The spectrum of the category of maximal Cohen–Macaulay modules

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Setting i

In this talk, **R** is a commutative complete Cohen-Macauly local ring with algebraic residue field **k**.

- All modules are **"finitely generated" R**-modules.
- $\mathcal{C}$ is the category of maximal Cohen-Macaulay (MCM) modules.

$$\mathcal{C} = \{M \mid \mathrm{Ext}^i_R(k, M) = 0 \text{ for } i < \dim R\}$$

Remark 1

- Since **R** is complete, $\mathcal{C}$ is a Krull-Schmidt category.

Setting ii

We consider the categories:

- $\mathbf{mod(\mathcal{C}) := \{F : \mathcal{C} \to Ab |} \begin{array}{l}\text{finitely presented} \\ \text{contravariant additive functors}\end{array} \}$.
- $\mathbf{\underline{mod}(\mathcal{C}) := \{F \in mod(\mathcal{C}) | F(R) = 0\}}$.

For $\forall \mathbf{F} \in \mathbf{mod(\underline{\mathcal{C}})}$, $\exists \mathbf{0 \to L \to M \to N \to 0}$ such that

$$\mathbf{0 \to Hom_R(\ ,L) \to Hom_R(\ ,M) \to Hom_R(\ ,N) \to F \to 0}$$

is exact in $\mathbf{mod(\mathcal{C})}$.

Auslander '86.

$\mathbf{mod(\mathcal{C})}$ and $\mathbf{\underline{mod}(\mathcal{C})}$ are abelian categories.

Remark 2

We denote by $\underline{\mathcal{C}}$ the stable category of $\mathcal{C}$. The objects of $\underline{\mathcal{C}}$ are the same as those of $\mathcal{C}$, and the morphisms $\underline{\mathrm{Hom}}_R(M, N) := \mathrm{Hom}_R(M, N)/\{M \to P \to N \text{ with } P \text{ free}\}$.

- The category $\underline{\mathrm{mod}}(\mathcal{C})$ is equivalent to $\mathrm{mod}(\underline{\mathcal{C}})$.

$$\mathrm{mod}(\underline{\mathcal{C}}) \to \underline{\mathrm{mod}}(\mathcal{C}); \quad F \mapsto F \circ \iota,$$

  where $\iota : \mathcal{C} \to \underline{\mathcal{C}}$.

- For $\forall F \in \underline{\mathrm{mod}}(\mathcal{C})$ with $\mathrm{Hom}_R(\ , M) \to \mathrm{Hom}_R(\ , L) \to F \to 0$, we have an exact sequence $\underline{\mathrm{Hom}}_R(\ , M) \to \underline{\mathrm{Hom}}_R(\ , L) \to F \to 0$.

In the rest of this slide, we denote $\mathrm{mod}(\underline{\mathcal{C}})$ instead of $\underline{\mathrm{mod}}(\mathcal{C})$.

We denote by $\mathbf{Sp}(\mathcal{C})$ the set of isomorphism classes of the indecomposable MCM $\mathbf{R}$-modules except $\mathbf{R}$ and $\mathbf{0}$.

$$\mathbf{Sp}(\mathcal{C}) := \{\text{the indecomp. MCM } \mathbf{R}\text{-modules except } \mathbf{R} \text{ and } \mathbf{0}\}/\cong$$

Definition 1 (Krause '97)

The assignments

$$\mathbf{\Sigma} : \mathfrak{P}(\mathbf{Sp}(\mathcal{C})) \to \mathbf{mod}(\underline{\mathcal{C}}), \quad \gamma : \mathbf{mod}(\underline{\mathcal{C}}) \to \mathfrak{P}(\mathbf{Sp}(\mathcal{C}))$$

are defined by

$$\mathbf{\Sigma}(\mathcal{X}) := \{\mathbf{F} \in \mathbf{mod}(\underline{\mathcal{C}}) \mid \mathbf{F}(\mathbf{X}) = \mathbf{0} \text{ for } \forall \mathbf{X} \in \mathcal{X}\}$$
$$\gamma(\mathcal{F}) := \{\mathbf{M} \in \mathbf{Sp}(\mathcal{C}) \mid \mathbf{F}(\mathbf{M}) = \mathbf{0} \text{ for } \forall \mathbf{F} \in \mathcal{F}\}.$$

Remark 3

In this talk, we consider only **finitely generated** (pure-injective) modules. Therefore $\mathcal{C}$ is not closed under arbitrary coproducts. In other words, $\mathcal{C}$ ($\underline{\mathcal{C}}$) is not compactly generated. The studies in [1, 2, 3, 4, 5] have considered categories that are compactly generated. In fact, they consider **infinitely generated** modules.

Lemma 2

For the assignments $\boldsymbol{\Sigma}$ *and* γ*, the following statements hold.*

1. $\mathcal{X} \subseteq \mathcal{Y} \Rightarrow \boldsymbol{\Sigma}(\mathcal{X}) \supseteq \boldsymbol{\Sigma}(\mathcal{Y})$.
2. $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \gamma(\mathcal{F}) \supseteq \gamma(\mathcal{G})$.
3. $\mathcal{X} \subseteq \gamma \circ \boldsymbol{\Sigma}(\mathcal{X})$.
4. $\mathcal{F} \subseteq \boldsymbol{\Sigma} \circ \gamma(\mathcal{F})$. *Moreover* $\gamma(\mathcal{F}) = \gamma \circ \boldsymbol{\Sigma} \circ \gamma(\mathcal{F})$.
5. $\forall \mathcal{X}$, $\boldsymbol{\Sigma}(\mathcal{X})$ *is a Serre subcategory in* $\mathbf{mod}(\underline{\mathcal{C}})$.

Proof.

(5) $\mathbf{ev_X} : \mathbf{mod}(\underline{\mathcal{C}}) \to \mathbf{AB}$; $\mathbf{F} \mapsto \mathbf{F(X)}$ is exact. □

Main Theorem

Theorem A

Suppose that $\mathbf{R}$ is ~~Gorenstein~~. The assignment $\mathcal{X} \mapsto \gamma \circ \boldsymbol{\Sigma}(\mathcal{X})$ is a Kuratowski closure operator. That is,

- (i) $\gamma \circ \boldsymbol{\Sigma}(\emptyset) = \emptyset$,
- (ii) $\mathcal{X} \subseteq \gamma \circ \boldsymbol{\Sigma}(\mathcal{X})$,
- (iii) $\gamma \circ \boldsymbol{\Sigma}(\mathcal{X} \cup \mathcal{Y}) = \gamma \circ \boldsymbol{\Sigma}(\mathcal{X}) \cup \gamma \circ \boldsymbol{\Sigma}(\mathcal{Y})$,
- (iv) $\gamma \circ \boldsymbol{\Sigma}(\gamma \circ \boldsymbol{\Sigma}(\mathcal{X})) = \gamma \circ \boldsymbol{\Sigma}(\mathcal{X})$

hold for $\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{P}(\mathbf{Sp}(\mathcal{C}))$.

The assertions (i), (ii), and (iv) follow from the definition and the lemma above. We give (the sketch of) the proof (iii):

Remark 4

Let $\mathbf{R}$ be a ~~Gorenstein~~ local ring. Then $\underline{\mathbf{Hom}}_{\mathbf{R}}(-, \mathbf{M}) \in \mathbf{mod}(\underline{\mathcal{C}})$ for $\forall \mathbf{M} \in \mathcal{C}$.

$\boldsymbol{\Sigma}(\mathcal{X}) := \{\mathbf{F} \in \mathbf{mod}(\underline{\mathcal{C}}) \mid \mathbf{F}(\mathbf{X}) = 0 \text{ for } \forall \mathbf{X} \in \mathcal{X}\}$
$\gamma(\mathcal{F}) := \{\mathbf{M} \in \mathbf{Sp}(\mathcal{C}) \mid \mathbf{F}(\mathbf{M}) = 0 \text{ for } \forall \mathbf{F} \in \mathcal{F}\}.$

(Proof of (iii) $\gamma \circ \boldsymbol{\Sigma}(\mathcal{X} \cup \mathcal{Y}) = \gamma \circ \boldsymbol{\Sigma}(\mathcal{X}) \cup \gamma \circ \boldsymbol{\Sigma}(\mathcal{Y})$)

The inclusion $\supseteq$ follows from $\boldsymbol{\Sigma}(\mathcal{X} \cup \mathcal{Y}) = \boldsymbol{\Sigma}(\mathcal{X}) \cap \boldsymbol{\Sigma}(\mathcal{Y})$.

To show $\subseteq$,

- take $\mathbf{M} \in \gamma \circ \boldsymbol{\Sigma}(\mathcal{X} \cup \mathcal{Y})$. Note that $\mathbf{M}$ is **indecomposable**.
- Assume that $\mathbf{M} \notin \gamma \circ \boldsymbol{\Sigma}(\mathcal{X}) \cup \gamma \circ \boldsymbol{\Sigma}(\mathcal{Y})$. Then $\exists \mathbf{F} \in \boldsymbol{\Sigma}(\mathcal{X}), \mathbf{G} \in \boldsymbol{\Sigma}(\mathcal{Y})$ such that $\mathbf{F}(\mathbf{M}) \neq \mathbf{0}$, $\mathbf{G}(\mathbf{M}) \neq \mathbf{0}$.

We construct the functor $\mathbf{H} \in \boldsymbol{\Sigma}(\mathcal{X} \cup \mathcal{Y})$ such that $\mathbf{H}(\mathbf{M}) \neq \mathbf{0}$ using $\mathbf{F}$ and $\mathbf{G}$.

$\Sigma(\mathcal{X}) := \{F \in \mathrm{mod}(\underline{\mathcal{C}}) \mid F(X) = 0 \text{ for } \forall X \in \mathcal{X}\}$

Construct $H \in \Sigma(\mathcal{X} \cup \mathcal{Y})$ s.t. $H(M) \neq 0$

- By Yoneda's Lemma,

$$\exists f : \underline{\mathrm{Hom}}_R(-, M) \to F, \quad \exists g : \underline{\mathrm{Hom}}_R(-, M) \to G.$$

- Taking pushout diagram in $\mathrm{mod}(\underline{\mathcal{C}})$:

$$\begin{array}{ccccc}
\underline{\mathrm{Hom}}_R(-, M) & \longrightarrow & \mathrm{Im}\, f & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\mathrm{Im}\, g & \longrightarrow & H & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
0 & & 0. & &
\end{array}$$

- Since $\mathrm{Im}\, f \in \Sigma(\mathcal{X})$, $\mathrm{Im}\, g \in \Sigma(\mathcal{Y})$, $H \in \Sigma(\mathcal{X} \cup \mathcal{Y})$.
- The exact sequence
  $\underline{\mathrm{Hom}}_R(-, M) \to \mathrm{Im}\, f \oplus \mathrm{Im}\, g \to H \to 0$ shows $H(M) \neq 0$.

$\square$

Corollary 3

Let $\mathbf{R}$ *be* ~~*Gorenstein*~~. *The assignment* $\mathcal{X} \mapsto \gamma \circ \boldsymbol{\Sigma}(\mathcal{X})$ *defines a topology on* $\mathbf{Sp}(\mathcal{C})$*: a subset* $\mathcal{X}$ *of* $\mathbf{Sp}(\mathcal{C})$ *is closed if and only if* $\gamma \circ \boldsymbol{\Sigma}(\mathcal{X}) = \mathcal{X}$.

Remark 5

For a locally coherent category $\mathscr{G}$, A bijective correspondence between **closed subsets** in $\mathbf{Sp}(\mathscr{G})$ and **Serre subcategories** in $\mathbf{mod}(\mathscr{G})$ is given in [1, 2]:

$$\mathcal{X} \mapsto \boldsymbol{\Sigma}(\mathcal{X}), \qquad \mathcal{F} \mapsto \gamma(\mathcal{F}).$$

In our setting, for a Serre subcategory $\mathcal{F} \in \mathbf{mod}(\underline{\mathcal{C}})$, $\mathcal{F} \neq \boldsymbol{\Sigma} \circ \gamma(\mathcal{F})$ in general.

Example 4

Let $\mathbf{R} = \mathbf{k}[[\mathbf{x}, \mathbf{y}]]/(\mathbf{x}^2)$. The indecomposable MCM $\mathbf{R}$-modules are

$$\mathbf{R},\ \mathbf{I} = (\mathbf{x})\mathbf{R} \text{ and } \mathbf{I_n} = (\mathbf{x}, \mathbf{y^n})\mathbf{R} \text{ for } \mathbf{n} > \mathbf{0}.$$

Since $\gamma(\underline{\mathbf{Hom}}_{\mathbf{R}}(-, \mathbf{I_n})) = \emptyset$,

$$\mathbf{\Sigma} \circ \gamma(\underline{\mathbf{Hom}}_{\mathbf{R}}(-, \mathbf{I_n})) = \mathbf{\Sigma}(\emptyset) = \mathbf{mod}(\underline{\mathcal{C}}).$$

However $\mathcal{S}(\underline{\mathbf{Hom}}_{\mathbf{R}}(-, \mathbf{I_n})) \neq \mathbf{mod}(\underline{\mathcal{C}})$.

- Since $\mathbf{KGdim}\ \underline{\mathbf{Hom}}_{\mathbf{R}}(-, \mathbf{I_n}) = 1$, $\mathbf{KGdim}\ \mathcal{S}(\underline{\mathbf{Hom}}_{\mathbf{R}}(-, \mathbf{I_n})) = 1$.
- Note that $\mathbf{KGdim}\ \underline{\mathbf{Hom}}_{\mathbf{R}}(-, \mathbf{I}) = 2$.
- Hence $\underline{\mathbf{Hom}}_{\mathbf{R}}(-, \mathbf{I}) \notin \mathcal{S}(\underline{\mathbf{Hom}}_{\mathbf{R}}(-, \mathbf{I_n}))$, so that $\mathcal{S}(\underline{\mathbf{Hom}}_{\mathbf{R}}(-, \mathbf{I_n})) \neq \mathbf{mod}(\underline{\mathcal{C}})$.

Remark 6

By using the lemma below, one can show that

$$\gamma \circ \mathbf{\Sigma}(\mathbf{X}) = \{\mathbf{X}\}$$

for $\forall \mathbf{X} \in \mathbf{Sp}(\mathcal{C})$. Hence $\mathbf{Sp}(\mathcal{C})$ is $\mathbf{T_1}$-space.

Lemma 5

Let $\mathbf{X}, \mathbf{Y} \in \mathbf{Sp}(\mathcal{C})$ with $\mathbf{X} \not\cong \mathbf{Y}$. Suppose that $\underline{\mathbf{Hom}}_{\mathbf{R}}(\mathbf{X}, \mathbf{Y}) \neq \mathbf{0}$. Then $\mathbf{Y} \notin \gamma \circ \mathbf{\Sigma}(\mathbf{X})$.

Proposition 7

Let $\mathbf{M} \in \mathbf{Sp}(\mathcal{C})$*.* $\mathbf{M}$ *is an isolated point, that is* $\{\mathbf{M}\}$ *is open, iff there exists an Auslander-Reiten (AR) sequence ending in* $\mathbf{M}$*.*

Proof.

($\Leftarrow$) Take the functor $\mathbf{S_M}$ obtain from the AR-sequence. Then $\gamma(\mathbf{S_M}) = \mathbf{Sp}(\mathcal{C}) \backslash \{\mathbf{M}\}$, which is closed.

($\Rightarrow$) It follows from the fact that $\mathbf{X}$ which appears in $\mathbf{Hom_R}(-, \mathbf{X}) \to \mathbf{F} \to \mathbf{0}$ is finitely generated.

$\square$

Corollary 6

Let $\mathbf{R}$ *be an isolated singularity. Then the topology of* $\mathbf{Sp}(\mathcal{C})$ *is discrete.*

Cantor-Bendixson rank

Definition 7 (Cantor-Bendixson rank)

$\mathcal{T}$ is a topological space.

- If $\mathbf{x} \in \mathcal{T}$ is an isolated point, then $\mathbf{CB(x) = 0}$.
- Put $\mathcal{T}' \subset \mathcal{T}$ is a set of the **non**-isolated point. Define the induced topology on $\mathcal{T}'$. Set

$$\mathcal{T}^{(0)} = \mathcal{T}, \mathcal{T}^{(1)} = \mathcal{T}^{(0)\prime}, \cdots, \mathcal{T}^{(n+1)} = \mathcal{T}^{(n)\prime}.$$

  We define $\mathbf{CB(x) = n}$ if $\mathbf{x} \in \mathcal{T}^{(n)} \backslash \mathcal{T}^{(n+1)}$
- If $\exists \mathbf{n}$ such that $\mathcal{T}^{(n+1)} = \emptyset$ and $\mathcal{T}^{(n)} \neq \emptyset$, then $\mathbf{CB}(\mathcal{T}) = \mathbf{n}$.
- If $\mathcal{T}^{\infty} := \bigcap \mathcal{T}^{(n)} \neq \emptyset$, then $\mathbf{CB}(\mathcal{T}) = \infty$.

Example 8

Let $\mathbf{R}$ be a DVR (e,g. $\mathbf{R} = \mathbf{k}[[\mathbf{x}]]$). Then $\mathbf{CB}(\mathbf{SpecR}) = \mathbf{1}$ concerning the Zariski topology. Note that $\mathbf{SpecR} = \{(0), \mathfrak{m}\}$. $(\mathbf{0})$ is an isolated point since $\mathbf{D}(\mathbf{f}) = \{(\mathbf{0})\}$ for some $\mathbf{f} \in \mathbf{R} \backslash \{\mathbf{0}\}$. Thus $\mathbf{SpecR}' = \{\mathfrak{m}\} = \mathbf{SpecR}^{(1)}$, and $\mathfrak{m}$ is isolated in the induced topology.

Corollary 9

Let $\mathbf{R}$ *be an isolated singularity. Then* $\mathbf{CB}(\mathbf{Sp}(\mathcal{C})) = \mathbf{0}$.

Proof.

$\mathbf{Sp}(\mathcal{C})$ is a discrete topology, □

Definition 10 ($\mathbf{CM}_+$-finite [Kobayashi, et al. 2020])

We say that a Cohen–Macaulay local ring $\mathbf{R}$ is $\mathbf{CM}_+$-finite if there exist only finitely many isomorphism classes of indecomposable MCM modules that are **not** locally free on the punctured spectrum.

Example 11

The following rings are $\mathbf{CM}_+$-finite.

1. A ring which is an isolated singularity. (Thus a ring which is of finite $\mathbf{CM}$-representation type.)
2. A hypersurface ring which is of countable $\mathbf{CM}$-representation type.

Here we say that $\mathbf{R}$ is of finite (countable) $\mathbf{CM}$-representation type if there exists only finitely (countably) many isomorphism classes of indecomposable MCM modules.

CB-rank of CM_+-finite representation type i

Theorem B

If $\mathbf{R}$ is CM_+-finite then $\mathrm{CB}(\mathbf{Sp}(\mathcal{C})) \leq \mathbf{1}$.

(Proof)

We denote by $\mathcal{C}_\mathbf{0}$ the subset of $\mathbf{Sp}(\mathcal{C})$ consisting of modules that are locally free on the punctured spectrum and put $\mathcal{C}_+ := \mathbf{Sp}(\mathcal{C}) \backslash \mathcal{C}_\mathbf{0}$.

- For $\forall \mathbf{M} \in \mathcal{C}_\mathbf{0}$, $\mathbf{M}$ is an isolated point since $\mathbf{M}$ admits an AR-sequence. Thus $\mathrm{CB}(\mathcal{C}_\mathbf{0}) = \mathbf{0}$.
- On the other hand, for $\forall \mathbf{M} \in \mathcal{C}_+$, $\mathbf{M}$ is not isolated.

CB-rank of CM_+-finite representation type ii

- Since $\mathbf{R}$ is CM_+-finite, $\mathcal{C}_+$ is a finite set. Hence, for $\forall \mathbf{M} \in \mathcal{C}_+$,
$$\mathbf{V_M} := \bigcup_{\substack{\mathbf{X} \neq \mathbf{M} \\ \mathbf{X} \in \mathcal{C}_+}}^{\text{finite}} \gamma \circ \mathbf{\Sigma(X)}$$
is closed in $\mathbf{Sp}(\mathcal{C})$.
- Thus
$$[\mathcal{C}_+] \bigcap [\mathbf{Sp}(\mathcal{C}) \backslash \mathbf{V_M}] = \{\mathbf{M}\}$$
is open in $\mathcal{C}_+ \cap \mathbf{Sp}(\mathcal{C})$.
- Therefore $\mathrm{C}\mathbf{B}(\mathbf{Sp}(\mathcal{C})) \leq \mathbf{1}$.

□

Thank you for your attention.