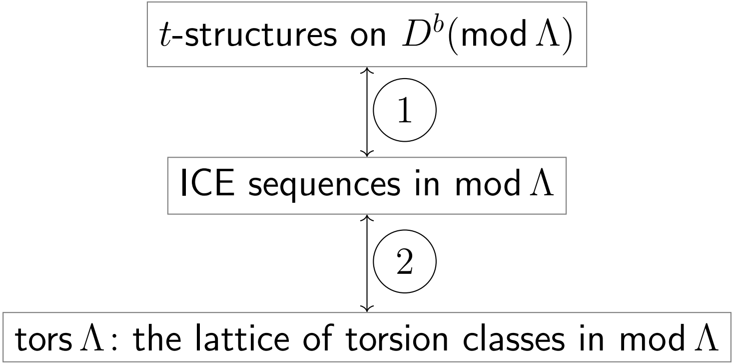
A classification of t-structures by a lattice of torsion classes

Arashi Sakai

Nagoya University
mail: m20019b@math.nagoya-u.ac.jp

The 55th Symposium on Ring Theory and Representation Theory,
5 September, 2023

Let Λ be a finite dimensional K-algebra.



Convention

- K : a field.
- Λ : a finite dimensional K-algebra.
- $\operatorname{mod}\Lambda$: the category of finitely generated right Λ-module.
- $D^b(\operatorname{mod}\Lambda)$: the bounded derived category of $\operatorname{mod}\Lambda$.
- All subcategories are full and additive.

Table of contents

Table of contents

Definition

A pair of subcategory $(\mathcal{U}, \mathcal{V})$ of $D^b(\text{mod}\,\Lambda)$ is a t-structure on $D^b(\text{mod}\,\Lambda)$ if it satisfies

1. $\mathrm{Hom}(\mathcal{U}, \mathcal{V}) = 0$,
2. $\mathcal{U} * \mathcal{V} = D^b(\text{mod}\,\Lambda)$,
3. $\mathcal{U}[1] \subseteq \mathcal{U}$.

- We call $\mathcal{U}$ the aisle of $(\mathcal{U}, \mathcal{V})$.
- $\mathcal{V} = \mathcal{U}^{\perp} := \{X \in D^b(\text{mod}\,\Lambda) \mid \mathrm{Hom}(\mathcal{U}, X) = 0\}$ holds.
- A t-structure is determined by its aisle.

Definition

A subcategory $\mathcal{U}$ of $D^b(\operatorname{mod}\Lambda)$ is a preaisle of $D^b(\operatorname{mod}\Lambda)$ if it is closed under extensions and positive shifts.

Aisles and thick subcategories are preaisles.

Proposition

TFAE for a subcategory $\mathcal{U}$ of $D^b(\operatorname{mod}\Lambda)$.

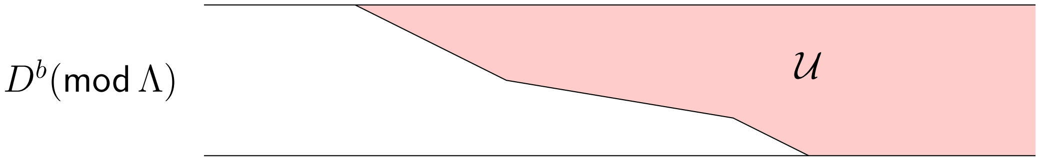
1. *$\mathcal{U}$ is an aisle of a t-structure on $D^b(\operatorname{mod}\Lambda)$.*
2. *$\mathcal{U}$ is a preaisle and the inclusion $\mathcal{U} \hookrightarrow D^b(\operatorname{mod}\Lambda)$ has a right adjoint functor.*

$H^k\colon D^b(\operatorname{mod}\Lambda) \to \operatorname{mod}\Lambda$: the k-th cohomology functor $(k \in \mathbb{Z})$

$\mathcal{U}$: a preaisle of $D^b(\operatorname{mod}\Lambda)$

$\rightsquigarrow H^k\mathcal{U} = \{H^kU \mid U \in \mathcal{U}\}$: a subcategory of $\operatorname{mod}\Lambda$

$\rightsquigarrow$ We consider a sequence $\{H^k\mathcal{U}\}_{k\in\mathbb{Z}}$ of subcategories of $\operatorname{mod}\Lambda$.



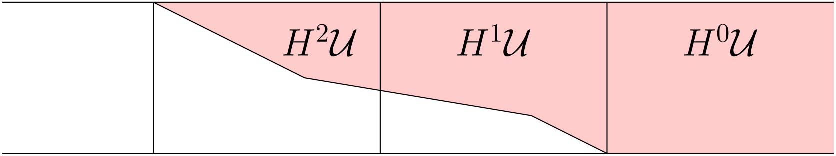
$H^k \colon D^b(\operatorname{mod}\Lambda) \to \operatorname{mod}\Lambda$: the k-th cohomology functor $(k \in \mathbb{Z})$

$\mathcal{U}$: a preaisle of $D^b(\operatorname{mod}\Lambda)$

$\rightsquigarrow H^k\mathcal{U} = \{H^k U \mid U \in \mathcal{U}\}$: a subcategory of $\operatorname{mod}\Lambda$

$\rightsquigarrow$ We consider a sequence $\{H^k\mathcal{U}\}_{k\in\mathbb{Z}}$ of subcategories of $\operatorname{mod}\Lambda$.

$D^b(\operatorname{mod}\Lambda)$



Definition

Let $\mathcal{C}$ be a subcategory of mod Λ.

1. $\mathcal{C}$ is a torsion class if $\mathcal{C}$ is closed under taking extensions and quotients.
2. $\mathcal{C}$ is a wide subcategory if $\mathcal{C}$ is closed under taking extensions, kernels and cokernels.

mod Λ		$D^b(\text{mod}\,\Lambda)$
Torsion classes	$\longleftrightarrow$	intermediate t-structures
wide subcategories	$\longleftrightarrow$	H^0-stable thick subcategories

Definition

A subcategory $\mathcal{C}$ of mod Λ is an ICE-closed subcategory if it is closed under Images, Cokernels and Extensions.

Torsion classes and wide subcategories are ICE-closed subcategories.

Proposition (Ingalls-Thomas, Enomoto)

Let $\mathcal{C}$ be an ICE-closed subcategory of mod Λ. *Then*

$$\alpha\mathcal{C} = \{X \in \mathcal{C} \mid {}^{\forall}(f : Y \to X) \in \mathcal{C} \ \operatorname{Ker} f \in \mathcal{C}\}$$

is a wide subcategory of mod Λ.

Definition

A sequence of subcategory $\{\mathcal{C}(k)\}_{k\in\mathbb{Z}}$ of mod Λ is an ICE sequnce if for any $k \in \mathbb{Z}$, it satisfies

1. $\mathcal{C}(k)$ is an ICE-closed subcategory of mod Λ,
2. $\mathcal{C}(k+1)$ is a torsion class in $\alpha(\mathcal{C}(k))$.

$$\cdots \subseteq \mathcal{C}(1) \subseteq \mathcal{C}(0) \subseteq \mathcal{C}(-1) \subseteq \cdots$$

This coincides with a narrow sequence introduced by Stanley and van-Roosmalen. They consider narrow sequences in hereditary abelian categories.

Theorem (S)

There is a bijective correspondence between

1. *the set of homology-determined preaisles of* $D^b(\mathrm{mod}\,\Lambda)$,
2. *the set of ICE sequences in* $\mathrm{mod}\,\Lambda$.

Λ: hereditary $\Rightarrow$ Every aisle is homology-determined.

Theorem (S)

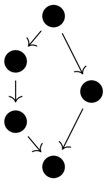
Let Λ be a τ-tilting finite algebra. Then there is a bijective correspondences between

1. *the set of bounded t-structures on $D^b(\operatorname{mod}\Lambda)$ whose aisles are homology-determined,*
2. *the set of full ICE sequences in $\operatorname{mod}\Lambda$.*

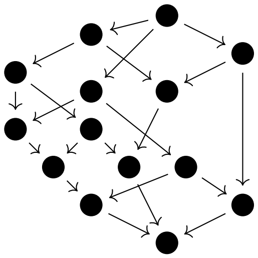
- An algebra Λ is τ-tilting finite if there are only finitely many torsion classes in $\operatorname{mod}\Lambda$.
- An ICE sequence $\{\mathcal{C}(k)\}_{k\in\mathbb{Z}}$ is full if there exist $m \leq n$ such that $\mathcal{C}(n) = 0$ and $\mathcal{C}(m) = \operatorname{mod}\Lambda$.

Table of contents

tors Λ : the set of torsion classes in mod Λ, a poset by inclusion.
Moreover tors Λ forms a lattice.
Hasse(tors Λ) : the Hasse quiver of tors Λ.



$\Lambda = K[1 \leftarrow 2]$



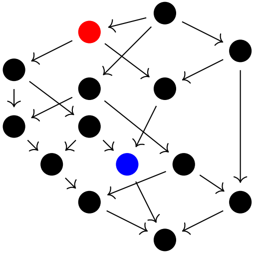
$\Lambda = K[1 \leftarrow 2 \leftarrow 3]$

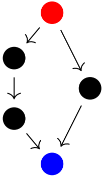
We give a description of full ICE sequences using the lattice tors Λ.

Proposition (Asai-Pfeifer, Enomoto-S)

Let $\mathcal{T}$ *be a torsion class in* $\operatorname{mod}\Lambda$. *Set* $\mathcal{T}^- := \mathcal{T} \cap \bigcap\{\mathcal{U} \mid \mathcal{T} \to \mathcal{U} \text{ in } \operatorname{Hasse}(\operatorname{tors}\Lambda)\}$. *Then*

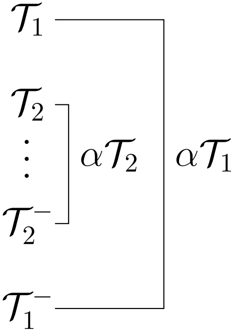
1. $\alpha\mathcal{T} = \mathcal{T} \cap (\mathcal{T}^-)^\perp$
2. $[\mathcal{T}^-, \mathcal{T}] := \{\mathcal{U} \mid \mathcal{T}^- \leq \mathcal{U} \leq \mathcal{T}\}$ *is isomorphic to* $\operatorname{tors}(\alpha\mathcal{T})$.





$$0 = \mathcal{C}(n) \subseteq \mathcal{C}(n-1) \subseteq \cdots \subseteq \mathcal{C}(1) \subseteq \mathcal{C}(0) = \mathsf{mod}\,\Lambda$$

: an ICE sequence in mod Λ.



Hasse(tors Λ)

$\rightsquigarrow \{[\mathcal{T}_i^-, \mathcal{T}_i]\}_{i=1}^{n-1}$: a decreasing sequence of maximal meet interval.

Theorem

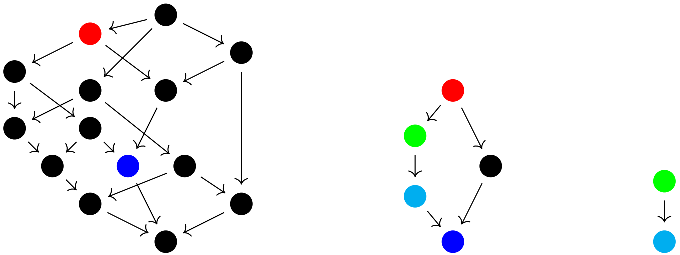
Let Λ be a τ-tilting finite algebra. Then there are bijective correspondences between

1. *the set of $(n+1)$-intermediate t-structures on $D^b(\operatorname{mod}\Lambda)$ whose aisles are homology-determined,*
2. *the set of ICE sequences in $\operatorname{mod}\Lambda$ of length $n+1$,*
3. *the set of decreasing sequences of maximal meet intervals in $\operatorname{tors}\Lambda$ of length n.*

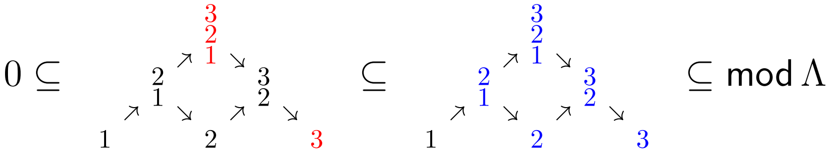
Every bounded t-structure on $D^b(\operatorname{mod}\Lambda)$ is $(n+1)$-intermediate for some $n \in \mathbb{Z}_{\geq 0}$, up to shifts.

Table of contents

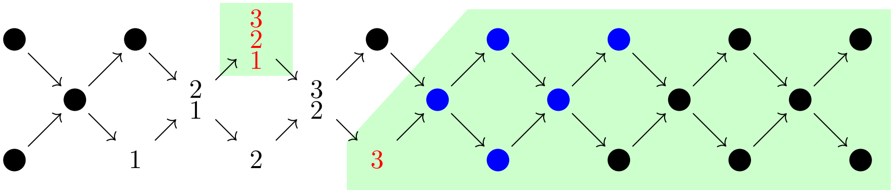
$$\Lambda = K[1 \leftarrow 2 \leftarrow 3]$$



:a decreasing sequence of maximal meet interval of length 2.



:an ICE sequence in mod Λ of length 3.



: an aisle of a 3-intermediate t-structure on $D^b(\mathrm{mod}\,\Lambda)$

References I

[AP] S. Asai, C. Pfeifer, *Wide subcategories and lattices of torsion classes*, Algebr. Represent. Theory 25 (2022), no. 6, 1611–1629.

[Eno] H. Enomoto, *Rigid modules and ICE-closed subcategories over path algebras*, J. Algebra 594 (2022), 364–388.

[ES] H. Enomoto, A. Sakai, *ICE-closed subcategories and wide τ-tilting modules*, Math. Z. 300 (2022), no. 1, 541–577.

References II

[IT] C. Ingalls, H. Thomas, *Noncrossing partitions and representations of quivers*, Compos. Math. 145 (2009), no. 6, 1533–1562.

[Sak] A. Sakai, *Classifying t-structures via ICE-closed subcategories and a lattice of torsion classes*, arXiv:2307.11347.

[SR] D. Stanley, A.-C. van Roosmalen, *t-structures on hereditary categories*, Math. Z. 293 (2019), no. 1–2, 731–766.