-1} \\ &+ \sum_{k=2}^{r_I}(-1)^k\binom{r_I+1}{k+1}\binom{n+d-k}{d-k} - \sum_{m=-1}^{r_I-1}\sum_{i=0}^{n}\ell_A(K^{(m)}_{i-m-1})\end{aligned}$$

for all $n \geq \max\{0, r_I - d\}$.

(2) $\sum_{m=-1}^{r_I-1} \ell_{T_{\mathcal{P}}}(K^{(m)}_{\mathcal{P}}) = r_I - n_I = |\Lambda|$ *where $\mathcal{P} = \mathfrak{m}T$.*

Now we get the following result of Sally and Rossi-Valla as a corollary.

Corollary 14. ([9, Corollary 2.4], [8, Theorem 2.6]) *Suppose that I is stretched. Then the equality $r_I = n_I$ holds true if and only if the associated graded ring G is Cohen-Macaulay. When this is the case the following assertions also follow.*

(1) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \binom{n_I}{2}$.

(2) $\mathrm{e}_k(I) = \binom{n_I+1}{k+1}$ *for $2 \leq k \leq d$.*

(3) *The Hilbert series $HS_I(z)$ of I is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \{\mathrm{e}_0(I) - \ell_A(A/I) - n_I + 1\}z + \sum_{2 \leq k \leq n_I} z^k}{(1-z)^d}.$$

The following proposition plays an important role for our proof of Theorem 1.

Proposition 15. *Suppose that I is stretched and assume that $r_I = n_I + 1$. We set $s = \min\{n \geq 1 \mid Q \cap I^{n+1} \neq QI^n\}$. Then the following conditions hold true:*

(1) $K^{(m)} \cong B(-s+m+1)$ *as graded T-modules and $K^{(n)} = (0)$ for all $n \neq m$ for either of $m = -1$ or $m = 0$.*

(2) $\operatorname{depth} G = d - 1$.

5. Applications

In this section let us introduce some applications of Theorem 1. We study the structure of stretched $\mathfrak{m}$-primary ideals with small reduction number.

Remark 16. Suppose that I is stretched. We notice that we have $n_I, r_I \geq 2$, and G is Cohen-Macaulay if $r_I = 2$.

We have the following proposition for the case where the reduction number is three.

Proposition 17. *Suppose that I is stretched and assume that $r_I = 3$. Then we have $\Lambda \subseteq \{2\}$ and the following condition holds true.*

(1) *Suppose $\Lambda = \emptyset$. Then*
- (*i*) $n_I = 3$, $\alpha_1 = 2$, $\alpha_2 = 1$,
- (*ii*) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 3$, $\mathrm{e}_2(I) = 4$ *if* $d \geq 2$, $\mathrm{e}_3(I) = 1$, *if* $d \geq 3$, *and* $\mathrm{e}_i(I) = 0$ *for* $4 \leq i \leq d$, *and*
- (*iii*) *G is Cohen-Macaulay.*

(2) *Suppose $\Lambda = \{2\}$. Then*
- (*i*) $n_I = 2$, $\alpha_1 = \alpha_2 = 1$,
- (*ii*) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 2$, $\mathrm{e}_2(I) = 3$ *if* $d \geq 2$, $\mathrm{e}_3(I) = 1$, *if* $d \geq 3$, *and* $\mathrm{e}_i(I) = 0$ *for* $4 \leq i \leq d$, *and*
- (*iii*) $\operatorname{depth} G = d - 1$.

Proof. Since $n_I = r_I - |\Lambda|$, the assertions (1) and (2) follow by Corollary 14 and Theorem 1 respectively. □

The following theorem determine the structure of stretched $\mathfrak{m}$-primary ideals with reduction number four.

Theorem 18. *Suppose that I is stretched and assume that $r_I = 4$. Then we have $\Lambda \subseteq \{2, 3\}$ and the following conditions hold true.*

(1) *Suppose $\Lambda = \emptyset$. Then*
- (*i*) $n_I = 4$, $\alpha_1 = 3$, $\alpha_2 = 2$, $\alpha_3 = 1$,
- (*ii*) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 6$, $\mathrm{e}_2(I) = 10$ *if* $d \geq 2$, $\mathrm{e}_3(I) = 5$, *if* $d \geq 3$, $\mathrm{e}_4(I) = 1$ *if* $d \geq 4$, *and* $\mathrm{e}_i(I) = 0$ *for* $5 \leq i \leq d$, *and*
- (*iii*) *G is Cohen-Macaulay.*

(2) *Suppose $\Lambda = \{2\}$. Then*
- (*i*) $n_I = 3$, $\alpha_1 = \alpha_2 = 2$, $\alpha_3 = 1$,
- (*ii*) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 5$, $\mathrm{e}_2(I) = 9$ *if* $d \geq 2$, $\mathrm{e}_3(I) = 5$, *if* $d \geq 3$, $\mathrm{e}_4(I) = 1$ *if* $d \geq 4$, *and* $\mathrm{e}_i(I) = 0$ *for* $5 \leq i \leq d$, *and*
- (*iii*) $\operatorname{depth} G = d - 1$

(3) *Suppose $\Lambda = \{3\}$. Then*
- (*i*) $n_I = 3$, $\alpha_1 = 2$, $\alpha_2 = \alpha_3 = 1$,
- (*ii*) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 4$, $\mathrm{e}_2(I) = 7$ *if* $d \geq 2$, $\mathrm{e}_3(I) = 4$, *if* $d \geq 3$, $\mathrm{e}_4(I) = 1$ *if* $d \geq 4$, *and* $\mathrm{e}_i(I) = 0$ *for* $5 \leq i \leq d$, *and*
- (*iii*) $\operatorname{depth} G = d - 1$.

(4) *Suppose $\Lambda = \{2, 3\}$. Then*
- (*i*) $n_I = 2$, $\alpha_1 = \alpha_2 = \alpha_3 = 1$,
- (*ii*) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 3$, $\mathrm{e}_2(I) = 6$ *if* $d \geq 2$, $\mathrm{e}_3(I) = 4$, *if* $d \geq 3$, $\mathrm{e}_4(I) = 1$ *if* $d \geq 4$, *and* $\mathrm{e}_i(I) = 0$ *for* $5 \leq i \leq d$, *and*
- (*iii*) $\operatorname{depth} G = d - 1$.

Proof. The assertions (1), (2), and (3) follow by Corollary 14 and Theorem 1. Suppose that $\Lambda = \{2, 3\}$ then we have $\alpha_1 = n_I - 1 = r_I - |\Lambda| - 1 = 1$. Thanks to [7, Theorem 2.1], [12, Theorem 3.1], and [4, Corollary 2.11], we obtain the assertion (4). □

We can get the following corollary by Proposition 17 and Theorem 18.

Corollary 19. *Suppose that I is stretched and assume that $I^5 = QI^4$ (i.e. $r_I \leq 4$), then* $\operatorname{depth} G \geq d-1$.

References

[1] A. Corso, *Sally modules of m-primary ideals in local rings*, Comm. Algebra, **37** (2009) 4503–4515.

[2] A. Corso, C. Polini, and W. V. Vasconcelos, *Multiplicity of the special fiber blowups*, Math. Proc. Camb. Phil. Soc., **140** (2006) 207–219.

[3] S. Goto, K. Nishida, and K. Ozeki, *Sally modules of rank one*, Michigan Math. J. **57** (2008) 359–381.

[4] S. Huckaba, *A d-dimensional extension of a lemma of Huneke's and formulas for the Hilbert coefficients*, Proc. AMS. **124** (1996) 1393–1401.

[5] K. Ozeki, *The equality of Elias-Valla and the associated graded ring of maximal ideals*, J. Pure and Appl. Algebra, 216 (2012) 1306–1317.

[6] ———, *The structure of Sally modules and Buchsbaumness of associated graded rings*, Nagoya Math. J., 212 (2013) 97–138.

[7] M. E. Rossi and G. Valla, *A conjecture of J. Sally*, Comm. Algebra, (13) **24** (1996), 4249—4261.

[8] ——— *Stretched $\mathfrak{m}$-primary ideals*, Beiträge Algebra und Geometrie Contributions to Algebra and Geometry, **42** (2001) 103–122.

[9] J. D. Sally, *Stretched Gorenstein rings*, J. Lond. Math. Soc. (2), **20** (1979) 19–26.

[10] W. V. Vasconcelos, *Hilbert Functions, Analytic Spread, and Koszul Homology*, Contemporary Mathematics, Vol **159** (1994) 410–422.

[11] M. Vaz Pinto, *Hilbert functions and Sally modules*, J. Algebra, **192** (1996) 504–523.

[12] H.-J. Wang, *On Cohen-Macaulay local rings with embedding dimension e+d − 2* J. Algebra **190** (1997), 226–240.

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THE SPECTRUM OF GROTHENDIECK MONOID

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Abstract. The Grothendieck monoid of an exact category is a monoid version of the Grothendieck group. We use it to classify Serre subcategories of an exact category and to reconstruct the topology of a noetherian scheme. We first construct bijections between (i) the set of Serre subcategories of an exact category, (ii) the set of faces of its Grothendieck monoid, and (iii) the monoid spectrum of its Grothendieck monoid. By using (ii), we classify Serre subcategories of exact categories related to a finite dimensional algebra and a smooth projective curve. For this, we determine the Grothendieck monoid of the category of coherent sheaves on a smooth projective curve. By using (iii), we introduce a topology on the set of Serre subcategories. As a consequence, we recover the topology of a noetherian scheme from the Grothendieck monoid.

Key Words: Grothendieck monoid; exact categories; reconstruction theorem.

2000 *Mathematics Subject Classification*: 18E10, 16G10, 14H60.

1962 年の論文 [2] において，Gabriel はネーター可換環の加群圏の Serre 部分圏を素スペクトラムを用いて分類した．以来，部分圏の分類問題は環の表現論における主要なテーマであり続けている．同論文 [2] において Gabriel はネーター・スキーム X をその上の準連層の圏 $\mathsf{Qcoh}\,X$ から復元した．これは幾何学的対象 X とアーベル圏 $\mathsf{Qcoh}\,X$ が等価であることを意味する．一連の Gabriel の仕事は幾何学的対象を圏を通して調べる非可換代数幾何学という分野に発展し，今日では環の表現論と代数幾何学の重要な合流地点となっている．本稿では [3] に基づいて，**Grothendieck モノイド**と呼ばれる圏の不変量を用いた**完全圏の Serre 部分圏の分類**と**ネーター・スキームの位相の復元**について紹介する．

第 1 節では，完全圏の簡単な復習と Grothendieck モノイドの定義を与える．第 2 節では，完全圏の部分圏と Grothendieck モノイドの部分集合の間に対応を作り，それが Serre 部分圏と良い性質を持つ部分モノイドの間に全単射を与えることを紹介する．第 3 節では，まず可換モノイドのスペクトラムについて復習し，連接層の圏の Grothendieck モノイドのスペクトラムを考えることで元のネーター・スキームの位相が復元できることを紹介する．

本稿において圏はすべて本質的に小であるとする．圏 $\mathcal{C}$ に対して $|\mathcal{C}|$ で $\mathcal{C}$ の対象の同型類の集合を表す．また $\mathcal{C}$ の対象 X の同型類も同じ記号 X で表す．任意の部分圏は充満部分圏であり同型で閉じているとする．モノイドは単位元を持つ**可換**半群を指し，加法的記法を用いる．非負整数の集合 $\mathbb{N}$ は通常の算術的演算によりモノイドと見なす．ネーター環 Λ に対して $\mathsf{mod}\,\Lambda$ で有限生成 (右)Λ 加群の圏を表す．ネーター・スキーム X に対して $\mathsf{coh}\,X$ で X 上の連接層の圏を表す．

The detailed version of this paper will be submitted for publication elsewhere.

1. 完全圏と Grothendieck モノイド

この節では，本稿の主な考察対象である完全圏と Grothendieck モノイドを紹介する．**完全圏 (exact category)** とは，**許容完全列 (conflation)** と呼ばれる射の列 $0 \to A \to B \to C \to 0$ が指定された加法圏であって，いくつかの条件を満たすものである．詳しい定義は [4, 第 5 章] を見よ．完全圏の例は以下のようなものがある：

Example 1.

(1) アーベル圏 $\mathcal{A}$ は許容完全列として短完全列を指定することで完全圏となる．
(2) 加法圏 $\mathcal{C}$ は許容完全列として分裂短完全列

$$0 \to A \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} A \oplus B \xrightarrow{[0\ 1]} B \to 0$$

を指定することで完全圏となる．とくに半単純でないアーベル圏には相異なる二つの完全圏構造が存在する．
(3) アーベル圏 $\mathcal{A}$ の部分圏 $\mathcal{E}$ が**拡大で閉じる (closed under extensions)** とは，任意の $\mathcal{A}$ の短完全列 $0 \to A \to B \to C \to 0$ に対して，$A, C \in \mathcal{E}$ ならば $B \in \mathcal{E}$ となるときに言う．$\mathcal{A}$ の短完全列で全ての項が $\mathcal{E}$ に入るものを許容完全列として指定することで $\mathcal{E}$ は完全圏になる．
(4) R をネーター可換整域とする．このとき，捩じれ自由加群 (torsionfree module) のなす $\mathsf{mod}\, R$ の部分圏 $\mathsf{tf}\, R$ は拡大で閉じた部分圏である．よって (3) より完全圏になる．短完全列

$$0 \to R \xrightarrow{\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]} R^{\oplus 2} \xrightarrow{[1\ 1]} R \to 0$$

は $\mathsf{tf}\, R$ の許容完全列だが，ゼロでない非単元 $x \in R$ に対して短完全列

$$0 \to R \xrightarrow{-\cdot x} R \to R/xR \to 0 \tag{1.1}$$

は $\mathsf{tf}\, R$ の許容完全列ではない．

次にアーベル圏の Grothendieck 群について復習しよう．アーベル圏 $\mathcal{A}$ の **Grothendieck 群** $\mathsf{K}_0(\mathcal{A})$ は，$\mathcal{A}$ の対象の同型類で生成される自由アーベル群を Euler 関係式で割ったアーベル群として定義される：

$$\mathsf{K}_0(\mathcal{A}) := \bigoplus_{X \in |\mathcal{A}|} \mathbb{Z}X \Big/ \langle A - B + C \mid 0 \to A \to B \to C \to 0 \text{ は短完全列} \rangle.$$

対象 $X \in \mathcal{A}$ の $\mathsf{K}_0(\mathcal{A})$ における剰余類を $[X]$ で表す．構成から明らかに任意の短完全列 $0 \to A \to B \to C \to 0$ に対して $\mathsf{K}_0(\mathcal{A})$ における等式 $[B] = [A] + [C]$ が成立する．以下で述べる様に Grothendieck 群はこのような性質を満たすアーベル群の中で最も普遍的なものである．

Definition 2. アーベル圏 $\mathcal{A}$ 上のアーベル群 G に値を持つ**加法的関数 (additive function)** とは，写像 $f\colon |\mathcal{A}| \to G$ であって次が成立するものを言う：

- 等式 $f(0) = 0$ が成立する．つまりゼロ対象は G の単位元に対応する．
- 任意の短完全列 $0 \to A \to B \to C \to 0$ に対して等式 $f(B) = f(A) + f(C)$ が成立する．

Proposition 3. アーベル圏 $\mathcal{A}$ に対して次が成立する．

(1) 自然な写像 $[-]\colon |\mathcal{A}| \to \mathsf{K}_0(\mathcal{A})$ は加法的関数である．

(2) 任意の $\mathcal{A}$ 上のアーベル群 G に値を持つ加法的関数 f は常に自然な加法的関数 $[-]\colon |\mathcal{A}| \to \mathsf{K}_0(\mathcal{A})$ を一意に経由する:

$$\begin{tikzcd} |\mathcal{A}| \arrow[r, "f"] \arrow[d, "{[-]}"'] & G \\ \mathsf{K}_0(\mathcal{A}) \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}.$$

この普遍性をもとに完全圏の Grothendieck モノイドを次のように定義する．

Definition 4. $\mathcal{E}$ を完全圏とする．

(1) $\mathcal{E}$ 上のモノイド M に値を持つ加法的関数とは，写像 $f\colon |\mathcal{E}| \to M$ であって次が成立するものを言う：
- 等式 $f(0) = 0$ が成立する．つまりゼロ対象は M の単位元に対応する．
- 任意の許容完全列 $0 \to A \to B \to C \to 0$ に対して等式 $f(B) = f(A) + f(C)$ が成立する．

(2) $\mathcal{E}$ の **Grothendieck モノイド** $\mathsf{M}(\mathcal{E}) = (\mathsf{M}(\mathcal{E}), \pi)$ とは，モノイド $\mathsf{M}(\mathcal{E})$ と加法的関数 $\pi\colon |\mathcal{E}| \to \mathsf{M}(\mathcal{E})$ の組であって次を満たすものである：
- 任意の $\mathcal{E}$ 上のモノイド M に値を持つ加法的関数 f は常に $\pi\colon |\mathcal{E}| \to \mathsf{K}_0(\mathcal{E})$ を一意に経由する:

$$\begin{tikzcd} |\mathcal{E}| \arrow[r, "f"] \arrow[d, "\pi"'] & M \\ \mathsf{M}(\mathcal{E}) \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}.$$

このとき，任意の対象 $X \in \mathcal{E}$ に対して $[X] := \pi(X)$ と表す．

Grothendieck モノイドは任意の完全圏に対して実際に存在する [1, Proposition 3.3]．写像 $\pi\colon |\mathcal{E}| \to \mathsf{M}(\mathcal{E})$ が加法的であることからモノイド $\mathsf{M}(\mathcal{E})$ において次の等式が常に成立する．

- $[0] = 0$．つまりゼロ対象は単位元に対応する．
- 任意の短完全列 $0 \to A \to B \to C \to 0$ に対して $[B] = [A] + [C]$.
- 任意の二つの対象 $A, B \in \mathcal{E}$ に対して $[A \oplus B] = [A] + [B]$.

最後の等式は，完全圏の公理より分裂短完全列 $0 \to A \to A \oplus B \to B \to 0$ が常に許容完全列であることから従う．Grothendieck モノイドの基本的な計算例は次のようなものがある．

Example 5.

(1) Λ を体上の有限次元代数とする．このとき次元ベクトルはモノイドの同型

$$\underline{\dim}\colon \mathsf{M}(\mathsf{mod}\,\Lambda) \xrightarrow{\cong} \mathbb{N}^{\oplus n}$$

を与える．ここで n は単純 Λ 加群の同型類の個数である．

(2) より一般に完全圏 $\mathcal{E}$ が Jordan-Hölder 性を満たすとき，次のモノイドの同型がある：

$$\mathsf{M}(\mathsf{mod}\,\Lambda) \xrightarrow{\cong} \mathbb{N}^{\oplus \operatorname{\mathsf{sim}} \mathcal{E}}.$$

ここで $\mathsf{sim}\,\mathcal{E}$ は完全圏 $\mathcal{E}$ の許容単純対象の同型類の集合である．詳細は [1] を参照せよ．

最後に Grothendieck 群と Grothendieck モノイドの関係を述べる．

Remark 6. アーベル圏の場合と同様にして完全圏 $\mathcal{E}$ の Grothendieck 群 $\mathsf{K}_0(\mathcal{E})$ を定義することができる．

(1) 一般にモノイド M からアーベル群 $\mathsf{gp}M$ を構成する**群完備化 (group completion)** という操作がある．これにより関手的群同型 $\mathsf{gp} \circ \mathsf{M}(\mathcal{E}) \cong \mathsf{K}_0(\mathcal{E})$ が得られる．

(2) Grothendieck モノイド $\mathsf{M}(\mathcal{E})$ と Grothendieck 群の正部分

$$\mathsf{K}_0^+(\mathcal{E}) := \{[X] \in \mathsf{K}_0(\mathcal{E}) \mid X \in \mathcal{E}\}$$

は一般に一致しない．実際，R をネーター可換整域とすれば (1.1) より

$$[R/xR] = [R] - [R] = 0$$

が $\mathsf{K}_0(\mathsf{mod}\, R)$ において成立する．一方，完全圏の Grothendieck モノイドの一般論からゼロでない対象は $\mathsf{M}(\mathsf{mod}\, R)$ においてゼロでない元を定める [1, Proposition 3.5]．

2. Grothendieck モノイドを用いた部分圏の分類

この節では，完全圏のある種の部分圏が Grothendieck モノイドの部分集合を用いて分類できることを紹介する．以下 $\mathcal{E}$ は完全圏とする．Grothendieck モノイド $\mathsf{M}(\mathcal{E})$ の部分集合と $\mathcal{E}$ の部分圏に対して次の対応を考える．

- 任意の $\mathcal{E}$ の部分圏 $\mathcal{D}$ に対して，$\mathsf{M}(\mathcal{E})$ の部分集合 $\mathsf{M}_{\mathcal{D}}$ を次で定める：

$$\mathsf{M}_{\mathcal{D}} := \{[X] \in \mathsf{M}(\mathcal{E}) \mid X \in \mathcal{D}\}.$$

- 任意の $\mathsf{M}(\mathcal{E})$ の部分集合 N に対して，$\mathcal{E}$ の部分圏 $\mathcal{D}_N$ を次で定める：

$$\mathcal{D}_N := \{X \in \mathcal{E} \mid [X] \in N\}.$$

上記の対応が全単射となるような部分圏のクラスを以下で導入する．

Definition 7. $\mathcal{E}$ の部分圏 $\mathcal{D}$ が **c-同値で閉じる (closed under c-equivalence)** あるいは **c-閉部分圏 (c-closed subcategory)** であるとは，任意の許容短完全列 $0 \to A \to B \to C \to 0$ に対して $B \in \mathcal{D}$ と $A \oplus C \in \mathcal{D}$ が同値であるときに言う．

c-閉部分圏の意味について補足しよう．

Remark 8. 任意の許容短完全列 $0 \to A \to B \to C \to 0$ に対して $\mathsf{M}(\mathcal{E})$ における次の等式が成立する：

$$[B] = [A] + [C] = [A \oplus C]$$

よって c-閉部分圏とは，Grothendieck モノイドの中で同じ元を定める対象を区別しない部分圏だと思える．実際，$\mathcal{E}$ の部分圏 $\mathcal{D}$ に対して次は同値である：

(1) $\mathcal{D}$ は c-閉部分圏である．

(2) 任意の対象 $X, Y \in \mathcal{E}$ に対して，$\mathsf{M}(\mathcal{E})$ において $[X] = [Y]$ であり $X \in \mathcal{D}$ ならば $Y \in \mathcal{D}$ である．

次の命題は，c-閉部分圏は Grothendieck モノイドの部分集合全体で分類されることを意味する．

Proposition 9. 対応 $\mathcal{D} \mapsto \mathsf{M}_{\mathcal{D}}$ と $N \mapsto \mathcal{D}_N$ は次の集合の間の全単射を与える．

(1) $\mathcal{E}$ の c-閉部分圏全体の集合．

(2) $\mathsf{M}(\mathcal{E})$ の部分集合全体の集合.

完全圏 $\mathcal{E}$ の部分圏 $\mathcal{D}$ が **Serre 部分圏**であるとは，任意の許容完全列 $0 \to A \to B \to C \to 0$ について $B \in \mathcal{D}$ と $A, C \in \mathcal{D}$ が同値であるときに言う．Serre 部分圏はアーベル圏の商圏との関係から古くより研究されてきた．Serre 部分圏と今回導入された c-閉部分圏の関係は次で与えられる．

Proposition 10. 完全圏の部分圏が Serre であることと直和，直和因子，c-同値で閉じることは同値である．

この命題からとくに Serre 部分圏は c-閉部分圏である．よって命題 9 の全単射は Serre 部分圏全体の集合に制限され，次の全単射を与える．

Proposition 11. 対応 $\mathcal{D} \mapsto \mathsf{M}_{\mathcal{D}}$ と $N \mapsto \mathcal{D}_N$ は次の集合の間の全単射を与える．

(1) $\mathcal{E}$ の Serre 部分圏全体の集合 $\mathsf{Serre}(\mathcal{E})$.

(2) $\mathsf{M}(\mathcal{E})$ の面全体の集合 $\mathsf{Face}(\mathsf{M}(\mathcal{E}))$.

ここでモノイド M の**面 (face)** とは，M の部分モノイド F であって，任意の $x, y \in M$ に対して $x + y \in F$ と $x, y \in F$ が同値であるものを言う．

Grothendieck モノイドが決定されているとき，その面を全て求めることは Serre 部分圏を全て求めることに比べてかなり容易である．よって完全圏の Serre 部分圏を分類するための次の戦略を提案したい：

(1) Grothendieck モノイド $\mathsf{M}(\mathcal{E})$ と抽象モノイド M を関連付ける．

(2) この抽象モノイド M の面をすべて決定する．

(3) (1) と (2) を用いて $\mathcal{E}$ の Serre 部分圏をすべて決定する．

この戦略に従って，非特異射影曲線に付随するいくつかの完全圏の Grothendieck モノイドを計算し，その Serre 部分圏を分類した．とくに次の結果を得た：

Corollary 12. 非特異射影曲線 C 上のベクトル束のなす完全圏 $\mathsf{vect}C$ は 0 と全体以外の非自明な Serre 部分圏を持たない．

3. Grothendieck モノイドのスペクトラムと位相の復元

この節では，まずモノイドのスペクトラムについて復習する．その後，前節で考察した Serre 部分圏との関係を述べる．最後に Grothendieck モノイドのスペクトラムに入る位相を用いて，ネーター・スキームの位相を連接層の圏の Grothendieck モノイドから復元する話を紹介する．

まずモノイドにおける素イデアルの概念を導入する．

Definition 13. M をモノイドとする．

(1) M の部分集合 I が**イデアル**であるとは，任意の $x \in I$ と $a \in M$ に対して $x + a \in I$ となるときに言う．

(2) 真のイデアル $\mathfrak{p} \subsetneq M$ が**素イデアル**であるとは，任意の $x, y \in M$ に対して，$x + y \in \mathfrak{p}$ のとき $x \in \mathfrak{p}$ または $y \in \mathfrak{p}$ が成立するときに言う．

(3) M の素イデアル全体の集合を $\operatorname{MSpec} M$ で表し，M の**モノイド・スペクトラム**と言う．

今定義した素イデアルと前節で考察した面の関係は次のようになっている．

Proposition 14. モノイド M に対して次が成立する．

(1) 任意の M の素イデアル $\mathfrak{p}$ に対して $\mathfrak{p}^c := M \setminus \mathfrak{p}$ は M の面である．
(2) 任意の M の面 F に対して $F^c := M \setminus F$ は M の素イデアルである．
(3) *(1)* と *(2)* の対応は $\mathsf{Face}(M)$ と $\operatorname{MSpec} M$ の間の全単射を与える．

とくに完全圏 $\mathcal{E}$ の Grothendieck モノイドにこの命題を適用することで次の全単射を得る：

$$\mathsf{Serre}(\mathcal{E}) \xrightarrow[\text{命題 } 11]{\cong} \mathsf{FaceM}(\mathcal{E}) \xrightarrow[\text{命題 } 14]{\cong} \operatorname{MSpec} \mathsf{M}(\mathcal{E}). \tag{3.1}$$

ところでモノイド・スペクトラム $\operatorname{MSpec} M$ は次の部分集合を閉集合とすることで位相空間になる：

$$V(S) := \{\mathfrak{p} \in \operatorname{MSpec} M \mid \mathfrak{p} \supseteq S\}, \quad S \text{ は } M \text{ の部分集合.}$$

全単射 (3.1) によって $\operatorname{MSpec} \mathsf{M}(\mathcal{E})$ の位相を誘導することで $\mathsf{Serre}(\mathcal{E})$ は位相空間となる．

ネーター・スキーム X に対して次の写像を考える：

$$j \colon X \to \mathsf{Serre}(\mathsf{coh}\, X), \quad x \mapsto \mathsf{coh}^x X := \{\mathcal{F} \in \mathsf{coh}\, X \mid \mathcal{F}_x = 0\}.$$

写像 j は（上で定めた $\mathsf{Serre}(\mathsf{coh}\, X)$ の位相に関して）位相空間として埋め込みになっている．実は j の埋め込みの像は $\mathsf{Serre}(\mathsf{coh}\, X)$ の位相構造から復元することが出来る．これにより位相空間として X は Grothendieck モノイド $\mathsf{M}(\mathsf{coh}\, X)$ から復元できることが結論付けられる．とくに次が成立する．

Proposition 15. ネーター・スキーム X と Y に関する次の条件を考える．

(1) スキームとしての同型 $X \cong Y$ がある．
(2) モノイドとしての同型 $\mathsf{M}(\mathsf{coh}\, X) \cong \mathsf{M}(\mathsf{coh}\, Y)$ がある．
(3) 位相空間としての同型 $X \cong Y$ がある．

このとき (1) $\Rightarrow$ (2) $\Rightarrow$ (3) が成立する．

References

[1] H. Enomoto, *The Jordan-Hölder property and Grothendieck monoids of exact categories*, Adv. Math, **396** (2022).
[2] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448.
[3] S. Saito, *The spectrum of Grothendieck monoid: classifying Serre subcategories and reconstruction theorem*, arXiv:2206.15271.
[4] 中岡宏行, **圏論の技法**, (2015), 日本評論社.

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ON IE-CLOSED SUBCATEGORIES

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Abstract. We study IE-closed subcategories of module categories, subcategories closed under taking images and extensions. The class of IE-closed subcategories contains that of torsion classes, torsion-free classes and wide subcategories, which are important objects in representation theory of algebras. We give a characterization of τ-tilting finiteness in terms of IE-closed subcategories. When we consider a hereditary algebra, we introduce the concept of twin rigid modules and give a classification result of IE-closed subcategories.

1. Introduction

Let Λ be a finite dimensional algebra over a field k and $\mathsf{mod}\Lambda$ the category of finitely generated right Λ-modules. It is one of the main subjects of representation theory of finite dimensional algebras to study subcategories of $\mathsf{mod}\Lambda$. For example, torsion classes are studied actively in connection with tilting and τ-tilting theory [1]. We focus on subcategories of $\mathsf{mod}\Lambda$ closed under some operations. In this note, we always assume that all subcategories are full and closed under isomorphisms.

Definition 1. Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\Lambda$.

(1) $\mathcal{C}$ is *closed under extensions* if for every short exact sequence
$$0 \to L \to M \to N \to 0$$
in $\mathsf{mod}\Lambda$ with $L, N \in \mathcal{C}$, we have $M \in \mathcal{C}$.

(2) $\mathcal{C}$ is *closed under quotients (resp. submodules) in* $\mathsf{mod}\Lambda$ if, for every object $C \in \mathcal{C}$, every quotient (resp. submodule) of C in $\mathsf{mod}\Lambda$ belongs to $\mathcal{C}$.

(3) $\mathcal{C}$ is a *torsion class (resp. torsion-free class) in* $\mathsf{mod}\Lambda$ if $\mathcal{C}$ is closed under extensions and quotients in $\mathsf{mod}\Lambda$ (resp. extensions and submodules).

(4) $\mathcal{C}$ is closed under *images (resp. kernels, cokernels)* if, for every map $\varphi\colon C_1 \to C_2$ with $C_1, C_2 \in \mathcal{C}$, we have $\mathsf{Im}\varphi \in \mathcal{C}$ (resp. $\mathsf{Ker}\varphi \in \mathcal{C}$, $\mathsf{Coker}\varphi \in \mathcal{C}$).

(5) $\mathcal{C}$ is a *wide subcategory of* $\mathsf{mod}\Lambda$ if $\mathcal{C}$ is closed under kernels, cokernels, and extensions.

(6) $\mathcal{C}$ is an *IE-closed subcategory of* $\mathsf{mod}\Lambda$ if $\mathcal{C}$ is closed under images and extensions.

It is easy to check that torsion classes, torsion-free classes and wide subcategories are IE-closed subcategories. The notion of *ICE-closed subcategories*, subcategories closed under images, cokernels and extensions is considered in [4].

For a collection $\mathcal{C}$ of Λ-modules in $\mathsf{mod}\Lambda$, we denote by $\mathsf{T}(\mathcal{C})$ (resp. $\mathsf{F}(\mathcal{C})$) the smallest torsion class (resp. torsion-free class) containing $\mathcal{C}$. The following proposition implies that

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The detailed version of this paper will be submitted for publication elsewhere.

an IE-closed subcategory is same as an intersection of a torsion class and a torsion-free class.

Proposition 2. [5, Proposition 2.3] *The following conditions are equivalent for a subcategory $\mathcal{C}$ of $\mathsf{mod}\Lambda$.*

(1) *$\mathcal{C}$ is an IE-closed subcategory of $\mathsf{mod}\Lambda$.*
(2) *There exist a torsion class $\mathcal{T}$ and a torsion-free class $\mathcal{F}$ in $\mathsf{mod}\Lambda$ satisfying $\mathcal{C} = \mathcal{T} \cap \mathcal{F}$.*

In this case, $\mathcal{C} = \mathsf{T}(\mathcal{C}) \cap \mathsf{F}(\mathcal{C})$ holds.

2. Functorial finiteness

In this section, we consider some finiteness conditions of subcategories and give implications among them. Using these, we characterize tau-tilting finite alegebras by fuctorial finiteness of IE-closed subcategories. We start introducing the concept of functorial finiteness.

Definition 3. Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\Lambda$ and M an object in $\mathsf{mod}\Lambda$.

(1) A morphism $f\colon M \to C$ in $\mathsf{mod}\Lambda$ is a *left $\mathcal{C}$-approximation* of M if C belongs to $\mathcal{C}$ and every morphism $f'\colon M \to C'$ with $C' \in \mathcal{C}$ factors through f. Dually, a *right $\mathcal{C}$-approximation* is defined.
(2) A subcategory $\mathcal{C}$ is *covariantly finite (resp. contravariantly finite) in* $\mathsf{mod}\Lambda$ if for any object M in $\mathsf{mod}\Lambda$, there exists a left (resp. right) $\mathcal{C}$-approximation of M.
(3) A subcategory is *functorially finite in* $\mathsf{mod}\Lambda$ if it is covariantly finite and contravariantly finite in $\mathsf{mod}\Lambda$.

Every torsion class $\mathcal{T}$ in $\mathsf{mod}\Lambda$ is contravariantly finite since it gives a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathsf{mod}\Lambda$. The notion of functorial finiteness appears in [2], which considers the existance of Auslander-Reiten sequences in subcategories of $\mathsf{mod}\Lambda$. Next we introduce the notion of Ext-projective.

Definition 4. Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\Lambda$ closed under extensions.

(1) An object P of $\mathcal{C}$ is Ext-*projective* if it satisfies $\mathrm{Ext}^1_\Lambda(P, C) = 0$ for any $C \in \mathcal{C}$.
(2) $\mathcal{C}$ *has enough* Ext-*projectives* if for any $C \in \mathcal{C}$, there exists a short exact sequence
$$0 \to C' \to P \to C \to 0$$
such that P is an Ext-projective object in $\mathcal{C}$ and $C' \in \mathcal{C}$.
(3) P is an Ext-*progenerator* of $\mathcal{C}$ if $\mathcal{C}$ has enough Ext-projectives and Ext-projective objects are precisely objects in $\mathsf{add}P$.
(4) If $\mathcal{C}$ has an Ext-progenerator, then $\mathbf{P}(\mathcal{C})$ denotes a unique basic Ext-progenerator of $\mathcal{C}$, that is, a direct sum of all indecomposable Ext-projective objects in $\mathcal{C}$ up to isomorphism.

Dually, the notions for Ext-injectives are defined, and $\mathbf{I}(\mathcal{C})$ denotes a unique basic Ext-injective cogenerator of $\mathcal{C}$ (if it exists).

It is well-known that the above notions are related to each other for torsion classes.

Proposition 5. *The following conditions are equivalent for a torsion class $\mathcal{T}$ in $\mathsf{mod}\Lambda$.*

(1) $\mathcal{T}$ *has an* Ext*-progenerator.*
(2) $\mathcal{T}$ *has a finite cover, that is, there is* $M \in \mathcal{T}$ *such that* $\mathcal{T} \subseteq \mathsf{Fac}M$*.*
(3) $\mathcal{T}$ *is covariantly finite in* $\mathsf{mod}\Lambda$*.*
(4) $\mathcal{T}$ *has enough* Ext*-projective objects.*

The following proposition gives relations among the above finiteness conditions for IE-closed subcategories.

Proposition 6. [5, Lemma 2.6, 2.9] *Consider the following conditions for an IE-closed subcategory* $\mathcal{C}$ *in* $\mathsf{mod}\Lambda$*.*

(1) $\mathcal{C}$ *is left finite, that is,* $\mathsf{T}(\mathcal{C})$ *is functorially finite in* $\mathsf{mod}\Lambda$*.*
(2) *There exist a torsion class* $\mathcal{T}$ *and a torsion-free class* $\mathcal{F}$ *such that* $\mathcal{C} = \mathcal{T} \cap \mathcal{F}$ *and* $\mathcal{T}$ *is functorially finite in* $\mathsf{mod}\Lambda$*.*
(3) $\mathcal{C}$ *has an* Ext*-progenerator.*
(4) $\mathcal{C}$ *has a finite cover.*
(5) $\mathcal{C}$ *is covariantly finite in* $\mathsf{mod}\Lambda$*.*
(6) $\mathcal{C}$ *has enough* Ext*-projective objects.*

The assertions (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5) $\Rightarrow$ (6) *hold. If* Λ *is hereditary, all conditions are equivalent.*

The notion of τ-tilting finiteness introduced in [3] is an analogue of representation finiteness in the perspective of τ-tilting theory. A finite dimensional algebra Λ is *τ-tilting finite* if the set of functorially finite torsion classes in $\mathsf{mod}\Lambda$ is a finite set. This definition coincides with the condition that there are only finitely many support τ-tilting Λ-modules up to isomorphisms, see Theorem 9. In [3], τ-tilting finiteness is characterized as follows:

Theorem 7. [3, Theorem 3.8] *Let* Λ *be a finite dimensional algebra. The following conditions are equivalent.*

(1) Λ *is τ-tilting finite.*
(2) *The set of torsion classes in* $\mathsf{mod}\Lambda$ *is a finite set.*
(3) *The set of torsion-free classes in* $\mathsf{mod}\Lambda$ *is a finite set.*
(4) *Every torsion class in* $\mathsf{mod}\Lambda$ *is functorially finite.*
(5) *Every torsion-free class in* $\mathsf{mod}\Lambda$ *is functorially finite.*

Now we give the following result analogous to the above[1].

Theorem 8. [5, Proposition 2.10] *Let* Λ *be a finite dimensional algebra. The following are equivalent.*

(1) Λ *is τ-tilting finite.*
(2) *The set of IE-closed subcategories of* $\mathsf{mod}\Lambda$ *is a finite set.*
(3) *Every IE-closed subcategory of* $\mathsf{mod}\Lambda$ *is functorially finite.*
(4) *Every IE-closed subcategory of* $\mathsf{mod}\Lambda$ *is covariantly finite.*
(5) *Every IE-closed subcategory of* $\mathsf{mod}\Lambda$ *is contravariantly finite.*

There is an analogous result for ICE-closed subcategories, see [4, Proposition 4.20].

[1]The author would like to thank Ryo takahashi and Haruhisa Enomoto for the conversation after the author's talk which gives the equivalence between (3) and (4).

3. Classification

In this section, we give the classification result of IE-closed subcategories. We start giving the following result which classify torsion classes in $\mathsf{mod}\Lambda$.

Theorem 9. [1, Theorem 2.7] *There exists bijective correspondences between:*

(1) *The set of functorially finite torsion classes in* $\mathsf{mod}\Lambda$*,*
(2) *The set of isomorphism classes of basic support* τ*-tilting modules.*

The correspondence from (1) to (2) is given by $\mathcal{T} \mapsto \mathbf{P}(\mathcal{T})$, and well-defined by Proposition 5.

Now we aim to classify IE-closed subcategories as an analogue of the above. Unfortunately, we need the assumption that Λ is hereditary. In the rest of this note, we assume it. We start introducing the concept of twin rigid modules.

Definition 10. A pair (P, I) of Λ-modules is a *twin rigid module* if it satisfies

- P and I are rigid, that is, $\mathrm{Ext}^1_\Lambda(P, P) = 0$ and $\mathrm{Ext}^1_\Lambda(I, I) = 0$.
- There are short exact sequences
$$0 \to P \to I^0 \to I^1 \to 0$$
$$0 \to P_1 \to P_0 \to I \to 0$$
with $P_0, P_1 \in \mathsf{add}P$ and $I^0, I^1 \in \mathsf{add}I$.

Two twin rigid pairs (P_1, I_1) and (P_2, I_2) are *isomorphic* if we have $P_1 \cong P_2$ and $I_1 \cong I_2$. A twin rigid module (P, I) is *basic* if P and I are basic.

The above concept is appered as a pair of an Ext-progenerator and an Ext-injective cogenerator of an IE-closed subcategory:

Theorem 11. [5, Theorem 2.14] *Let Λ be a hereditary finite dimensional algebra. Then there exist bijective correspondences between:*

(1) *The set of functorially finite IE-closed subcategories of* $\mathsf{mod}\Lambda$*,*
(2) *The set of isomorphism classes of basic* twin rigid Λ-modules.

The correspondence from (1) to (2) is given by $\mathcal{C} \mapsto (\mathbf{P}(\mathcal{C}), \mathbf{I}(\mathcal{C}))$, and well-defined by Proposition 6 and its dual.

Next we give the property of twin rigid modules, which gives a connection between twin rigid modules and tilting modules.

Proposition 12. [5, Proposition 3.4 (1)] *Assume that Λ is hereditary. Let (P, I) be a twin rigid module and set $\Gamma_P = \mathsf{End}_\Lambda(P)$. Then*

(1) $\mathsf{Hom}_\Lambda(P, I)$ *is a tilting Γ_P-module.*
(2) *The equality $|P| = |I|$ holds.*

The equality (2) gives a partial anwer to the question raised by Auslander and Smalø in [2], see [5, Remark 3.8]. Thanks to the previous proposition, we obtatin the following bijection.

Proposition 13. [5, Proposition 3.4 (2)] *Assume that Λ is hereditary. Let (P, I) be a twin rigid module. Then the functor* $\mathrm{Hom}_\Lambda(P, -) \colon \mathsf{mod}\Lambda \to \mathsf{mod}\Gamma_P$ *induces a bijective correspondence between:*

(1) *the set of isomorphism classes of twin rigid modules* (P, I),
(2) *the set of isomorphism classes of tilting* Γ_P*-modules contatined in* $\mathsf{Sub}(\mathrm{Hom}_k(P, k))$.

In [5], the notion of completion and mutation of twin rigid modules is introduced taking advantage of that of tilting modules through the above proposition. All twin rigid modules are obtained by mutation in the case that Λ is a representation-finite hereditary algebra. By Theorem 11, we obtain all IE-closed subcategories.

References

[1] T. Adachi, O. Iyama, I. Reiten, *τ-tilting theory*, Compos. Math. 150 (2014), no. 3, 415–452.
[2] M. Auslander, S.O. Smalø, *Almost split sequences in subcategories*, J. Algebra 69 (1981), no. 2, 426–454.
[3] L. Demonet, O. Iyama, G. Jasso, *τ-tilting finite algebras, bricks, and g-vectors*, Int. Math. Res. Not. IMRN 2019, no. 3, 852–892.
[4] H. Enomoto, A. Sakai, *ICE-closed subcategories and wide τ-tilting modules*, Math. Z. 300 (2022), no. 1, 541–577.
[5] H. Enomoto, A. Sakai, *Image-extension-closed subcategories of module categories of hereditary algebras*, arXiv:2208.13937.

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ON GENERALIZED NAKAYAMA-AZUMAYA'S LEMMA AND THE FINAL REPORT ON WARE'S PROBLEM
中山・東屋の補題の一般化および WARE の問題解決の最終報告

MASAHISA SATO (佐藤眞久)

This paper is dedicated to
Professor Goro Azumaya and Professor Hiroyuki Tachikawa
as the memory of them, who were greatly contributed to ring theory.
環論の発展に多大な貢献をされた東屋五郎先生および太刀川弘幸先生を偲んでこの論文を捧げます

Abstract. Let R be an associative ring and $J(R)$ its Jacobson radical. The following Nakayama-Azumaya's Lemma is well known.

Nakayama-Azumaya's Lemma *Let M be a finitely generated right R-module. If M satisfies $MJ(R) = M$, then $M = 0$.*

This property holds also for a projective module M. We show the following Generalized Nakayama-Azumaya's Lemma as the unified theorem of the above two results.

Generalized Nakayama-Azumaya's Lemma *Let M be a direct summand of a direct sum of finitely generated right R-modules. If M satisfies $MJ(R) = M$, then $M = 0$.*

In the beginning of this article, we give the final report on Ware's problem proposed in [9] by R.Ware, i.e., the following Ware's problem is affirmative.

Ware's problem *Assume a projecive module P has a unique maximal submodule, then this submodule is the largest submodule of P.*

Key Words: Nakayama-Azumaya's Lemma, NAS-modules, Ware's problem, Maximal submodules

2010 *Mathematics Subject Classification*: Primary 08A05Gxx, 16S90; Secondary 16D10, 16N20, 16U99.

1. Ware の問題の最終報告

第 51 回環論および表現論シンポジューム (2018 年 9 月 岡山理科大学) において、下記の Ware の問題 [9] が肯定的に解けたことを報告した所、[2] に反例があるとの指摘を受けた。（詳細は [5] に記載）

Ware の問題 環 R 上の右射影 R 加群 P が唯一つの極大部分加群 L を持てば、 L は P の最大の部分加群である。よって、P は R の直和因子と同型である。

しかし、この論文 [2] は直接反例を作っているのでなく、quasi-small という条件を考え、quasi-small でない加群の特徴付けを [2] の定理 5.3 で与え、このような加群の存在は [3] にあるので、Ware の問題は否定的であると結論を出していた。

This article is the Japanese explanations of the two papers [1, 5].

この論文 [3] が入手できず、この条件を満たす例を別途構成してみた（[6] Example 5 参照)。この例を検証した所、[2] の結果が成り立たないことが分かった。この結果、シンポジュームでの報告内容 [5] が否定された訳ではないので、証明を記した論文を Proc. AMS に投稿しレフリーの判定を受け論文 [6] が掲載された。これにより、最終的に Ware 問題は肯定的に解決された。

2. 一般化された中山・東屋の補題

本稿では以下 R を環、$J(R)$ を R の Jacobson Radical とする。中山・東屋の補題とは良く知られた次の命題である。

中山・東屋の補題 M を有限生成加群あるいは射影加群とする。
このとき、M が $MJ(R) = M$ を満たせば $M = 0$ である。

前章で述べた Ware の問題の証明で、中山・東屋の補題が射影加群で成立することを使用した。この事実が成立することは [1] 等に記載がある。なお、[6] での証明はこれらとは異なる新しいものである。

この 2 つの結果を統合するものとして、次の一般化された中山・東屋の補題が成立することを示すのが本稿の主目的である。

一般化された中山・東屋の補題 M を有限生成加群の直和の直和因子とする。このとき、M が $MJ(R) = M$ を満たせば $M = 0$ が成立する。

Ware の問題の [6] での考察では、非零射影加群は極大部分加群を含むことを最初に示し、$P = PJ(R)$ を満たす射影加群は零加群であるという方向で射影加群に対する中山・東屋の補題の別証明を与えた。しかし、射影加群でない場合は、一般に極大部分加群を含まないので、この論法は使えない。射影加群 P が極大部分加群を含まないと仮定して $P = PJ(R)$ を示した後の矛盾を導いている部分の論法も、射影性の仮定をなくし、単に $M = MJ(R)$ を満たす加群 M とした場合は、証明中で用いた Zorn の補題を適用するための極小な短完全列の分離性が言えないので、やはり一般には適用できない。
このようないきさつから、全面的に方針を変え、次の 2 段階の考察を通じて一般化された中山・東屋の補題を証明した。

第一段階： 一般化された中山・東屋の補題が成立しないとすると、**中山-東屋特異加群 (NAS-加群)** と呼ばれる、次の性質を持つ自明でない加群 M が存在することを示す [7]。

(1) M の有限生成部分加群の列 $M_1 \subset M_2 \subset \cdots$ で、
$M = \bigcup_{i\in\mathbb{N}} M_i$ かつ各自然数 i について $M_i \subset M_{i+1}J(R)$ となる。
（結果的に $M = MJ(R)$ である。）

(2) 各 $(x_i) \in \oplus_{i\in\mathbb{N}} M_i$ に対し $f((x_i)) = \sum_{i\in\mathbb{N}} x_i$ と準同形 $f:\ \oplus_{i\in\mathbb{N}} M_i \to M$ を定義する。このとき、ある準同形 $g:\ M \to \oplus_{i\in\mathbb{N}} M_i$ で $fg = 1_M$ を満たすものがある。

(3) 各自然数 i について $M_i \cap g(M) = M_i \cap \ker f = 0$ となる。

第二段階： 上記の条件のうち、(1) と (2) を満たす加群 (**WNAS 加群**と呼ぶ) は零加群のみであることを示す [8]。

3. NAS 加群

この節では、第一段階で述べたように、一般化された中山・東屋の補題が成立しないとすると非零 NAS 加群が存在することの証明を与える。具体的には、有限生成加群の直和の直和因子である非零加群 M で $MJ(R) = M$ を満たす加群から NAS 加群を構成する

ことが本質的な内容である。なお、この節の内容は日中韓環論国際シンポジューム報告集[7] に記載されているが、[7] には構成方法の他に例なども載っているので併せて参照願いたい。

3.1. **準備と設定.** 加群 F を有限生成加群 $\{F_\delta\}_{\delta\in\Delta}$ の直和 $F=\oplus_{\delta\in\Delta}F_\delta$ とする。F の直和分解 $F=M\oplus N$ があり $M=MJ(R)$ であると仮定する。このとき、I. Kaplansky[4] より、M は可算生成加群の直和である。したがって、M の可算生成の直和因子 M' を取ると $M'=M'J(R)$ を満たすので、M 自身が可算生成と仮定して良い。このとき、M は可算個の F_δ の直和に入り、したがって、その直和因子になる。これにより Δ も可算集合と仮定して良い。そこで、本稿では Δ として自然数の集合 $\mathbb{N}$ を用いることにする。

次に、直和因子であることを写像の条件で記述する。$f:F\to M$ を分離全射準同形、$g:M\to F$ を $fg=1_M$ となる単射準同形とする。これに付随して、準同形 $g':F\to N$ および $f':N\to F$ があり、$1_F=gf+f'g',\ g'f'=1_N, ff'=0,\ g'g=0$ となるものがある。

3.2. **条件 (3) を満たす非零加群の構成 (Non-redundant Process).** ここでは、NAS 加群の条件 (3) を満たす非零加群で $MJ(R)=M$ を満たす加群が存在することを調べる。

そこで、$F=M\oplus N$ の商加群について、次の等式を考える。

$$\oplus_{i\in\mathbb{N}}(F_i/\oplus_{i\in\mathbb{N}}((M\cap F_i)\oplus(N\cap F_i)))=M/\oplus_{i\in\mathbb{N}}(M\cap F_i)\oplus N/\oplus_{i\in\mathbb{N}}(N\cap F_i)$$

もし、$M/\oplus_{i\in\mathbb{N}}(M\cap F_i)$ が零なら $M=\oplus_{i\in\mathbb{N}}(M\cap F_i)$ となり、$F=M\oplus N=\oplus_{i\in\mathbb{N}}(M\cap F_i)\oplus N$ であることから、各自然数 n についてモジュラー則を適用して

$$F_n=F\cap F_n=(M\cap F_n)\oplus(((\oplus_{n\neq i}(M\cap F_i))\oplus N)\cap F_n)$$

となる。したがって、$M\cap F_n$ は有限生成で $(M\cap F_n)J(R)=M\cap F_n$ から $M\cap F_n=0$ である。よって、$M=0$ となり矛盾する。

これにより、F,M,N 各々を上記の各商群と置き換えても、前提条件は成り立っているので、置き換えた加群を改めて F,M,N として良い。

3.3. **部分加群の構成（Rearrangement Process）.** ここでは、加群 $\{F_n\}$ を NAS 加群の条件 (1) を満たす M の有限生成部分加群 $\{M_n\}$ に取ることができることを調べる。

各 F_n は有限生成より $F_n=\sum_{s=1}^{i_n}a_{ns}R\ (a_{ns}\in F_n)$ とし、$x_{ns}=f(a_{ns})\in M, y_{ns}=g'(a_{ns})\in N$ とおくと、$a_{ns}=g(x_{ns})+f'(y_{ns})$ である。さらに、$p_{ns}=g(x_{ns})$ とおくと、$f(a_{ns})=f(p_{ns})=x_{ns}$ である。そこで、a_{ns} は $M\oplus N$ の元として $a_{ns}=\binom{x_{ns}}{y_{ns}}$ とかける。また、$G_n=\{p_{ns}\,|\,s=1,2,\cdots,i_n\}$ とすると、$\bigcup_{n\in\mathbb{N}}G_n$ は $g(M)$ の生成元である。ここでは、$F=M\oplus N$ の直和の元としての M の元は x_{ns}、F の元としての M の元を p_{ns} と区別して使う。

$MJ(R)=M$ より、$p_{ns}\in g(M)=g(M)J(R)\subset FJ(R)=\sum_{n\in\mathbb{N}}\oplus F_nJ(R)$ から、

$$p_{ns}=\sum_{j=1}^{t_{ns}}\sum_{k=1}^{i_{(ns)_j}}a_{(ns)_jk}r_{(ns)_jk},\ (r_{(ns)_jk}\in J(R))$$

と表される。この式に f を施すと、

$$x_{ns}=f(p_{ns})=\sum_{j=1}^{t_{ns}}\sum_{k=1}^{i_{(ns)_j}}f(a_{(ns)_jk})r_{(ns)_jk}=\sum_{j=1}^{t_{ns}}\sum_{k=1}^{i_{(ns)_j}}x_{(ns)_jk}r_{(ns)_jk}$$

となる。さらに g を施すと $p_{ns}=\sum_{j=1}^{t_{ns}}\sum_{k=1}^{i_{(ns)_j}}p_{(ns)_jk}r_{(ns)_jk}$ となる。特に、$n=s=1$ と取り $(1,1)_1=1$ と番号を入れ替えると $p_{1,1}=\sum_{j=1}^{t_{1,1}}\sum_{k=1}^{i_{(1,1)_j}}p_{(1,1)_jk}r_{(1,1)_jk}$ となる。

$r_{(1,1)_11}=r_{1,1}=0$ なら、G_1 から $p_{1,1}$ を取り去ることができる。

$r_{(1,1)_1 1} = r_{1,1} \neq 0$ なら、$p_{1,1}(1 - r_{1,1}) = \sum_{k=2}^{i_1} p_{1k} r_{1k} + \sum_{j=2}^{t_{1,1}} \sum_{k=1}^{i_{(1,1)_j}} p_{(1,1)_j k} r_{(1,1)_j k}$ となり、$r_{1,1} \in J(R)$ であるので $1 - r_{1,1}$ は可逆元である。このことから、G_1 から $p_{1,1}$ を取り去ることができる。

G_1 の元に順次この議論を繰り返し、ある自然数 s で $G_1 \subset \sum_{n=2}^{s} \sum_{k=1}^{i_n} p_{nk} R$ となる。さらに、ある自然数 s' で $\sum_{n=1}^{s} \sum_{k=1}^{i_n} p_{nk} R \subset \oplus_{1 \leq n \leq s'} F_n$ となる。
ここで、$m \leq s$ となる m について、$f'(y_{mk}) = a_{mk} - g(x_{mk}) \in \oplus_{1 \leq n \leq s'} F_n$ であるので、$\{f'(y_{nk}) \mid n = 1, \cdots s,\ k = 1, \cdots, i_n\} \subset \oplus_{1 \leq n \leq s'} F_n$ に注意すると、改めて $\oplus_{2 \leq n \leq s'} F_n$ を F_2 と置き換えて良い。

この議論を繰り返し、帰納的に $G_n \subset F_{n+1}$ となるように F_{n+1} を置き換える。さらに、これにプロセス 3.2 を適用しておく。

3.4. **NAS 加群の構成.** 上記のことから、$\binom{x}{y} \in F_i$ となる $x \in M, y \in N$ に対し、$\binom{x}{y'} \in F_{i+1}$ となる $y' \in N$ があることがわかる。この y' はただ一つ決まることに注意する。なぜなら、$\binom{x}{y''} \in F_{i+1}$ があれば、両者の差を取り $\binom{0}{y'-y''} \in F_{i+1}$、すなわち、$y' - y'' \in N \cap F_{i+1}$ より、プロセス 3.2 から $y' - y'' = 0$ である。同様に、$x \in M$ 対し、$\binom{x}{y} \in F_i$ なら、この $y \in N$ もただ一つ決まるので、これを $y_x^{(i)}$ とおく。これから、$x, x' \in M, r \in R$ に対し、

$$(1)\ y_x^{(i)} r = y_{xr}^{(i)}, \quad (2)\ y_x^{(i)} + y_{x'}^{(i)} = y_{x+x'}^{(i)}$$

が成立する。よって、$f_i = f | F_i : F_i \to M,\ f_i\binom{x}{y} = x,\ M_i = \mathrm{Im} f_i$ とすると $f_i : F_i \to M_i$ は同型写像を与えることがわかる。実際、$\epsilon_i = f_i^{-1} : M_i \to F_i$ は $\epsilon_i(x) = \binom{x}{y_x^{(i)}}$ で与えられる。このことから、F_i と M_i を同一視する。

また、N についても $g_i' = g' | F_i : F_i \to N,\ N_i = \mathrm{Im}\ g_i'$ とする。このとき、上記 (1), (2) から $\alpha_i : M_i \to N_i,\ \alpha_i(x) = y_x^{(i)}$ は準同形を与えることがわかる。

先の対応 $F_i \to F_{i+1} : \binom{x}{y_x^{(i)}} \mapsto \binom{x}{y_x^{(i+1)}}$ は単射準同形 $\beta_i : M_i \to M_{i+1},\ \beta_i(x) = x$ および $\gamma_i : N_i \to N_{i+1},\ \gamma_i(y_x^{(i)}) = y_x^{(i+1)}$ を与えるので、これにより $M_i \subset M_{i+1},\ N_i \subset N_{i+1}$ と考える。また、構成から $G_i \subset F_{i+1} J(R)$ であったので、$M_i \subset M_{i+1} J(R)$ である。また、$\bigcup_{j \in \mathbb{N}} M_j = M$ も構成方法から成立している。

4. WNAS 加群

この節では第二段階として WNAS 加群は零加群のみであることを示す。これにより、一般化された中山・東屋の補題が成立することがわかる。

4.1. **第二段階の設定.** $F = \oplus_{i \in \mathbb{N}} M_i = M \oplus N$ について、各 M_i は M の有限生成部分加群で、$M_i \subset M_{i+1} J(R)$ かつ $\bigcup_{i \in \mathbb{N}} M_i = M$ を満たすとする。必然的に $M = MJ(R)$ である。また、第一段階で用いた次の準同形の記号を用いる。
$f :\ F \to M$ と $g :\ M \to F$ は $f((x_i)) = \sum_{i \in \mathbb{N}} x_i,\ fg = 1_M$ で定義された準同形、
$g' : F \to N$ と $f' : N \to F$ は F が M と N の直和であることを示す準同形、
すなわち、$1_F = gf + f'g',\ g'f' = 1_N, ff' = 0,\ g'g = 0$ を満たし、次の完全列を与える。

$$0 \longrightarrow N \xrightarrow{f'} F \xrightarrow{f} M \longrightarrow 0.$$

$$0 \longleftarrow N \xleftarrow{g'} F \xleftarrow{g} M \longleftarrow 0.$$

さらに、第一段階の議論から各 i について、次のように仮定してよい。

$$g(M_i) \subset M_1 \oplus \cdots \oplus M_i \oplus M_{i+1},\ M \cap M_i = N \cap M_i = 0.$$

これを元に次のような設定をする。

$M_0=0$ として $F[1]=M_0\oplus F$ とおく。各自然数 i について、$\alpha_i\colon M_i\to M_i/M_{i-1}$ を自然な全射準同形とし、これを直和への準同形 $\alpha=(\alpha_i)\colon F\to F/F[1]=M_1\oplus M_2/M_1\oplus\cdots$ に拡張する。また、$p_i\colon F\to M_i$ を第 i 成分への射影とする。

$p_{i+1}g|M_i\colon M_i\to F\to M_{i+1}\subset M$ を考える。$g(M_{i-1})\subset M_1\oplus\cdots\oplus M_i$ であることに注意すると、$p_{i+1}g(M_{i-1})=0$ である。よって、$p_{i+1}g|M_i$ は $\overline{f}_i\ :M_i/M_{i-1}\to M$ を導き、$p_{i+1}g|M_i=\overline{f}_i\alpha_i$ となる。そこで、$\overline{f}\colon F/F[1]\to M$ を $\overline{f}(\alpha(x))=\sum_{i\in\mathbb{N}}\overline{f}_i(\alpha_ip_i(x))$、$h$ を $h=\overline{f}\alpha g\colon M\to F\to F/F[1]\to\mathrm{Im}(\overline{f})$ と決める。

4.2. **生成元の挙動.** この節では F,M,N,M_i $(i\in\mathbb{N})$ の生成元の関係を調べる。

最初に次の補題を挙げておく。これは $M_i\subset M_{i+1}J(R)$ より中山・東屋の補題を用いて容易にわかる。

補題１ $M_i/M_{i-1}=\alpha_i(y_{i1})R+\cdots+\alpha_i(y_{it_i})R$ とすると $M_i=y_{i1}R+\cdots+y_{it_i}R$ となる。

そこで、$M_i=a_{i1}R+\cdots+a_{it_i}R,\ (i\in\mathbb{N})$ かつ $\alpha_i(a_{ij})\neq\alpha_i(0)$ となる生成元 $\{a_{i1},\cdots,a_{it_i}\}$ が取れる。また、$g(a_{ij})$ の F での成分表示を次のように定義する。

$$g(a_{ij})=(y_1^{(ij)},\cdots,y_i^{(ij)},y_{i+1}^{(ij)},0,\cdots)\in F=\bigoplus_{i\in\mathbb{N}}M_i.\qquad\text{(A)}$$

これらについて、次のような関係式がある。

補題２ 次のことが成立する。ただし、$i\in\mathbb{N},1\le j\le t_i$ である。

(1) $a_{ij}=y_1^{(ij)}+\cdots+y_i^{(ij)}+y_{i+1}^{(ij)}$.
(2) $g(a_{ij})=g(y_1^{(ij)})+\cdots+g(y_i^{(ij)})+g(y_{i+1}^{(ij)})$.
(3) $\alpha_i(y_i^{(ij)})=\alpha_i(a_{ij})-\alpha_i(y_{i+1}^{(ij)})$.
(4) $x\in M_i$ とし、$x=a_{i1}r_1+\cdots+a_{it_i}r_{t_i}$ $(r_1,\cdots,r_{t_i}\in R)$ を $F=\bigoplus_{i\in\mathbb{N}}M_i$ の生成元で表した関係式とする。各 $1\le k\le i+1$ について $p_kg(x)=y_k^{(i1)}r_1+\cdots+y_k^{(it_i)}r_{t_i}$.
(5) $y_{i+1}^{(ij)}\in M_i$.

証明 (1) は $fg=1_M$ より従い、(2) はこの式に g を施したものである。

(3) $y_1^{(ij)},\cdots,y_{i-1}^{(ij)}\in M_{i-1}$ より、$\alpha_i(y_1^{(ij)}),\cdots,\alpha_i(y_{i-1}^{(ij)})$ は全て $\alpha_i(0)$ と一致するので

$$\begin{aligned}\alpha_i(y_i^{(ij)})&=\alpha_i(y_1^{(ij)})+\cdots+\alpha_i(y_{i-1}^{(ij)})+\alpha_i(y_i^{(ij)})\\&=\alpha_i(y_1^{(ij)}+\cdots+y_{i-1}^{(ij)}+y_i^{(ij)}+y_{i+1}^{(ij)})-\alpha_i(y_{i+1}^{(ij)})=\alpha_i(a_{(ij)})-\alpha_i(y_{i+1}^{(ij)})\end{aligned}$$

となるので (3) が成立する。

(4) x の関係式に p_kg を施すと

$$p_kg(x)=p_k(g(a_{i1}))r_1+\cdots+p_k(g(a_{it_i}))r_{t_i}=y_k^{(i1)}r_1+\cdots+y_k^{(it_i)}r_{t_i}$$

であるので (4) が成立する。

(5) 上記 (1) より $y_{i+1}^{(ij)}=a_{ij}-(y_1^{(ij)}+\cdots+y_i^{(ij)})$ である。
一方、$a_{ij}\in M_i$ かつ $y_k^{(ij)}\in M_k\subset M_i$ $(1\le k\le i)$ なので (5) が成立する。 □

上記の式 (A) において、$y_s^{(ij)}\in M_s$ であるので、$g(y_s^{(ij)})$ の F での成分表示を

$$g(y_s^{(ij)})=(z_{s1}^{(ij)},\cdots,z_{s\,s+1}^{(ij)},0,\cdots)$$

とおく。補題２ (5) より $y_{i+1}^{(ij)}\in M_i$ なので、最終項については $z_{i+1\,i+2}^{(ij)}=0$ であることを注意しておく。

次の補題は、準同形 $\overline{f}\alpha g$ に関して、具体的に関係式を記したものである。

補題３ 次のことが成立する。

(1) $\overline{f}(\alpha(g(a_{ij}))) = \sum_{k=1}^{i} z_{k\,k+1}^{(ij)}$ である。

(2) $0 \neq x \in M$ に対し、$x \in M_i$ を満たす最小の自然数を i とし、x を M_i の生成元を用いて $x = a_{i1}r_1 + \cdots + a_{it_i}r_{t_i}$ $(r_1, \cdots, r_{t_i} \in R)$ と表す。このとき、
$\overline{f}(\alpha(g(x)) = \sum_{j=1}^{t_i}(\sum_{k=1}^{i} z_{k\,k+1}^{(ij)})r_j$ である。

(3) 上記 x について、$g(x) = (y_1^{(x)}, \cdots, y_i^{(x)}, y_{i+1}^{(x)}, 0, \cdots)$ を $F = \bigoplus_{i\in\mathbb{N}} M_i$ での成分表示とする。このとき、
$\overline{f}(\alpha(g(x)))$ は、$1 \leq k \leq i$ に対する $g(y_k^{(x)})$ の第 $k+1$ 成分の総和である。

証明 (1) $\alpha(g(a_{ij})) = (\alpha_1(y_1^{(ij)}), \cdots, \alpha_i(y_i^{(ij)}), \alpha_{i+1}(y_{i+1}^{(ij)}), 0, \cdots) \in F/F[1]$ であったので、$\overline{f}(\alpha(g(a_{ij}))) = \overline{f}_1(\alpha_1(y_1^{(ij)})) + \cdots + \overline{f}_i(\alpha_i(y_i^{(ij)})) + \overline{f}_{i+1}(\alpha_{i+1}(y_{i+1}^{(ij)}))$ となる。
また、定義から $\overline{f}_k(\alpha_k(y_k^{(ij)})) = z_{k\,k+1}^{(ij)}$ $(1 \leq k \leq i+1)$ であり、特に $k = i+1$ のときは $z_{i\,i+1}^{(ij)} = 0$ より (1) が成立する。

(2) $\overline{f}(\alpha(g(x)) = \sum_{j=1}^{t_i} \overline{f}(\alpha(g(a_{ij})))r_j$ に (1) の式を代入して得られる。

(3) $x \in M_i$ を生成元を用いて $x = a_{i1}r_1 + \cdots + a_{it_i}r_{t_i}$ $(r_1, \cdots, r_{t_i} \in R)$ と表し g を施した式 $g(x) = g(a_{i1})r_1 + \cdots + g(a_{it_i})r_{t_i}$ を考える。$g(a_{ij}) = (y_1^{(ij)}, \cdots, y_i^{(ij)}, y_{i+1}^{(ij)}, 0, \cdots)$ なので、$y_k^{(x)} = p_k(g(x)) = \sum_{j=1}^{t_i} y_k^{(ij)} r_k$ となる。したがって、$g(y_k^{(x)}) = \sum_{j=1}^{t_i} g(y_k^{(ij)})r_k$ であることから、$g(y_k^{(x)})$ の第 $k+1$ 成分は $p_{k+1}(g(y_k^{(x)})) = \sum_{j=1}^{t_i} p_{k+1}(g(y_k^{(ij)}))r_k = \sum_{j=1}^{t_i} z_{k\,k+1}^{(ij)} r_k$ となる。このことから、 $g(y_1^{(x)}), \cdots, g(y_i^{(x)})$ の第 $k+1$ 成分の総和は $\sum_{k=1}^{i}\sum_{j=1}^{t_i} z_{k\,k+1}^{(ij)} r_k$ で、和の順序を入れ替え $\sum_{j=1}^{t_i}(\sum_{k=1}^{i} z_{k\,k+1}^{(ij)})r_j$ となる。したがって、 (2) の式から、これは $\overline{f}(\alpha(g(x))$ と一致し (3) が成立する。 □

5. 一般化された中山・東屋の補題の証明

この章では WNAS 加群は零加群であることを証明する。これにより、第３章の結果から一般化された中山・東屋の補題が成立することがわかる。

5.1. 加群 N の生成元と写像 $\overline{f}\alpha g$.

まず、$f'(N) \cong N$ の F での生成元を調べる。

補題４ 集合 $\{g(a_{ij}) - (0_1, \cdots, 0_{i-1}, a_{ij}, 0, \cdots) \mid i \in \mathbb{N}, 1 \leq j \leq t_i\}$ は加群 $f'(N)$ の生成元をなす。

証明 ほぼ一般論であるが証明を与えておく。
$N = \ker f = g'(F)$ より、$f'(N) = f'g'(F)$ である。任意の元 $x = (x_1, \cdots, x_n, 0, \cdots) \in F$ に対し、$x = f'g'(x) + gf(x) = f'g'(x) + \sum_{i=1}^{n} g(x_i)$ となるので、

$$f'g'(x) = \sum_{i=1}^{n}((0_1, \cdots, 0_{i-1}, x_i, 0, \cdots) - g(x_i))$$

である。x_i は $a_{i1}, \cdots, a_{it_i}$ で生成されるので、目的の集合が $f'(N)$ の生成元になる。 □

補題５ 次のことが成立する。

(1) $\overline{f}\alpha f' = 0.$

(2) $hf = \overline{f}\alpha.$

(3) $\mathrm{Im}(h) = \mathrm{Im}(\overline{f})$ である。したがって、$h\colon\ M \to \mathrm{Im}(\overline{f})$ は全射準同形である。

証明 (1) $f'(N)$ は $\{g(a_{ij}) - (0_1, \cdots, 0_{i-1}, a_{ij}, 0, \cdots)\}$ で生成されているので、
$\overline{f}\alpha(g(a_{ij}) - (0_1, \cdots, 0_{i-1}, a_{ij}, 0, \cdots)) = 0$ を示せばよい。
補題 3(1) より $\overline{f}\alpha(g(a_{ij})) = \sum_{k=1}^{i} z_{k\,k+1}^{(ij)}$ である。

一方、$\overline{f}\alpha((0_1,\cdots,0_{i-1},a_{ij},0,\cdots))=\overline{f}_i\alpha_i(a_{ij})=y_{i+1}^{(ij)}$ であり、
$g(y_{i+1}^{(ij)})=(z_{i+1\,1}^{(ij)},\cdots,z_{i+1\,i+2}^{(ij)},0,\cdots)$ から $y_{i+1}^{(ij)}=fg(y_{i+1}^{(ij)})=\sum_{s=1}^{i+1}z_{i+1\,s}^{(ij)}=\sum_{s=1}^{i}z_{i+1\,s}^{(ij)}$ となり、(1) が成立する。

(2) (1) より $hf=\overline{f}\alpha gf=\overline{f}\alpha gf+(\overline{f}\alpha f')g'=\overline{f}\alpha(gf+f'g')=\overline{f}\alpha 1_F=\overline{f}\alpha$ となり (2) が成立する。

(3) $F=f'(N)\oplus g(M)$ が直和であることと、(1) より $\overline{f}\alpha(f'(N))=0$ であることから $\overline{f}\alpha(F)=\overline{f}\alpha(g(M))$ である。$\overline{f}\alpha\colon F\to F/F[1]\to \mathrm{Im}(\overline{f})$ が全射準同形より $h=\overline{f}\alpha g$ も全射準同形となる。 □

5.2. **WNAS 加群は零加群.** 最終目的である次の定理を示す。

定理 6 非零な WNAS 加群は存在しない。特に、非零な NAS 加群は存在しない。

証明 非零な WNAS 加群 M が存在するとする。自然な単射準同形を $\beta_1\colon F[1]\to F$ および $\beta_2\colon \ker h\to M$ とすると、補題 5 より短完全列よりなる次の可換な図式をうる。

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F[1] & \xrightarrow{\beta_1} & F & \xrightarrow{\alpha} & F/F[1] & \longrightarrow & 0\\
 & & & & {\scriptstyle f}\big\downarrow & & {\scriptstyle \overline{f}}\big\downarrow & & \\
0 & \longrightarrow & \ker h & \xrightarrow{\beta_2} & M & \xrightarrow{h} & \mathrm{Im}(\overline{f}) & \longrightarrow & 0.
\end{array}$$

したがって、$\beta_2(f|F[1])=f\beta_1$ である。一方、$f|F[1](F[1])=M$ より $\ker h=M$、すなわち $\mathrm{Im}(\overline{f})=0$ となるので、各自然数 i について、$\overline{f}(a_{ij})=y_{i+1}^{(ij)}=0$ である。これにより各 $x\in M_i$ で $g(x)\in M_1\oplus\cdots\oplus M_i$ となる。$g(a_{ij})=g(y_1^{(ij)})+\cdots+g(y_{i-1}^{(ij)})+g(y_i^{(ij)})$ および $g(y_1^{(ij)})+\cdots+g(y_{i-1}^{(ij)})\in M_1\oplus\cdots\oplus M_{i-1}$ から

$$y_i^{(ij)}=p_i(g(a_{(ij)}))=p_i(g(y_i^{(ij)}))=z_{ii}^{(ij)} \qquad (B)$$

をうる。よって、$\alpha_i(a_{ij}-y_i^{(ij)})=0$ で、全ての自然数 i で $M_i/M_{i-1}=\alpha_i(y_i^{(i1)})R+\cdots+\alpha_i(y_i^{(it_i)})R$ となる。これから、補題 1 より M_i は $\{y_i^{(i1)},\cdots,y_i^{(it_i)}\}$ によって生成されることがわかる。そこで、改めて生成元 $\{a_{ij}\}$ として $\{y_i^{(ij)}\}$ を取ることにする。このとき、上記の関係式 (B) より $p_i(g(a_{ij}))=a_{ij}$ が成立する。

いま、$a_{i-1\,1}$ を M_i の生成元で表した式 $a_{i-1\,1}=a_{i1}r_1+\cdots+a_{it_i}r_{t_i}$ $(r_1,\cdots,r_{t_i}\in R)$ に準同形 p_ig を施すと、$p_ig(a_{i-1\,1})=p_i(g(a_{i1}))r_1+\cdots+p_i(g(a_{it_i}))r_{t_i}=a_{i1}r_1+\cdots+a_{it_i}r_{t_i}=a_{i-1\,1}$ となる。一方、$a_{i-1\,1}\in M_{i-1}$ であることから $p_i(g(a_{i-1\,1}))=0$ より、$a_{i-1\,1}=0$ となり矛盾が生じる。 □

最後に主結果とその直接の帰結を述べておく。

定理 7(一般化された中山・東屋の補題) M を有限生成加群の直和の直和因子とする。このとき、M が $MJ(R)=M$ を満たせば $M=0$ が成立する。

系 8 非零加群 M の部分加群 $M_i(i\in\mathbb{N})$ が各 i で $M_i\subset M_{i+1}J(R)$ を満たし、かつ、$M=\bigcup_{i\in\mathbb{N}}M_i$ とする。このとき、$(x_i)\in\bigoplus_{i\in\mathbb{N}}M_i$ に対し、$f((x_i))=\sum_{i\in\mathbb{N}}x_i$ で与えられる自然な準同形 $f\colon \bigoplus_{i\in\mathbb{N}}M_i\to M$ は分離しない。

6. 極大部分加群の存在に関する応用

ここでは一般化された中山・東屋の補題から得られる極大部分加群の存在に関する結果を挙げておく。証明は [7] を参照されたい。

定理９ 加群 F を有限生成加群 $\{M_\delta\}_{\delta\in\Delta}$ の直和 とし、M を F の自明でない部分加群で $M \not\subset \mathrm{rad}\ F$ とする。このとき、 M は極大部分加群を含む。

系１０ 自由加群 F の自明でない部分加群を M とする。$M \not\subset FJ(R)$ なら M は極大部分加群を含む。特に零でない射影加群は極大部分加群を含む。

上記で条件 $M \not\subset FJ(R)$ は必須である。詳細は [7] Example 2.1 を参照願いたい。

定理１１ M を有限生成加群の直和の零でない直和因子で $MJ(R) = \mathrm{rad}\ M$ とすると、M は極大部分加群を含む。特に、$M_\delta J(R) = \mathrm{rad}\ M_\delta$ を満たす有限生成加群 M_δ の直和 $F = \oplus_{\delta\in\Delta} M_\delta$ の任意の直和因子は極大部分加群を含む。

7. 一般化された中山・東屋の補題の証明の短縮化

一般化された中山・東屋の補題の主張は明確で分かりやすい。しかし、それに比して、ここで与えた証明は長く複雑である。そこで、短く明快な証明が見出されることを期待したい。

References

[1] F.W. Anderson, K.R. Fuller, *Rings and Categories of Modules*, GTM **13**, Springer-Verlag (1992).

[2] A. Facchini, D. Herbera, I. Sakhajev, *Finitely Generated Flat Modules and a Characterization of Semiperfect Rings*, Comm. in Algebra, Vol.**31** No.9(2003), 4195–4214.

[3] V.N. Gerasimov, I.I. Sakhaev, *A Counterexample to two Hypotheses on Projective and Flat Modules*, (Russian). Sibirsk Math. Zh. 25(6), 31-35. English Translation: Sibirsk Math. J. **24** , 855–859, 1984.

[4] I. Kaplansky, *Projective modules*, Ann. of Math. **68**, 372–377 (1958).

[5] M. SATO, *On Projective Modules with Unique Maximal submodules* , Proc. of the 51st Symposium on Ring Theory and Representation Theory, pp.135-145, 2019.

[6] ———, *Projective Modules with Unique Maximal submodules are cyclic*, Proc. of Amer. Math. Soc, **148**(9), pp.3673-3684, 2020.

[7] ———, *On Generalized Nakayama-Azumaya Lemma and NAS-Modules*, Ring theory 2019(Proc. of the 8th China-Japan-Korea International Conference) , pp.188–202, World Scientific Publishing (2021).

[8] ———, *On Generalized Nakayama-Azumaya's Lemma*, Comm. in Algebra, Vol.**50** No.5, 2037–2044, 2022.

[9] R. Ware, *Endomorphism rings of projective modules*, Trans. Amer. Math. Soc. **155** , 233-256, 1971.

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DIMITROV–HAIDEN–KATZARKOV–KONTSEVICH COMPLEXITIES FOR SINGULARITY CATEGORIES

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Abstract. Dimitrov, Haiden, Katzarkov and Kontsevich have introduced the notion of complexities for arbitrary triangulated categories. This paper deals with complexities for singularity categories.

1. Preliminaries

In this section, we work on a general triangulated category.

Setup 1. Throughout this section, let $\mathcal{T}$ be a triangulated category. All subcategories of $\mathcal{T}$ are assumed to be strictly full. We may omit a subscript if it is clear from the context.

We introduce the operation $\star$ for subcategories of $\mathcal{T}$, which plays a central role throughout the paper.

Definition 2. Let $\mathcal{X}$ and $\mathcal{Y}$ be subcategories of $\mathcal{T}$.
(1) We denote by $\mathcal{X} \star \mathcal{Y}$ the subcategory of $\mathcal{T}$ consisting of objects $T \in \mathcal{T}$ such that there exists an exact triangle $X \to T \to Y \rightsquigarrow$ in $\mathcal{T}$ such that $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.
(2) When $\mathcal{X}, \mathcal{Y}$ consist of single objects X, Y respectively, we simply write $X \star Y$ to denote $\mathcal{X} \star \mathcal{Y}$.

In the following lemma, we make a list of several fundamental properties of the operation $\star$. The first assertion says that the operation $\star$ satisfies associativity. The second and third assertions state that the operation $\star$ is compatible with taking finite direct sums and shifts. The proof is standard.

Lemma 3. (1) *For subcategories $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ of $\mathcal{T}$ one has $(\mathcal{X} \star \mathcal{Y}) \star \mathcal{Z} = \mathcal{X} \star (\mathcal{Y} \star \mathcal{Z})$. Hence, there is no ambiguity in writing $\bigstar_{i=1}^{n} \mathcal{X}_i = \mathcal{X}_1 \star \cdots \star \mathcal{X}_n$ for subcategories $\mathcal{X}_1, \dots, \mathcal{X}_n$ of $\mathcal{T}$ or $\mathcal{X}^{\star n} = \underbrace{\mathcal{X} \star \cdots \star \mathcal{X}}_{n}$.*
(2) *Let $\{X_{ij}\}_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n}$ and $\{M_i\}_{1 \leqslant i \leqslant m}$ be families of objects of $\mathcal{T}$. Suppose that $M_i \in \bigstar_{j=1}^{n} X_{ij}$ for each $1 \leqslant i \leqslant m$. Then it holds that $\bigoplus_{i=1}^{m} M_i \in \bigstar_{j=1}^{n} (\bigoplus_{i=1}^{m} X_{ij})$.*
(3) *Let $X_1, \dots, X_n \in \mathcal{T}$. Then the following statements hold true.*
  (a) *If $M \in \bigstar_{i=1}^{n} X_i$, then $M[s] \in \bigstar_{i=1}^{n} X_i[s]$ for all integers s, $M^{\oplus m} \in \bigstar_{i=1}^{n} X_i^{\oplus m}$ for all positive integers m, and $M \oplus (\bigoplus_{i=1}^{n} Y_i) \in \bigstar_{i=1}^{n} (X_i \oplus Y_i)$ for all objects $Y_1, \dots, Y_n \in \mathcal{T}$.*
  (b) *One has the containment $\bigoplus_{i=1}^{n} X_i \in \bigstar_{i=1}^{n} X_i$.*

Here we recall the definition of split generators, which are used to define complexities and entropies.

The detailed version [10] of this paper has been submitted for publication elsewhere.

Definition 4. (1) A *thick subcategory* of $\mathcal{T}$ is by definition a triangulated subcategory of $\mathcal{T}$ closed under direct summands, i.e., a subcategory closed under shifts, mapping cones and direct summands.
(2) For an object $X \in \mathcal{T}$ we denote by $\mathsf{thick}_{\mathcal{T}} X$ the *thick closure* of T, that is to say, the smallest thick subcategory of $\mathcal{T}$ to which X belongs.
(3) A *split generator* of $\mathcal{T}$, which is also called a *thick generator* of $\mathcal{T}$, is defined to be an object of $\mathcal{T}$ whose thick closure coincides with $\mathcal{T}$.

Now we can state the definitions of complexities and entropies introduced in [5].

Definition 5 (Dimitrov–Haiden–Katzarkov–Kontsevich)**.**
(1) Let $X, Y \in \mathcal{T}$ and $t \in \mathbb{R}$. We denote by $\delta_t(X,Y)$ the infimum of the sums $\sum_{i=1}^{r} \mathrm{e}^{n_i t}$, where r runs through the nonnegative integers and n_i run through the integers such that there exist a sequence

$$\begin{tikzcd}
0 \arrow[r, equal] & Y_0 \arrow[rr] & & Y_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & Y_{r-1} \arrow[rr] & & Y_r \arrow[r, equal] \arrow[dl] & Y \oplus Y' \\
& & X[n_1] \arrow[ul, squiggly] & & \cdots & & X[n_r] \arrow[ul, squiggly] & &
\end{tikzcd}$$

of exact triangles $\{Y_{i-1} \to Y_i \to X[n_i] \rightsquigarrow\}_{i=1}^{r}$ in $\mathcal{T}$. The function $\mathbb{R} \ni t \mapsto \delta_t(X,Y) \in \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ is called the *complexity* of Y relative to X. When $Y = 0$, one can take $r = 0$, and hence $\delta_t(X,Y) = 0$.
(2) Let $F : \mathcal{T} \to \mathcal{T}$ be an exact functor and $t \in \mathbb{R}$. The *entropy* $\mathrm{h}_t(F)$ of F is defined by

$$\mathrm{h}_t(F) = \lim_{n\to\infty} \tfrac{1}{n} \log \delta_t(G, F^n(G)),$$

where G is a split generator of $\mathcal{T}$. This is independent of the choice of G; see [5, Lemma 2.6].

The following proposition gives an equivalent definition of a complexity.

Proposition 6. *Let $X, Y \in \mathcal{T}$ and $t \in \mathbb{R}$. One then has the equality*

$$\delta_t(X,Y) = \inf\{\textstyle\sum_{i=1}^{r} \mathrm{e}^{n_i t} \mid Y \oplus Y' \in \bigstar_{i=1}^{r} X[n_i] \text{ for some } Y' \in \mathcal{T}\}.$$

We give a couple of statements concerning complexities. Recall that $\mathcal{T}$ is said to be *periodic* if there exists an integer $n > 0$ such that the nth shift functor $[n]$ is isomorphic to the identity functor $\mathrm{id}_{\mathcal{T}}$ of $\mathcal{T}$.

Proposition 7. *Let X and Y be objects of $\mathcal{T}$. Then the following statements hold.*
(1) *Let $t \in \mathbb{R}$. Then $\delta_t(X,Y) < \infty$ if and only if $Y \in \mathsf{thick}_{\mathcal{T}} X$.*
(2) *There is an equality $\delta_0(X,Y) = \inf\{r \in \mathbb{Z}_{\geqslant 0} \mid Y \oplus Y' \in \bigstar_{i=1}^{r} X[n_i]$ for some $Y' \in \mathcal{T}\}$.*
(3) *Let $t \in \mathbb{R}$. Suppose that $\mathcal{T}$ is periodic and $\delta_t(X,Y) < \infty$. Then $\delta_t(X,Y) = 0$ unless $t = 0$.*

Remark 8. The equality in Proposition 7(2) may remind the reader of the notion of a *level* introduced by Avramov, Buchweitz, Iyengar and Miller [2]. Namely, $\delta_0(X,Y)$ looks closely related to the X-level $\mathrm{level}_{\mathcal{T}}^{X}(Y)$ of Y. The difference is that an X-level ignores finite direct sums of copies of X. This is similar to the difference between the lengths of a composition series and a Loewy series of a module over a ring. The complexity $\delta_t(X,Y)$ can also be regarded as a weighted version of $\delta_0(X,Y)$ with respect to shifts.

The following lemma comes from [5, Proposition 2.2]. In this proposition, neither $\delta_t(X,Y)$ nor $\delta_t(Y,Z)$ is assumed to be finite, but in its proof both $\delta_t(X,Y)$ and $\delta_t(Y,Z)$ seem to be assumed to be finite. In fact, without this assumption, we would need to clarify what $0\cdot\infty$ and $\infty\cdot 0$ mean.

Lemma 9. *Let t be a real number. Let X, Y and Z be objects of $\mathcal{T}$. Suppose that both $\delta_t(X,Y)$ and $\delta_t(Y,Z)$ are finite. Then there is an inequality $\delta_t(X,Z)\leqslant\delta_t(X,Y)\cdot\delta_t(Y,Z)$.*

2. Main results

In this section, we shall investigate complexities and entropies for the singularity category of a commutative noetherian local ring, which is a triangulated category.

Setup 10. Throughout this section, let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. The triangulated category considered in this section is the *singularity category* $\mathsf{D}_{\mathsf{sg}}(R)$ of R, which is by definition the Verdier quotient of the bounded derived category of finitely generated R-modules by perfect complexes (i.e., bounded complexes of finitely generated projective R-modules).

We recall several fundamental notions from commutative algebra, whose details can be found in [1, 3].

Definition 11. (1) We say that R is a *singular* local ring if it is not a regular local ring. Note that R is singular if and only if the category $\mathsf{D}_{\mathsf{sg}}(R)$ is nonzero.

(2) The *codimension* and the *codepth* of R are defined by $\operatorname{codim} R = \operatorname{edim} R - \dim R$ and $\operatorname{codepth} R = \operatorname{edim} R - \operatorname{depth} R$. Here, $\operatorname{edim} R$ and $\operatorname{depth} R$ stand for the embedding dimension of R and the depth of R, respectively. Note that $\operatorname{codim} R = \operatorname{codepth} R$ if (and only if) R is Cohen–Macaulay.

(3) The local ring R is said to be a *hypersurface* provided the inequality $\operatorname{codepth} R \leqslant 1$ holds. According to Cohen's structure theorem, this condition is equivalent to saying that the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is isomorphic to the residue ring $S/(f)$ of some regular local ring S by some principal ideal (f).

(4) The local ring R is called a *complete intersection* if the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is isomorphic to the residue ring $S/(\boldsymbol{f})$ of a regular local ring $(S,\mathfrak{n})$ by the ideal $(\boldsymbol{f})$ generated by a regular sequence $\boldsymbol{f}=f_1,\dots,f_c$. One can choose $\boldsymbol{f}=f_1,\dots,f_c$ so that $c=\operatorname{codim} R$, and in this case, $f_i\in\mathfrak{n}^2$ for all i.

(5) The *Koszul complex* K^R of R is defined to be the Koszul complex $\mathrm{K}(\boldsymbol{x},R)$ on R of a minimal system of generators $\boldsymbol{x}=x_1,\dots,x_n$ of $\mathfrak{m}$. This complex is uniquely determined up to isomorphism; see [3, the part following Remark 1.6.20]. Each homology $\mathrm{H}_i(\mathrm{K}^R)$ is a finite-dimensional k-vector space.

(6) We say that R has an *isolated singularity* if $R_\mathfrak{p}$ is a regular local ring for all $\mathfrak{p}\in\operatorname{Spec} R\setminus\{\mathfrak{m}\}$.

(7) Let $\mathrm{e}(R)$ and $\mathrm{r}(R)$ be the *(Hilbert–Samuel) multiplicity* and *type* of R, respectively. One has R is singular if and only if $\mathrm{e}(R)>1$, and R is Gorenstein if and only if R is Cohen–Macaulay and $\mathrm{r}(R)=1$.

(8) Let M be a finitely generated R-module. Let n be a nonnegative integer. Then we denote by $\Omega_R^n M$ the nth *syzygy* of M over R, that is, the image of the nth differential

map in a minimal free resolution of the R-module M. Note that the module $\Omega_R^n M$ is uniquely determined up to isomorphism. We denote by $\beta_n^R(M)$ the nth *Betti number* of M, namely, the minimal number of generators of $\Omega_R^n M$.

(9) For an R-module M, we denote by $\ell(M)$ the *length* of (a composition series of) M.

What we want to consider in this section is the following conjecture.

Conjecture 12. Let G be a split generator of $\mathsf{D}_{\mathsf{sg}}(R)$. Then one has the equality $\delta_t(G, X) = 0$ for all objects X of $\mathsf{D}_{\mathsf{sg}}(R)$ and for all nonzero real numbers t.

In the case where R is a hypersurface, it is easy to see that Conjecture 12 holds true.

Example 13. If R is a hypersurface, then $\delta_t(G, X) = 0$ for all split generators G of $\mathsf{D}_{\mathsf{sg}}(R)$, for all $X \in \mathsf{D}_{\mathsf{sg}}(R)$ and for all $0 \neq t \in \mathbb{R}$. Indeed, in this case, there exists an isomorphism $\widehat{R} \cong S/(f)$, where S is a regular local ring and $f \in S$. The singularity category $\mathsf{D}_{\mathsf{sg}}(\widehat{R})$ of the completion $\widehat{R}$ is equivalent as a triangulated category to the homotopy category of matrix factorizations of f over S, which is periodic of periodicity two; we refer the reader to [4, 6, 7, 8, 11] for the details. It is easy to see that $\mathsf{D}_{\mathsf{sg}}(R)$ is also periodic of periodicity two, and the assertion follows from (1) and (3) of Proposition 7.

We introduce a condition on an object of the singularity category, which is essential in our theorems.

Definition 14. We say that an object X of $\mathsf{D}_{\mathsf{sg}}(R)$ is *locally zero on the punctured spectrum of R* if for each nonmaximal prime ideal $\mathfrak{p}$ of R the localized complex $X_\mathfrak{p}$ is isomorphic to 0 in the singularity category $\mathsf{D}_{\mathsf{sg}}(R_\mathfrak{p})$ of the local ring $R_\mathfrak{p}$. This condition is equivalent to saying that $X_\mathfrak{p}$ is isomorphic to a perfect complex over $R_\mathfrak{p}$ in the bounded derived category of finitely generated $R_\mathfrak{p}$-modules.

Remark 15. Suppose that R has an isolated singularity. Then every object of $\mathsf{D}_{\mathsf{sg}}(R)$ is locally zero on the punctured spectrum of R, since $\mathsf{D}_{\mathsf{sg}}(R_\mathfrak{p}) = 0$ for all nonmaximal prime ideals $\mathfrak{p}$ of R.

We establish a lemma, whose proof is done by [9, Corollary 4.3(3)], Proposition 7(1) and Lemma 9.

Lemma 16. *Let $t \in \mathbb{R}$. Let X be an object of $\mathsf{D}_{\mathsf{sg}}(R)$ such that k belongs to $\mathsf{thick}_{\mathsf{D}_{\mathsf{sg}}(R)} X$. Let Y be an object of $\mathsf{D}_{\mathsf{sg}}(R)$ which is locally zero on the punctured spectrum of R. If $\delta_t(k, k) = 0$, then $\delta_t(X, Y) = 0$.*

Now we shall state three theorems, all of which support Conjecture 12. The proofs use Lemma 3, Lemma 16, [1, Theorem 8.1.2], and fundamental properties of Koszul complexes and multiplicities stated in [3]. For the details of the proofs of the theorems, we refer the reader to [10].

Theorem 17. *Let R be a complete intersection. Let $X \in \mathsf{D}_{\mathsf{sg}}(R)$ be such that k belongs to $\mathsf{thick}_{\mathsf{D}_{\mathsf{sg}}(R)} X$. Let $Y \in \mathsf{D}_{\mathsf{sg}}(R)$ be locally zero on the punctured spectrum of R. Then $\delta_t(X, Y) = 0$ for all $t \neq 0$.*

Theorem 18. *Let R be singular and Cohen–Macaulay. Assume that the residue field k is infinite. Let X be an object of $\mathsf{D}_{\mathsf{sg}}(R)$ such that $k \in \mathsf{thick}_{\mathsf{D}_{\mathsf{sg}}(R)} X$. Let Y be an* *object of* $\mathsf{D_{sg}}(R)$ *which is locally zero on the punctured spectrum of* R*. Put* $u = \mathrm{e}(R)$ *and* $r = \mathrm{r}(R)$*. Then* $\delta_t(X,Y) = 0$ *for all* $t < -\log(u-1)$ *and for all* $t > \log(u-r)$*. Therefore,* $\delta_t(X,Y) = 0$ *for all* $|t| > \log(u-1)$ *provided that* R *is Gorenstein.*

Theorem 19. *Suppose* R *is singular. Set* $c = \operatorname{codepth} R$ *and* $m = \max_{1\leqslant i\leqslant c}\{\dim_k \mathrm{H}_i(\mathrm{K}^R)\}$*. Let* X *be an object of* $\mathsf{D_{sg}}(R)$ *such that* k *belongs to* $\mathsf{thick}_{\mathsf{D_{sg}}(R)} X$*, and let* Y *be an object of* $\mathsf{D_{sg}}(R)$ *which is locally zero on the punctured spectrum of* R*. Then* $\delta_t(X,Y) = 0$ *for all* $|t| > \frac{\log c + \log m}{2}$*.*

Remark 20. (1) Put $n = \operatorname{edim} R$. Cohen's structure theorem shows that there exist an n-dimensional regular local ring $(S, \mathfrak{n}, k)$ and an ideal I of S such that the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is isomorphic to the residue ring S/I. Choose a minimal system of generators $\boldsymbol{x} = x_1, \dots, x_n$ of $\mathfrak{n}$. It holds that

$$\mathrm{H}_i(\mathrm{K}^R) = \mathrm{H}_i(\boldsymbol{x}, R) \cong \mathrm{H}_i(\boldsymbol{x}, R) \otimes_R \widehat{R} \cong \mathrm{H}_i(\boldsymbol{x}, \widehat{R}) \cong \mathrm{H}_i(\mathrm{K}(\boldsymbol{x}, S) \otimes_S \widehat{R}) \cong \mathrm{Tor}_i^S(k, \widehat{R})$$

for each integer i, where the first isomorphism holds since the R-module $\mathrm{H}_i(\boldsymbol{x}, R)$ has finite length, while the last isomorphism follows from the fact that the Koszul complex $\mathrm{K}(\boldsymbol{x}, S)$ is a free resolution of k over S. Hence, the number $\dim_k \mathrm{H}_i(\mathrm{K}^R)$ is equal to the ith Betti number $\beta_i^S(\widehat{R})$ of $\widehat{R}$ over S.

(2) Let R be a singular hypersurface. Let G be a split generator of $\mathsf{D_{sg}}(R)$, and let X be an object of $\mathsf{D_{sg}}(R)$ which is locally zero on the punctured spectrum of R. The following two statements hold.
   (a) As R is a complete intersection, Theorem 17 implies that $\delta_t(G,X) = 0$ for all $0 \neq t \in \mathbb{R}$.
   (b) Put $c = \operatorname{codepth} R$ and $m = \max_{1\leqslant i\leqslant c}\{\dim_k \mathrm{H}_i(\mathrm{K}^R)\}$. Then $c = 1$. We have $\widehat{R} \cong S/(f)$ for some regular local ring $(S, \mathfrak{n})$ and some element $f \in \mathfrak{n}^2$. The sequence $0 \to S \xrightarrow{f} S \to \widehat{R} \to 0$ gives a minimal free resolution of the S-module $\widehat{R}$, and the equalities $\dim_k \mathrm{H}_1(\mathrm{K}^R) = \beta_1^S(\widehat{R}) = 1$ hold by (1). Hence $m = 1$. We get $\frac{\log c + \log m}{2} = 0$, and $\delta_t(G,X) = 0$ for all $t \neq 0$ by Theorem 19.

   Thus, each of Theorems 17 and 19 recovers Example 13 in the case where X is locally zero on the punctured spectrum of R (e.g., in the case where R has an isolated singularity by Remark 15).

Combining the above three theorems with Remark 15, we obtain the corollary below on entropies.

Corollary 21. *Let* R *be singular with an isolated singularity. Let* F *be an exact endofunctor of* $\mathsf{D_{sg}}(R)$*.*

(1) *Put* $c = \operatorname{codepth} R$ *and* $m = \max_{1\leqslant i\leqslant c}\{\dim_k \mathrm{H}_i(\mathrm{K}^R)\}$*. Then* $\delta_t(G,X) = 0$ *for all split generators* $G \in \mathsf{D_{sg}}(R)$*, all* $X \in \mathsf{D_{sg}}(R)$ *and all* $|t| > \frac{\log c + \log m}{2}$*. Thus* $\mathrm{h}_t(F)$ *is not defined if* $|t| > \frac{\log c + \log m}{2}$*.*

(2) *Assume that* R *is Gorenstein and* k *is infinite. Then* $\delta_t(G,X) = 0$ *for all split generators* $G \in \mathsf{D_{sg}}(R)$*, all* $X \in \mathsf{D_{sg}}(R)$ *and all* $|t| > \log(\mathrm{e}(R) - 1)$*. Thus* $\mathrm{h}_t(F)$ *is not defined for* $|t| > \log(\mathrm{e}(R) - 1)$*.*

(3) *Suppose that R is a complete intersection. Then $\delta_t(G, X) = 0$ for all split generators $G \in \mathsf{D}_{\mathsf{sg}}(R)$, all $X \in \mathsf{D}_{\mathsf{sg}}(R)$ and all nonzero real numbers t. Therefore, the entropy $\mathrm{h}_t(F)$ is defined only for $t = 0$.*

We close this section by mentioning that examples are constructed in [10], which say that the bounds $\frac{\log c + \log m}{2}$ and $\log(\mathrm{e}(R) - 1)$ for the real numbers t given in Theorems 18, 19 and Corollary 21(1)(2) are not necessarily best possible.

References

[1] L. L. Avramov, Infinite free resolutions, *Six lectures on commutative algebra*, 1–118, Mod. Birkhäuser Class., *Birkhäuser Verlag, Basel*, 2010.

[2] L. L. Avramov; R.-O. Buchweitz; S. B. Iyengar; C. Miller, Homology of perfect complexes, *Adv. Math.* **223** (2010), no. 5, 1731–1781.

[3] W. Bruns; J. Herzog, Cohen–Macaulay rings, revised edition, Cambridge Studies in Advanced Mathematics **39**, *Cambridge University Press, Cambridge*, 1998.

[4] R.-O. Buchweitz, Maximal Cohen–Macaulay modules and Tate cohomology, Mathematical Surveys and Monographs **262**, *American Mathematical Society, Providence, RI*, 2021.

[5] G. Dimitrov; F. Haiden; L. Katzarkov; M. Kontsevich, Dynamical systems and categories, *The influence of Solomon Lefschetz in geometry and topology*, 133–170, Contemp. Math. **621**, *Amer. Math. Soc., Providence, RI*, 2014.

[6] T. Dyckerhoff, Compact generators in categories of matrix factorizations, *Duke Math. J.* **159** (2011), no. 2, 223–274.

[7] D. Eisenbud, Homological algebra on a complete intersection, with an application to group representations, *Trans. Amer. Math. Soc.* **260** (1980), no. 1, 35–64.

[8] D. O. Orlov, Triangulated categories of singularities and D-branes in Landau–Ginzburg models, *Tr. Mat. Inst. Steklova* **246** (2004), Algebr. Geom. Metody, Svyazi i Prilozh., 240–262; translation in *Proc. Steklov Inst. Math.* 2004, no. 3(246), 227–248.

[9] R. Takahashi, Reconstruction from Koszul homology and applications to module and derived categories, *Pacific J. Math.* **268** (2014), no. 1, 231–248.

[10] R. Takahashi, Remarks on complexities and entropies for singularity categories, preprint 2022.

[11] Y. Yoshino, Cohen–Macaulay modules over Cohen–Macaulay rings, London Mathematical Society Lecture Note Series **146**, *Cambridge University Press, Cambridge*, 1990.

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TWISTED SEGRE PRODUCTS AND NONCOMMUTATIVE QUADRIC SURFACES

KENTA UEYAMA

Abstract. In this paper, we study twisted Segre products $A \circ_\psi B$ of noetherian Koszul AS-regular algebras A and B. We state that if $A \circ_\psi B$ is noetherian, then the noncommutative projective scheme $\mathsf{qgr}\, A \circ_\psi B$ has finite global dimension. We also state that if $A = k[u, v]$, $B = k[x, y]$, and ψ is a diagonal twisting map, then $A \circ_\psi B$ is a noncommutative quadric surface.

1. Introduction

Throughout this paper, let k be an algebraically closed field of characteristic 0. All algebras and vector spaces considered in this paper are over k, and all unadorned tensor products $\otimes$ are taken over k. For an algebra S, we write $\mathsf{Mod}\, S$ for the category of right S-modules. For a graded algebra S, we write $\mathsf{GrMod}\, S$ for the category of graded right S-modules.

Denote by $\mathbb{P}^d$ the d-dimensional projective space over k. Let $\Phi : \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \to \mathbb{P}^{nm-1}$ be the map defined by

$$((a_1, \dots, a_n), (b_1, \dots, b_m)) \mapsto (a_1b_1, a_2b_1, \dots, a_{n-1}b_m, a_nb_m).$$

Note that Φ is injective. It is called the *Segre embedding*. Let $A = \bigoplus_{i\in\mathbb{Z}} A_i$ and $B = \bigoplus_{i\in\mathbb{Z}} B_i$ be (not necessarily commutative) $\mathbb{Z}$-graded algebras. The *Segre product* of A and B is the $\mathbb{Z}$-graded algebra defined by

$$A \circ B := \bigoplus_{i\in\mathbb{Z}} (A_i \otimes B_i).$$

It is well-known that if $X \subset \mathbb{P}^{n-1}$ and $Y \subset \mathbb{P}^{m-1}$ are projective varieties with the homogeneous coordinate rings A and B, respectively, then $A \circ B$ is the homogeneous coordinate ring for the image of $X \times Y$ in $\mathbb{P}^{nm-1}$ under the Segre embedding Φ.

Let us consider the simplest case. The Segre embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3; \quad ((a_1, a_2), (b_1, b_2)) \mapsto (a_1b_1, a_2b_1, a_1b_2, a_2b_2).$$

embeds $\mathbb{P}^1 \times \mathbb{P}^1$ as a smooth quadric surface $Q = V(XW - YZ)$ in $\mathbb{P}^3$, and the Segre product

$$k[u, v] \circ k[x, y] = k[X, Y, Z, W]/(XW - YZ)$$

is the homogeneous coordinate ring of Q, where $X = u \otimes x,\ Y = v \otimes x,\ Z = u \otimes y,\ W = v \otimes y$.

This is a summary of joint work with Ji-Wei He (Hangzhou Normal University). The detailed version of this paper was published as [4].

To develop the study of noncommutative quadric surfaces (in the sense of [3]), it is natural to consider noncommutative generalizations of the Segre product $k[u,v]\circ k[x,y]$. Since noetherian Koszul AS-regular algebras are considered as nice noncommutative generalizations of polynomial algebras in noncommutative algebraic geometry, one of the natural noncommutative generalizations of $k[u,v]\circ k[x,y]$ is to replace $k[u,v]$ and $k[x,y]$ by 2-dimensional noetherian Koszul AS-regular algebras. However, this is not so interesting in the following sense.

Proposition 1 ([5, Lemma 2.12])**.** *If C and D are 2-dimensional noetherian Koszul AS-regular algebras, then we have an equivalence* $\mathsf{GrMod}\, C\circ D \cong \mathsf{GrMod}\, k[x,y]\circ k[u,v]$.

To obtain a proper noncommutative generalization (up to equivalence of graded module categories), in this paper, we discuss the notion of twisted Segre product. In particular, we focus on the study of twisted Segre products of noetherian Koszul AS-regular algebras.

2. Twisted Segre products

In this section, we give the definition of a twisted Segre product.

Definition 2. Let A,B be $\mathbb{Z}$-graded algebras. A bijective linear map $\psi : B\otimes A\to A\otimes B$ is called a *twisting map* if

(1) $\psi(B_i\otimes A_j)\subset A_j\otimes B_i$ for all $i,j\in\mathbb{Z}$,
(2) $\psi(1\otimes a)=a\otimes 1$ for all $a\in A$,
(3) $\psi(b\otimes 1)=1\otimes b$ for all $b\in B$,
(4) the following diagrams commute:

$$\begin{tikzcd} B\otimes B\otimes A \arrow[r, "\mathrm{id}_B\otimes\psi"] \arrow[d, "m_B\otimes\mathrm{id}_A"'] & B\otimes A\otimes B \arrow[r, "\psi\otimes\mathrm{id}_B"] & A\otimes B\otimes B \arrow[d, "\mathrm{id}_A\otimes m_B"] \\ B\otimes A \arrow[rr, "\psi"] & & A\otimes B, \end{tikzcd}$$

$$\begin{tikzcd} B\otimes A\otimes A \arrow[r, "\psi\otimes\mathrm{id}_A"] \arrow[d, "\mathrm{id}_B\otimes m_A"'] & A\otimes B\otimes A \arrow[r, "\mathrm{id}_A\otimes\psi"] & A\otimes A\otimes B \arrow[d, "m_A\otimes\mathrm{id}_B"] \\ B\otimes A \arrow[rr, "\psi"] & & A\otimes B, \end{tikzcd}$$

where m_A and m_B are the multiplications of A and B, respectively.

Definition 3. Let A,B be $\mathbb{Z}$-graded algebras, and let $\psi : B\otimes A\to A\otimes B$ be a twisting map. Then the *twisted Segre product* of A and B with respect to ψ, denoted by $A\circ_\psi B$, is the $\mathbb{Z}$-graded algebra defined as follows:

- $A\circ_\psi B = A\circ B$ as a graded vector space,
- the multiplication of $A\circ_\psi B$ is defined by

$$(a\otimes b)(c\otimes d)=a\psi(b\otimes c)d$$

for $a\in A_n, c\in A_m, b\in B_n, d\in B_m$.

Note that this definition is well-defined. If ψ is the flip map, i.e., $\psi(b\otimes a)=a\otimes b$ for all $a\in A$ and $b\in B$, then $A\circ_\psi B$ is the usual Segre product of A and B.

Remark 4. Let A and B be $\mathbb{Z}$-graded algebras, and let $\psi : B \otimes A \to A \otimes B$ be a twisting map. Write $A \otimes_\psi B$ for the graded vector space $A \otimes B$ with the multiplication defined by $(a \otimes b)(c \otimes d) = a\psi(b \otimes c)d$ for $a, c \in A$ and $b, d \in B$. By [2, Theorem 2.5], $A \otimes_\psi B$ is also a $\mathbb{Z}$-graded algebra. We say that $A \otimes_\psi B$ is a *twisted tensor product* of A and B. It is easy to see that $A \circ_\psi B$ is an ungraded subalgebra of $A \otimes_\psi B$.

3. Dade's theorem for densely bigraded algebras

In this section, as a preparation for discussing twisted Segre products, we give a version of Dade's theorem for densely bigraded algebras.

Definition 5. Let $S = \bigoplus_{i \in \mathbb{Z}} S_i$ be a $\mathbb{Z}$-graded algebra.

(1) S is called a *strongly graded algebra* if $S_{i+j} = S_i S_j$ for all $i, j \in \mathbb{Z}$.
(2) S is called a *densely graded algebra* if $\dim_k(S_{i+j}/S_i S_j) < \infty$ for all $i, j \in \mathbb{Z}$.

Clearly a strongly graded algebra is a densely graded algebra. If S is a strongly graded algebra, then

$$(-)_0 : \mathsf{GrMod}\, S \to \mathsf{Mod}\, S_0; \qquad M \mapsto M_0$$

is an equivalence. This is well-known as Dade's theorem.

Let $S = \bigoplus_{i,j \in \mathbb{Z}} S_{(i,j)}$ be a $\mathbb{Z}^2$-bigraded algebra. Let $S_i = \bigoplus_{j \in \mathbb{Z}} S_{(i,j)}$ for all $i \in \mathbb{Z}$. Then $S_{\mathbb{Z}} := \bigoplus_{i \in \mathbb{Z}} S_i$ is a $\mathbb{Z}$-graded algebra. Note that S and $S_{\mathbb{Z}}$ are the same as ungraded algebras. However, they have different gradings.

Definition 6. Let $S = \bigoplus_{i,j \in \mathbb{Z}} S_{(i,j)}$ be a $\mathbb{Z}^2$-bigraded algebra. Then S is called a *densely bigraded algebra* if $S_{\mathbb{Z}}$ is a densely graded algebra.

Definition 7. Let S be a noetherian $\mathbb{Z}$-graded algebra.

(1) An element $m \in M \in \mathsf{GrMod}\, S$ is called *torsion* if $\dim_k mS < \infty$.
(2) A graded module $M \in \mathsf{GrMod}\, S$ is called *torsion* if every homogeneous element of M is torsion.
(3) Let $\mathsf{Tors}\, S$ be the full subcategory of $\mathsf{GrMod}\, S$ consisting of torsion modules.
(4) Let $\mathsf{QGr}\, S$ be the Serre quotient category $\mathsf{GrMod}\, S / \mathsf{Tors}\, S$.
(5) Let $\mathsf{grmod}\, S$ be the full subcategory of $\mathsf{GrMod}\, S$ consisting of finitely generated modules.
(6) Let $\mathsf{tors}\, S = \mathsf{grmod}\, S \cap \mathsf{Tors}\, S$.
(7) Let $\mathsf{qgr}\, S$ be the Serre quotient category $\mathsf{grmod}\, S / \mathsf{tors}\, S$.

The categories $\mathsf{QGr}\, S$ and $\mathsf{qgr}\, S$ are often called *noncommutative projective schemes* and play an important role in noncommutative algebraic geometry; see [1].

Definition 8. Let S be a densely bigraded algebra satisfying the conditions

(D1) $S_0 := \bigoplus_{j \in \mathbb{Z}} S_{(0,j)}$ is a noetherian $\mathbb{Z}$-graded algebra, and
(D2) $S_i := \bigoplus_{j \in \mathbb{Z}} S_{(i,j)}$ is a finitely generated graded right and left S_0-module for every $i \in \mathbb{Z}$.

We use the following terminology and notation:

(1) Let $\mathsf{BiGrMod}\, S$ be the category of bigraded right S-modules.
(2) An element $m \in M \in \mathsf{BiGrMod}\, S$ is called *locally torsion* if $\dim_k mS_0 < \infty$.

(3) A bigraded module $M \in \mathsf{BiGrMod}\, S$ is called *locally torsion* if every homogeneous element of M is locally torsion.
(4) Let $\mathsf{BiLTors}\, S$ be the full subcategory of $\mathsf{BiGrMod}\, S$ consisting of locally torsion modules.
(5) Let $\mathsf{QBiGr_L}\, S$ be the Serre quotient category $\mathsf{BiGrMod}\, S/\, \mathsf{BiLTors}\, S$.
(6) Let $\mathsf{lbigrmod}\, S$ be the full subcategory of $\mathsf{BiGrMod}\, S$ consisting of bigraded right S-modules $M = \bigoplus_{i,j\in\mathbb{Z}} M_{(i,j)}$ such that $M_i := \bigoplus_{j\in\mathbb{Z}} M_{(i,j)}$ is finitely generated as a graded right S_0-module.
(7) Let $\mathsf{lbiltors}\, S = \mathsf{lbigrmod}\, S \cap \mathsf{BiLTors}\, S$.
(8) Let $\mathsf{qlbigr_l}\, S$ be the Serre quotient category $\mathsf{lbigrmod}\, S/\, \mathsf{lbiltors}\, S$.

Then we have the following theorem.

Theorem 9 ([4, Theorem 4.10])**.** *Let S be a densely bigraded algebra satisfying the conditions* (D1) *and* (D2) *in Definition 8. Then the functor*

$$(-)_0 : \mathsf{BiGrMod}\, S \to \mathsf{GrMod}\, S_0; \qquad M \mapsto M_0 \ (= \bigoplus_{j\in\mathbb{Z}} M_{(0,j)})$$

induces equivalences

$$\mathsf{QBiGr_L}\, S \cong \mathsf{QGr}\, S_0 \quad \textit{and} \quad \mathsf{qlbigr_l}\, S \cong \mathsf{qgr}\, S_0.$$

This theorem can be regarded as a version of Dade's theorem for densely bigraded algebras.

4. Twisted Segre products of noetherian Koszul AS-regular algebras

In this section, we study twisted Segre products of noetherian Koszul AS-regular algebras.

Definition 10. Let $A = \bigoplus_{i\in\mathbb{N}} A_i$ be a noetherian $\mathbb{N}$-graded algebra with $A_0 = k$.
(1) A is called *an AS-Gorenstein algebra* of dimension d if
  (a) $\operatorname{injdim}_A A = \operatorname{injdim}_{A^{\mathrm{op}}} A = d < \infty$, and
  (b) $\operatorname{Ext}^i_A(k, A) \cong \operatorname{Ext}^i_{A^{\mathrm{op}}}(k, A) \cong \begin{cases} k(\ell) \text{ for some } \ell \in \mathbb{Z} & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$
(2) A is called *an AS-regular algebra* if A is an AS-Gorenstein algebra of $\operatorname{gldim} A = d$.
(3) A is called *Koszul* if $k \in \mathsf{GrMod}\, A$ has a free resolution

$$\cdots \to A(-3)^{r_3} \to A(-2)^{r_2} \to A(-1)^{r_1} \to A \to k \to 0.$$

Let A and B be noetherian Koszul AS-regular algebras with a twisting map $\psi : B \otimes A \to A \otimes B$. Let $S = A \otimes_\psi B$ be the twisted tensor product; see Remark 4. Here we endow S with a bigraded structure as follows:

$$S_{(i,j)} = A_{i+j} \otimes B_j$$

for $i, j \in \mathbb{Z}$. Note that $S_0 = \bigoplus_{j\in\mathbb{Z}} S_{(0,j)}$ is equal to the twisted Segre product $A \circ_\psi B$ as graded algebras.

Proposition 11 ([4, Proposition 4.11])**.** *Let A and B be noetherian Koszul AS-regular algebras with a twisting map $\psi : B \otimes A \to A \otimes B$. Then $A \otimes_\psi B$ is a densely bigraded algebra satisfying the condition* (D2) *in Definition 8.*

By Proposition 11, if $A \circ_\psi B$ is noetherian, then $A \otimes_\psi B$ is a densely bigraded algebra satisfying the conditions (D1) and (D2) in Definition 8. Therefore, Theorem 9 yields the following result.

Theorem 12 ([4, Theorem 4.13])**.** *Let A and B be noetherian Koszul AS-regular algebras with a twisting map $\psi : B \otimes A \to A \otimes B$. Assume that $A \circ_\psi B$ is noetherian. Then there exist equivalences*

$$\mathsf{QBiGr_L}\, A \otimes_\psi B \cong \mathsf{QGr}\, A \circ_\psi B \quad \textit{and} \quad \mathsf{qlbigr_l}\, A \otimes_\psi B \cong \mathsf{qgr}\, A \circ_\psi B.$$

By [6, Theorem 2], it follows that $A \otimes_\psi B$ has finite global dimension, so we see that $\mathsf{qlbigr_l}\, A \otimes_\psi B$ in Theorem 12 has finite global dimension. Thus, we obtain the following consequence.

Corollary 13 ([4, Theorem 4.16])**.** *Let A and B be noetherian Koszul AS-regular algebras with a twisting map $\psi : B \otimes A \to A \otimes B$. Assume that $A \circ_\psi B$ is noetherian. Then $\mathsf{qgr}\, A \circ_\psi B$ has finite global dimension.*

The above corollary can be regarded as a noncommutative analogue of the fact that $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ is smooth.

5. Twisted Segre products of $k[u, v]$ and $k[x, y]$

Let $A = k[u, v], B = k[x, y]$ be standard graded polynomial rings in two variables. In this section, we consider certain twisted Segre products of $A = k[u, v]$ and $B = k[x, y]$. If $\psi : B \otimes A \to A \otimes B$ is a twisting map, then

$$\begin{aligned} \psi(x \otimes \begin{pmatrix} u \\ v \end{pmatrix}) &= N_{11} \begin{pmatrix} u \\ v \end{pmatrix} \otimes x + N_{12} \begin{pmatrix} u \\ v \end{pmatrix} \otimes y, \\ \psi(y \otimes \begin{pmatrix} u \\ v \end{pmatrix}) &= N_{21} \begin{pmatrix} u \\ v \end{pmatrix} \otimes x + N_{22} \begin{pmatrix} u \\ v \end{pmatrix} \otimes y, \end{aligned} \tag{5.1}$$

where $N_{11}, N_{12}, N_{21}, N_{22} \in M_2(k)$.

Definition 14. Let $A = k[u, v], B = k[x, y]$ be standard graded polynomial rings in two variables. A twisting map $\psi : B \otimes A \to A \otimes B$ is called *diagonal* if $N_{12} = N_{21} = 0$ in (5.1).

Then we have the following theorem.

Theorem 15 ([4, Theorem 6.4, Corollary 6.12])**.** *Let $A = k[u, v], B = k[x, y]$ be standard graded polynomial rings in two variables, and let $\psi : B \otimes A \to A \otimes B$ be a diagonal twisting map. Then the following statements hold.*

1. *The twisted Segre product $A \circ_\psi B$ is a noncommutative quadric surface, i.e., there exist a 4-dimensional noetherian Koszul AS-regular algebra S with Hilbert series $H_S(t) = (1-t)^{-4}$ and a regular normal homogeneous element $f \in S$ of degree 2 such that $A \circ_\psi B \cong S/(f)$. In particular, $A \circ_\psi B$ is a 3-dimensional noetherian*

Koszul AS-Gorenstein algebra, and hence $\mathsf{qgr}\, A \circ_\psi B$ *has finite global dimension (by Corollary 13).*

(2) *There exists an equivalence of triangulated categories*

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(A \circ_\psi B) \cong \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, k \times k),$$

where $\underline{\mathsf{CM}}^{\mathbb{Z}}(A \circ_\psi B)$ *is the stable category of graded maximal Cohen-Macaulay modules over* $A \circ_\psi B$ *and* $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, k \times k)$ *is the bounded derived category of finite dimensional modules over* $k \times k$*.*

It turns out that a twisted Segre product $A \circ_\psi B$ appearing in the above theorem has a nice property similar to the usual Segre product $A \circ B = k[u, v] \circ k[x, y] = k[X, Y, Z, W]/(XW - YZ)$, which is the homogeneous coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$. On the other hand, one can check that there exists a twisted Segre product $A \circ_\psi B$ such that $\mathsf{GrMod}\, A \circ_\psi B \not\cong \mathsf{GrMod}\, A \circ B = \mathsf{GrMod}\, k[u, v] \circ k[x, y]$. This is different from the situation in Proposition 1.

References

[1] M. Artin and J. J. Zhang, *Noncommutative projective schemes*, Adv. Math. **109** (1994), no. 2, 228–287.

[2] S. Caenepeel, B. Ion, G. Militaru, and S. Zhu, *The factorization problem and the smash biproduct of algebras and coalgebras*, Algebr. Represent. Theory **3** (2000), no. 1, 19–42.

[3] S. P. Smith and M. Van den Bergh, *Noncommutative quadric surfaces*, J. Noncommut. Geom. **7** (2013), no. 3, 817–856.

[4] J.-W. He and K. Ueyama, *Twisted Segre products*, J. Algebra **611** (2022), 528–560.

[5] K. Van Rompay, *Segre product of Artin-Schelter regular algebras of dimension 2 and embeddings in quantum* $\mathbb{P}^3$*'s*, J. Algebra **180** (1996), no. 2, 483–512.

[6] X. Wang and Y. Shen, *Artin-Schelter regularity of twisted tensor products*, Front. Math. China **13** (2018), no. 5, 1141–1167.

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CHARACTERIZATION OF EVENTUALLY PERIODIC MODULES AND ITS APPLICATIONS

SATOSHI USUI

Abstract. For a left Noetherian ring, its singularity category makes only the modules of finite projective dimension vanish among the modules. Thus the singularity categories are expected to characterize homological properties of modules with infinite projective dimension. In this paper, we focus on eventually periodic modules over a left artin ring, and we characterize them in terms of morphisms in the singularity category. As applications, we prove that, for the class of finite dimensional algebras over a field, being eventually periodic is preserved under singular equivalence of Morita type with level. Moreover, we determine which finite dimensional connected Nakayama algebras are eventually periodic when the ground field is algebraically closed.

1. Introduction

Throughout this paper, let k be a field, and we assume that all rings are associative and unital. By a module, we mean a left module.

The *singularity category* $\mathcal{D}_{\mathrm{sg}}(R)$ of a left Noetherian ring R is defined to be the Verdier quotient of the bounded derived category $\mathcal{D}^{\mathrm{b}}(R\text{-mod})$ of finitely generated R-modules by the full subcategory of perfect complexes (see [2]), and it provides a homological measure of singularity of finitely generated R-modules M in the following sense: M has finite projective dimension if and only if M is isomorphic to 0 in $\mathcal{D}_{\mathrm{sg}}(R)$. From this point of view, it is expected that the singularity categories capture homological properties of modules of infinite projective dimension.

In this paper, we work with left artin rings and focus on *eventually periodic modules*, that is, finitely generated modules whose minimal projective resolutions have infinite length and eventually become periodic. Recently, the author has proved in [12, Proposition 3.4] that a finitely generated Λ-module M is eventually periodic if and only if there exists an invertible homogeneous element of positive degree in the Tate cohomology ring of M, where Λ is a finite dimensional Gorenstein algebra over an algebraically closed field.

We first give our main result, showing that the above result also holds for any left artin rings. We then state two applications: the first is that being eventually periodic is invariant under singular equivalence of Morita type with level for the class of finite dimensional k-algebras; the second is that, when the ground field k is algebraically closed, connected Nakayama algebras that are eventually periodic are precisely those of infinite global dimension.

The detailed version of this paper will be submitted for publication elsewhere.

2. Tate cohomology rings

In this subsection, we recall from [2, 14] the definition of Tate cohomology and some related facts.

Definition 1 ([2, Definition 6.1.1] and [14, page 11])**.** Let i be an integer.

(1) Let R be a left Noetherian ring and M and N two finitely generated R-modules. We define the *i-th Tate cohomology group of M with coefficients in N* by

$$\widehat{\mathrm{Ext}}_R^i(M,N) := \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(R)}(M, N[i]).$$

(2) Let Λ be a finite dimensional k-algebra. The *i-th Tate-Hochschild cohomology group* of Λ is defined by $\widehat{\mathrm{HH}}^i(\Lambda) := \widehat{\mathrm{Ext}}_{\Lambda^{\mathrm{e}}}^i(\Lambda,\Lambda)$, where we put $\Lambda^{\mathrm{e}} := \Lambda \otimes_k \Lambda^{\mathrm{op}}$.

The graded abelian group

$$\widehat{\mathrm{Ext}}_R^{\bullet}(M,M) := \bigoplus_{i\in\mathbb{Z}} \widehat{\mathrm{Ext}}_R^i(M,M)$$

carries a structure of a graded ring given by the *Yoneda product*. We call this graded ring the *Tate cohomology ring* of M. It follows from the definition of singularity categories that $\widehat{\mathrm{Ext}}_R^{\bullet}(M,M)$ is the zero ring if and only if the projective dimension of M is finite.

The Tate cohomology ring $\widehat{\mathrm{Ext}}_{\Lambda^{\mathrm{e}}}^{\bullet}(\Lambda,\Lambda)$ for a finite dimensional k-algebra Λ is called the *Tate-Hochschild cohomology ring* of Λ and denoted it by $\widehat{\mathrm{HH}}^{\bullet}(\Lambda)$. It was proved by Wang [14, Proposition 4.7] that $\widehat{\mathrm{HH}}^{\bullet}(\Lambda)$ is graded commutative.

3. Main result

In this section, we give our main result. Let R be a left artin ring. We denote by $\Omega_R^n(M)$ the n-th syzygy of a finitely generated R-module M. Recall that a finitely generated R-module M is called *periodic* if $\Omega_R^p(M) \cong M$ as R-modules for some $p > 0$. The least such p is called the *period* of M. We say that M is *eventually periodic* if $\Omega_R^n(M)$ is non-zero and periodic for some $n \geq 0$. Remark that our eventually periodic modules are those of infinite projective dimension in the sense ever before. Following Küpper [7], we define *eventually periodic algebras* as finite dimensional algebras that are eventually periodic as regular bimodules.

We now characterize the existence of invertible homogeneous elements in Tate cohomology rings. In the following result, Ω_R denotes the loop space functor on the stable module category R-$\underline{\mathrm{mod}}$.

Theorem 2. *Let R be a left Noetherian ring, and let p be a positive integer. Then the following conditions are equivalent for a finitely generated R-module M.*

(1) *There exists an isomorphism $\Omega_R^{n+p}(M) \cong \Omega_R^n(M)$ in R-$\underline{\mathrm{mod}}$ for some $n \geq 0$.*

(2) *The Tate cohomology ring $\widehat{\mathrm{Ext}}_R^{\bullet}(M,M)$ has an invertible homogeneous element of degree p.*

The main result of this paper can be obtained from the above theorem.

Corollary 3. *Let R be a left artin ring. Then the following conditions are equivalent for a finitely generated R-module M.*

(1) *M is eventually periodic.*
(2) *The Tate cohomology ring of M has a non-zero invertible homogeneous element of positive degree.*

In this case, there exists an invertible homogeneous element in the Tate cohomology ring of M whose degree equals the period of some periodic syzygy $\Omega_R^n(M)$ with $n \geq 0$.

Remark 4. For a left Noetherian local ring R, one can define eventually periodic R-modules by using minimal free resolutions. Then Theorem 2 enables us to characterize eventually periodic R-modules as in Corollary 3.

We end this section by giving a result on Tate-Hochschild cohomology rings, which will be used in the next section.

Proposition 5. *Let Λ be a finite dimensional k-algebra. Then Λ is eventually periodic if and only if $\widehat{\mathrm{HH}}^{\bullet}(\Lambda)$ has a non-zero invertible homogeneous element of positive degree. In this case, there exists an isomorphism of graded rings*

$$\widehat{\mathrm{HH}}^{\bullet}(\Lambda) \cong \widehat{\mathrm{HH}}^{\geq 0}(\Lambda)[\chi^{-1}],$$

where we set $\widehat{\mathrm{HH}}^{\geq 0}(\Lambda) := \bigoplus_{i \geq 0} \widehat{\mathrm{HH}}^{i}(\Lambda)$, and the degree of the invertible homogeneous element χ is equal to the period of some periodic syzygy $\Omega_{\Lambda^{\mathrm{e}}}^n(\Lambda)$ with $n \geq 0$.

Note that the proposition generalizes [4, Corollary 6.4] and [12, Theorem 3.5] (in a proper sense).

4. Applications

There are two aims of this section. The first is to show that being eventually periodic is preserved under singular equivalence of Morita type with level. The second is to give a criterion for a Nakayama algebra to be eventually periodic. Throughout this section, algebras will mean finite dimensional k-algebras.

4.1. **Eventually periodic algebras and singular equivalences of Morita type with level.** This subsection is devoted to showing that singular equivalences of Morita type with level preserve eventual periodicity of algebras. We start with the definition of *singular equivalences of Morita type with level.*

Definition 6 ([13, Definition 2.1])**.** Let Λ and Γ be two algebras, and let $l \geq 0$ be an integer. We say that a pair $({}_\Lambda M_\Gamma, {}_\Gamma N_\Lambda)$ of bimodules *defines a singular equivalence of Morita type with level l* (and that Λ and Γ are *singularly equivalent of Morita type with level l*) if the following conditions are satisfied.

(1) The one-sided modules ${}_\Lambda M, M_\Gamma, {}_\Gamma N$ and N_Λ are finitely generated and projective.
(2) There exist isomorphisms $M \otimes_\Gamma N \cong \Omega^l_{\Lambda^{\mathrm{e}}}(\Lambda)$ and $N \otimes_\Lambda M \cong \Omega^l_{\Gamma^{\mathrm{e}}}(\Gamma)$ in Λ^{e}-$\underline{\mathrm{mod}}$ and Γ^{e}-$\underline{\mathrm{mod}}$, respectively.

A lot of invariants under singular equivalence of Morita type with level have been discovered by Skartsæterhagen [11], Qin [8] and Wang [13, 15]. We now give a new invariant, using Corollary 5 and Wang's result [15].

Theorem 7. *Assume that Λ and Γ are singularly equivalent of Morita type with level. If Λ is eventually periodic, then so is Γ. In particular, the periods of their periodic syzygies coincide.*

Recall that two algebras Λ and Γ are *derived equivalent* if there exists a triangle equivalence between $\mathcal{D}^{\mathrm{b}}(\Lambda\text{-mod})$ and $\mathcal{D}^{\mathrm{b}}(\Gamma\text{-mod})$. It was proved by Wang [13, Theorem 2.3] that any two derived equivalent algebras are singularly equivalent of Morita type with level. Thus Theorem 7 generalizes a result of Erdmann and Skowroński [6, Theorem 2.9].

We end this subsection with examples of eventually periodic algebras. Note that Γ and Σ below can be found in [3, Example 4.3 (2)] and [12, Example 3.2 (1)], respectively.

Example 8. Let Λ, Γ and Σ be the k-algebras given by the following quivers with relations

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \end{tikzcd} \qquad \alpha^2 = 0,$$

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & 2 \end{tikzcd} \qquad \alpha^2 = 0 = \beta\alpha$$

and

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \end{tikzcd} \qquad \alpha\beta\alpha = 0,$$

respectively. Λ is a self-injective Nakayama and hence a periodic algebra (see [5, Lemma in Section 4.2]). Note that Λ has period 1 if the characteristic of k is 2 and 2 otherwise. By [11, Example 7.5], Λ and Γ are singularly equivalent of Morita type with level 1, so that Γ is eventually periodic by Theorem 7. Also, using the APR-tiling Γ-module corresponding to the vertex 2, one sees that Γ is derived equivalent to Σ, and hence Σ is eventually periodic by Theorem 7 again. On the other hand, a direct computation shows that $\Omega^3_{\Gamma^{\mathrm{e}}}(\Gamma)$ and $\Omega^2_{\Sigma^{\mathrm{e}}}(\Sigma)$ are the first periodic syzygies. Thus we conclude from Theorem 7 that the periodic bimodules Λ, $\Omega^3_{\Gamma^{\mathrm{e}}}(\Gamma)$ and $\Omega^2_{\Sigma^{\mathrm{e}}}(\Sigma)$ have the same period.

4.2. **Eventually periodic Nakayama algebras.** Throughout, we suppose that the field k is algebraically closed. Then the global dimension of an algebra is equal to the projective dimension of the algebra as a regular bimodule over itself. We denote by $\mathcal{CN}$ the class of connected Nakayama algebras. The aim of this subsection is to decide which algebras from $\mathcal{CN}$ are eventually periodic.

Let Λ be in $\mathcal{CN}$ and $J(\Lambda)$ its Jacobson radical. Then Λ is Morita equivalent to a bound quiver algebra whose ordinary quiver is given by either

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow e-1 \longrightarrow e$$

or

$$\begin{tikzcd}[column sep=small] & & 1 \arrow[r] & 2 \arrow[dr] & \\ Z_e: & e \arrow[ur] & & & 3 \arrow[dl] \\ & & e-1 \arrow[ul] & \cdots \arrow[l] & \end{tikzcd}$$

where $e \geq 1$. Note that the global dimension of Λ is finite in the first case. Moreover, Λ is non-simple and self-injective if and only if it is Morita equivalent to the bound quiver algebra kZ_e/R^N for some $e \geq 1$ and $N \geq 2$, where R denotes the arrow ideal of the path algebra kZ_e.

Using results of Asashiba [1], Qin [8] and Shen [9, 10], we can classify $\mathcal{CN}$ up to singular equivalence of Morita type with level. We note that $\mathcal{CN}$ is not closed under singular equivalence of Morita type with level (see Example 8).

Theorem 9. *The algebras k and kZ_e/R^N with $e \geq 1$ and $N \geq 2$ form a complete set of representatives of pairwise different equivalence classes of finite dimensional connected Nakayama k-algebras under singular equivalence of Morita type with level.*

It was proved by Erdmann and Holm [5, Lemma in Section 4.2] that kZ_e/R^N is periodic for all $e \geq 1$ and $N \geq 2$. Hence we obtain the following consequence of Theorems 7 and 9, which is the main result of this subsection.

Corollary 10. *Let Λ be a connected Nakayama algebra. Then Λ is eventually periodic if and only if the global dimension of Λ is infinite.*

References

[1] H. Asashiba, *The derived equivalence classification of representation-finite selfinjective algebras*, J. Algebra **214(1)** (1999), 182–221.

[2] R.-O. Buchweitz, *Maximal cohen-macaulay modules and Tate-cohomology over Gorenstein rings*, reprint (1986). available at http://hdl.handle.net/1807/16682.

[3] X.-W. Chen, *Singularity categories, Schur functors and triangular matrix rings*, Algebr. Represent.Theory **12(2-5)** (2009), 181–191.

[4] V. Dotsenko, V. Gélinas, and P. Tamaroff, *Finite generation for Hochschild cohomology of Gorenstein monomial algebras*, (2019). arXiv:1909.00487.

[5] K. Erdmann and T. Holm, *Twisted bimodules and Hochschild cohomology for self-injective algebras of class A_n*, Forum Math. **11(2)** (1999), 177–201.

[6] K. Erdmann and A. Skowroński, *Periodic algebras*, Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep. (2008), 201–251.

[7] S. Küpper, *Two-sided Projective Resolutions, Periodicity and Local Algebras*, Logos Verlag Berlin, (2010).

[8] Y. Qin, *Reduction techniques of singular equivalences*, (2021). arXiv:2103.10393v1.

[9] D. Shen, *The singularity category of a Nakayama algebra*, J. Algebra **429** (2015), 1–18.

[10] ———, *A note on singular equivalences and idempotents*, Proc. Amer. Math. Soc. **149(10)** (2021), 4067–4072.

[11] Ø. Skartsæterhagen, *Singular equivalence and the (fg) condition*, J. Algebra **452** (2016), 66–93.

[12] S. Usui, *Tate-Hochschild cohomology rings for eventually periodic Gorenstein algebras*, SUT J. Math. **57(2)** (2021), 133–146.

[13] Z. Wang, *Singular equivalence of Morita type with level*, J. Algebra **439** (2015), 245–269.

[14] ———, *Gerstenhaber algebra and Deligne's conjecture on the Tate–Hochschild cohomology*, Trans. Amer. Math. Soc. **374(7)** (2021), 4537–4577.

[15] ———, *Invariance of the Gerstenhaber algebra structure on Tate-Hochschild cohomology*, J. Inst. Math. Jussieu **20(3)** (2021), 893–928.

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