On quiver Heisenberg algebras and the algebra ${}^{v}B(Q)$

joint work with Martin Herschend

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§1. Throughout this talk,
k is an algebraically closed field and
Q is a finite acyclic quiver.

For kQ-module M, the dimension vector <u>dim</u>M is regarded as an element of $\mathbf{k}Q_0 = \mathbf{k} \times \cdots \times \mathbf{k}$ (not of $\mathbb{Z}Q_0$).

For $v \in \mathbf{k}Q_0$, we set

$$^{v}\dim M := \sum_{i \in Q_0} v_i \dim e_i M.$$

<u>Def.</u> $v \in \mathbf{k}Q_0$ is called **regular** if

$$^{v}\dim M \neq 0 \quad (\forall M \in \operatorname{ind} Q)$$

inside k.

Example. Let Q be a directed A_3-quiver.

$$Q: 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 .$$

The dimension vectors of indecomposable modules are

$$\begin{matrix}1\\0\\0\end{matrix}, \quad \begin{matrix}0\\1\\0\end{matrix}, \quad \begin{matrix}0\\0\\1\end{matrix}, \quad \begin{matrix}1\\1\\0\end{matrix}, \quad \begin{matrix}0\\1\\1\end{matrix}, \quad \begin{matrix}1\\1\\1\end{matrix}.$$

Thus, regularity of a weight $v = (v_1, v_2, v_3)^t$ is

$$v_1 \neq 0,\ v_2 \neq 0,\ v_3 \neq 0,$$
$$v_1 + v_2 \neq 0,\ v_2 + v_3 \neq 0,\ v_1 + v_2 + v_3 \neq 0.$$

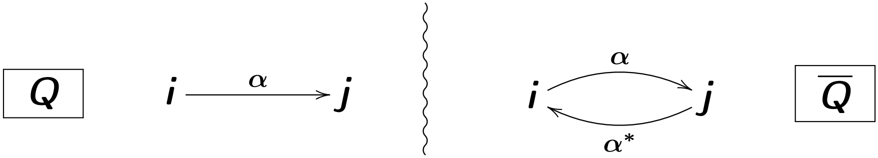
Lem. v : regular $\Rightarrow v_i \neq 0\ (\forall i \in Q_0)$.

Definition 1

The quiver Heisenberg algebra ${}^v\Lambda(Q)$ with the weight $v \in \mathbf{k}Q_0$ is defined to be

$$ {}^v\Lambda(Q) := \frac{\mathbf{k}[z]\overline{Q}}{(\rho_i - v_i z e_i \mid i \in Q_0)} $$

$\overline{Q}$ denotes the double of Q.



For $i \in Q_0$, ρ_i denotes the mesh relation at i

$$ \rho_i := \sum_{\alpha \in Q_1 : t(\alpha) = i} \alpha\alpha^* - \sum_{\alpha \in Q_1 : h(\alpha) = i} \alpha^*\alpha. $$

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$$ {}^{v}\Lambda(Q) := \frac{\mathbf{k}[z]\overline{Q}}{(\rho_i - v_i z e_i \mid i \in Q_0)} $$

If $v_i \neq 0$ $(\forall i \in Q_0)$ (e.g, v is regular), then ${}^{v}\Lambda(Q)$ is isomorphic to the previous definition:

$$ {}^{v}\Lambda(Q) \cong \frac{\mathbf{k}\overline{Q}}{([a, {}^{v}\rho] \mid a \in \overline{Q}_1)}, \quad z \mapsto {}^{v}\rho $$

where ${}^{v}\rho := \sum_i v_i^{-1}\rho_i$ "weighted mesh relation" and $[a, {}^{v}\rho] = a{}^{v}\rho - {}^{v}\rho a$ is the commutator.

Theorem 2

If $\mathbf{v}$ *is regular, then as* $\mathbf{k}Q$*-modules*

$$^{v}\Lambda(Q) \cong \bigoplus_{M:\ \text{indec.pp.mod.}} M^{\dim M}.$$

In particular, if Q is Dynkin and v is regular, then $\dim {}^{v}\Lambda(Q) < \infty$.

Theorem 3

Assume that $\mathbf{Q}$ *is Dynkin. Then,*

$$v \text{ is regular} \iff \dim {}^{v}\Lambda(Q) < \infty$$

We explain the proof of $\Longleftarrow$ by contraposition.

• In the case $\exists i \in Q_0$ s.t. $v_i = 0$,
we can directly show that $\dim {}^v\Lambda(Q) = \infty$.

• Assume $v_i \neq 0 (\forall i \in Q_0)$.
QHA ${}^v\Lambda(Q)$ acquires the grading
that counts the numbers of extra arrows α^*.

Theorem 4

Let $M \in \operatorname{ind} Q$ with ${}^v\dim M = 0$.
Then, for all $n \geq 1$,
${}^v\Lambda(Q)_n \otimes_{kQ} M$ contains M as a direct summand.
In particular ${}^v\Lambda(Q)_n \neq 0$ $(\forall n)$ and $\dim {}^v\Lambda(Q) = \infty$.

$\because \exists$ ex.tri. $\mathrm{AR}_M : M \xrightarrow{{}^v\rho} {}^v\widetilde{\Lambda}_1 \otimes^{\mathbb{L}}_{kQ} M \to \widetilde{\Pi}_1 \otimes^{\mathbb{L}}_{kQ} M$
"trace formula" & ${}^v\dim M = 0 \Rightarrow \mathrm{AR}_M$ splits.

- In the case $\exists i \in Q_0$ s.t. $v_i = 0$,

we can directly show that $\dim {}^{v}\Lambda(Q) = \infty$.

- Assume $v_i \neq 0 (\forall i \in Q_0)$.

QHA ${}^{v}\Lambda(Q)$ **acquires the grading that counts the numbers of extra arrows** α^*.

Theorem 4

Let $M \in \mathrm{ind}\, Q$ *with* ${}^{v}\mathbf{dim}\, M = \mathbf{0}$.
Then, for all $n \geq 1$,
${}^{v}\Lambda(Q)_n \otimes_{kQ} M$ *contains* M *as a direct summand.*
In particular ${}^{v}\Lambda(Q)_n \neq 0\ (\forall n)$ *and* $\dim {}^{v}\Lambda(Q) = \infty$.

$\because$ $n \geq 2$. **Induction, with the ex.tri.**

$$\widetilde{\Pi}_1 \otimes^{\mathbb{L}} {}^{v}\widetilde{\Lambda}_{n-2} \otimes^{\mathbb{L}} M \to {}^{v}\widetilde{\Lambda}_1 \otimes^{\mathbb{L}} {}^{v}\widetilde{\Lambda}_{n-1} \otimes^{\mathbb{L}} M \to {}^{v}\widetilde{\Lambda}_n \otimes^{\mathbb{L}} M$$

§2. Now we discuss symmetric property of ${}^{\mathbf{v}}\Lambda(Q)$.

Theorem 5

Assume that Q is Dynkin and $\mathbf{v}$ is regular.
Then the algebra ${}^{\mathbf{v}}\boldsymbol{\Lambda}(Q)$ is symmetric.

In the case $\operatorname{char} k = 0$, Etingof-Latour-Rains showed that ${}^{v}\Lambda(Q)$ is symmetric for a generic weight v.

Main ingredients of our proof are
(i) A general result about derived preprojective algebra of d-representation finite (RF) algebra.
(ii) The algebra ${}^{v}B(Q)$.
(iii) A direct computation of the cohomology algebra of derived QHA.

(i) Let A be a d-RF algebra.
Iyama-Oppermann showed that
the $d+1$-ppa $\Pi := \Pi_{d+1}(A)$ is Frobenius.
Let ν be the Nakayama automorphism of Π,
i.e., $\Pi \cong {}_{\nu}\mathrm{D}(\Pi)$ as Π-bimodules.

Theorem 6

Let $\widetilde{\boldsymbol{\Pi}}$ be the $\boldsymbol{d+1}$-derived ppa of $\boldsymbol{A}$.
Then, as cohomologically graded algebras,

$$\mathrm{H}(\widetilde{\boldsymbol{\Pi}}) \cong \boldsymbol{\Pi}[\boldsymbol{u};\boldsymbol{\nu}]$$

where $\boldsymbol{u}$ is a formal variable of coh.degree $\boldsymbol{-d}$ and

$$\boldsymbol{au = u\nu(a)} \quad (\forall \boldsymbol{a} \in \boldsymbol{\Pi}).$$

(ii)

Definition 7

For a quiver Q and a regular weight v, we define

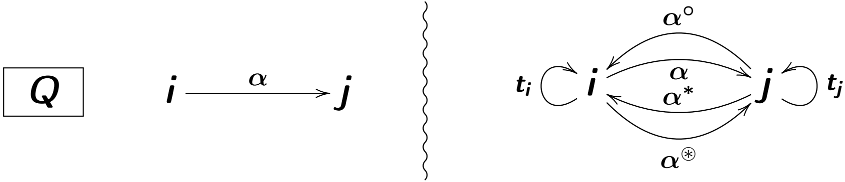
$$ {}^v B(Q) := \begin{pmatrix} \mathbf{k}Q & {}^v\Lambda(Q)_1 \\ 0 & \mathbf{k}Q \end{pmatrix} $$

the bypath algebra (a.k.a., $\mathbf{2}$-path algebra) of Q.

Theorem 8

${}^v B(Q)$ is $\mathbf{2}$-RF if and only if Q is Dynkin.
${}^v B(Q)$ is $\mathbf{2}$-RI if and only if Q is non-Dynkin.

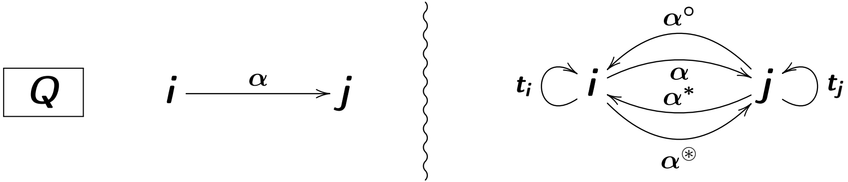
Recall that the derived QHA ${}^{\mathsf{v}}\widetilde{\Lambda}(Q)$ is a DGA explicitly defined by the quiver



the differential is defined by

$$d(\alpha) := 0, d(\alpha^*) := 0,$$
$$d(\alpha^{\circ}) := -[\alpha^*, {}^{\mathsf{v}}\rho], d(\alpha^{\circledast}) := [\alpha, {}^{\mathsf{v}}\rho],$$
$$d(t_i) := \sum_{\alpha \in Q_1} e_i[\alpha, \alpha^{\circ}]e_i + \sum_{\alpha \in Q_1} e_i[\alpha^*, \alpha^{\circledast}]e_i.$$

Recall that the derived QHA ${}^{v}\widetilde{\Lambda}(Q)$ is a DGA explicitly defined by the quiver



If char k $\neq 2$,
${}^{v}\widetilde{\Lambda}(Q)$ is the Ginzburg dg-algebra $\mathcal{G}(\overline{Q}, W)$ where

$$W := -\frac{1}{2}{}^{v}\rho\rho = -\frac{1}{2}\sum_{i \in Q_0} v_i^{-1}\rho_i^2.$$

The 3-derived preprojective algebra of ${}^{v}B(Q)$ and the 2-ed quasi-Veronese algebra of ${}^{v}\widetilde{\Lambda}(Q)$ are isomorphic

$$\widetilde{\Pi}_3({}^{v}B(Q)) \cong {}^{v}\widetilde{\Lambda}(Q)^{[2]}.$$

(iii) By (more or less) direct computation we have

Theorem 9

Assume that Q is Dynkin and $\mathbf{v}$ is regular. Then,

$$\mathrm{H}({}^{v}\widetilde{\Lambda}(Q)) \cong {}^{v}\Lambda(Q)[u]$$

where u is a formal variable of coh.degree -2.

Combining this with the previous theorem, we conclude that $\nu_{\Lambda} = \mathrm{id}_{\Lambda}$ (up to inner aut).

§3. Now we discuss tilting theory of ${}^v\Lambda = {}^v\Lambda(Q)$.

Let $i \in Q_0$. We define a complex $T^{(i)}$ over ${}^v\Lambda$ to be

$$T^{(i)} := {}^v\Lambda(1-e_i) \oplus \left[{}^v\Lambda e_i \xrightarrow{(\pm a^*)_{a\in h^{-1}(i)}} \bigoplus_{a\in h^{-1}(i)} {}^v\Lambda e_{t(a)} \right]$$

where the right factor is a complex placed in $-1, 0$-th coh. deg.

This complex is a "family version" of the tilting complex of Crawley-Boevey-Kimura.

The reduction $\Pi \otimes_{{}^v\Lambda} T^{(i)}$ is the tilting complex introduced by Baumann-Kamniter and Buan-Iyama-Reiten-Scott.

Let $r : W_Q \curvearrowright \mathbf{k}Q_0$ be the dual action.
Let r_i be the action of the Coxeter generator s_i.

Theorem 10

The complex $T^{(i)}$ is a tilting complex and

$$\mathrm{End}_{{}^v\Lambda}(T^{(i)})^{\mathrm{op}} \cong {}^{r_i(v)}\Lambda.$$

From now we assume that Q is Dynkin.
Then,

$$T^{(i)} = \mu_i^+({}^v\Lambda) \quad \text{(left silting mutation).}$$

Thus, taking iterated mutations

$${}^{w(v)}\Lambda \cong \mathrm{End}_{{}^v\Lambda}(\mu_{i_n}^+ \cdots \mu_{i_1}^+({}^v\Lambda))^{\mathrm{op}}$$

where $w = s_{i_n} \cdots s_{i_1}$.

$\exists$ bijections, the first is established by Mizuno, the second is a consequence of a general result due to Adachi-Iyama-Reiten

$$W_Q \xrightarrow{1:1} \mathsf{s\tau\,tilt}\,\Pi(Q) \xrightarrow{1:1} 2\,\mathsf{silt}\,\Pi(Q).$$

The weighted mesh rel. ${}^{v}\rho$ is central in ${}^{v}\Lambda(Q)$ and ${}^{v}\Lambda(Q)/({}^{v}\rho) \cong \Pi(Q)$. Applying a general result by Eisele-Janssens-Raedschelders, we obtain bijections

$$W_Q \xrightarrow{1:1} \mathsf{s\tau\,tilt}\,{}^{v}\Lambda(Q) \xrightarrow{1:1} 2\,\mathsf{silt}\,{}^{v}\Lambda(Q)$$

which is given by

$$w = s_{i_n} \cdots s_{i_1} \mapsto \mu^{+}_{i_n} \cdots \mu^{+}_{i_1}({}^{v}\Lambda)$$

By general criteria due to Aihara-Mizuno, we conclude that ${}^{v}\Lambda(Q)$ is silting discrete.

We note silt ${}^v\Lambda$ = tilt ${}^v\Lambda$ by Theorem 5.

Theorem 11

Assume that Q is Dynkin and v is regular.

1. *The algebra ${}^v\Lambda(Q)$ is silting discrete.*
2. *A silting complex T is a tilting complex and*

$$\mathrm{End}_{{}^v\Lambda(Q)}(T)^{\mathrm{op}} \cong {}^{w(v)}\Lambda(Q)$$

for some $w \in W_Q$.

Theorem 12

Let B_Q be the braid group of Q. There is a bijection

$$B_Q \xrightarrow{1:1} \mathrm{silt}\, {}^v\Lambda(Q), \; b \mapsto \mu_b({}^v\Lambda(Q)).$$

Thank you for your browsing.