On Interval Global Dimension of Posets: a Characterization of Case 0

多田 駿介

神戸大学　人間発達環境学研究科

$$1 \longleftrightarrow \cdots \longleftrightarrow n$$

$$\begin{array}{ccccc} & & 1 \to 2 \to \cdots \to n & & \\ & \nearrow & & \searrow & \\ \hat{0} & & & & \hat{1} \\ & \searrow & & \nearrow & \\ & & 1' \to 2' \to \cdots \to m' & & \end{array}$$

Joint work with

青木 利隆 氏(神戸) エスカラ エマソン ガウ 氏(神戸)

Preprint Summand-injectivity of interval approximations and monotonicity of interval global dimension. Toshitaka Aoki, Emerson G. Escolar, Shunsuke Tada. arXiv:2308.14979.

発表の流れ

■位相的データ解析とは？

■パーシステンス加群(隣接代数の加群)

■得られた結果

発表の流れ

■位相的データ解析とは？

■パーシステンス加群(隣接代数の加群)

■得られた結果

位相的データ解析(TDA)とは？

Topological Data Analysis

トポロジーを用いたデータ解析手法

位相的データ解析(TDA)とは？

Topological Data Analysis

トポロジーを用いたデータ解析手法

・パーシステントホモロジー解析

・Mapper解析

・topological flow analysis

位相的データ解析(TDA)とは？

Topological Data Analysis

トポロジーを用いたデータ解析手法

- パーシステントホモロジー解析
- Mapper解析
- topological flow analysis

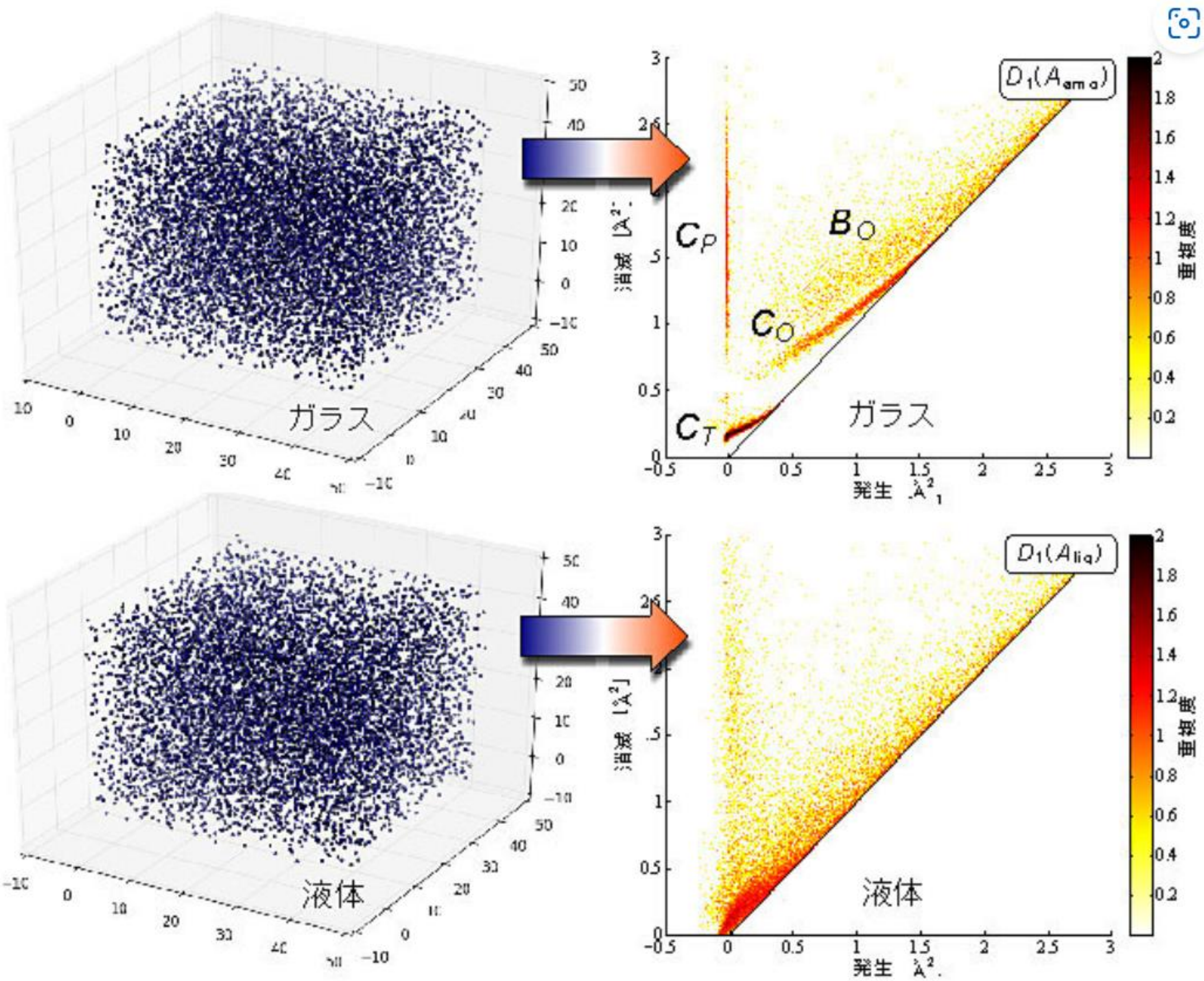


図１　ＳｉＯ２の原子配置（左）とそのパーシステントホモロジー（右）

[共同発表：ガラスの「形」を数学的に解明～トポロジーで読み解く無秩序の中の秩序～ (jst.go.jp)]

Hiraoka, Yasuaki, et al. "Hierarchical structures of amorphous solids characterized by persistent homology." *Proceedings of the National Academy of Sciences* 113.26 (2016): 7035-7040.

応用例

- 材料科学
- 進化生物学
- 物理(宇宙)
- スポーツ科学

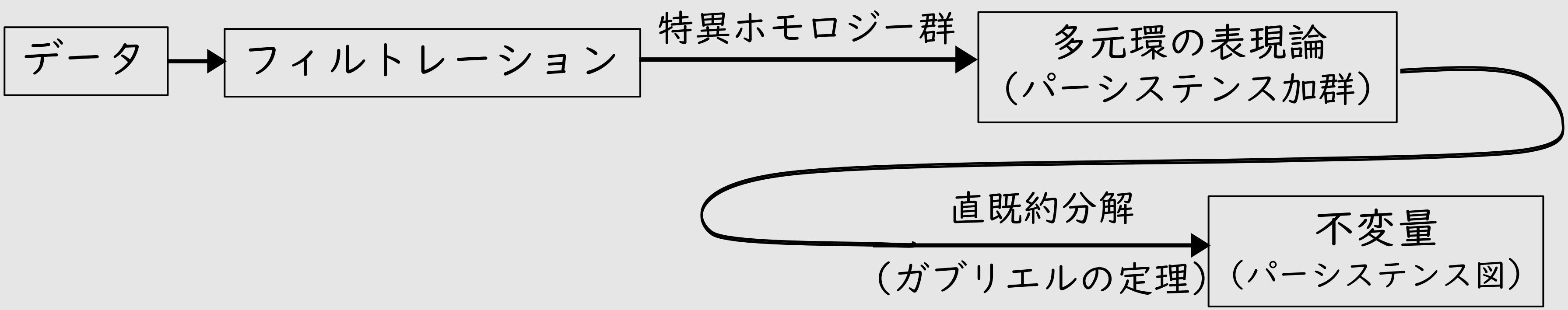
(see Zotero | Groups > TDA-Applications)

パーシステントホモロジー解析

データの形(穴や空洞)の
「パーシステンス」(持続性)
に着目

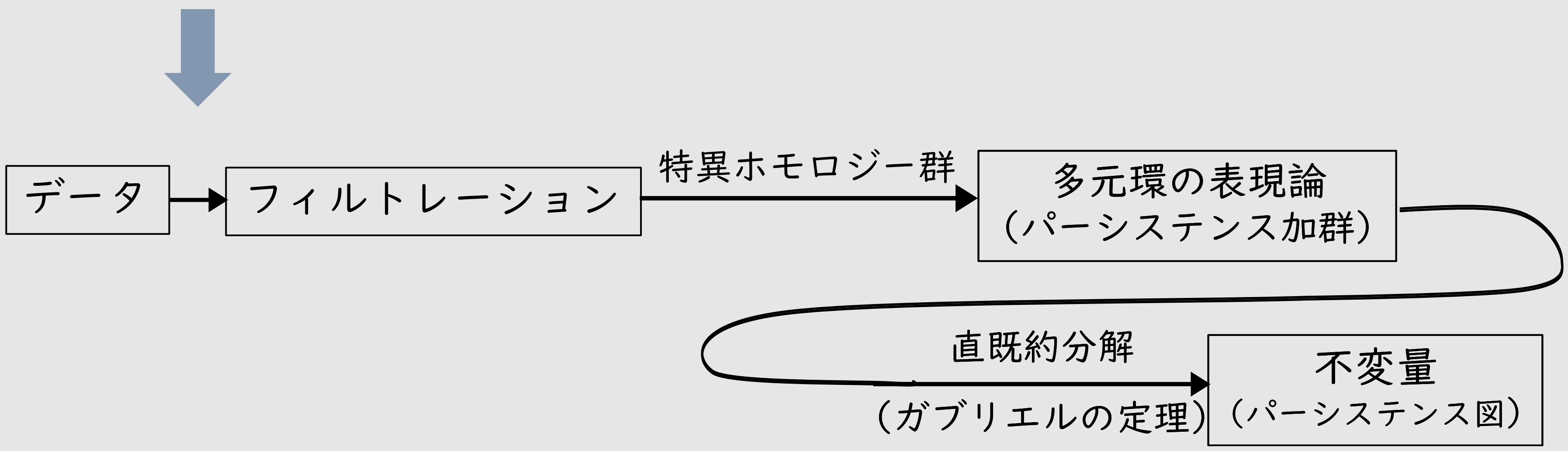
パーシステントホモロジー解析

データの形(穴や空洞)の「パーシステンス」(持続性)に着目



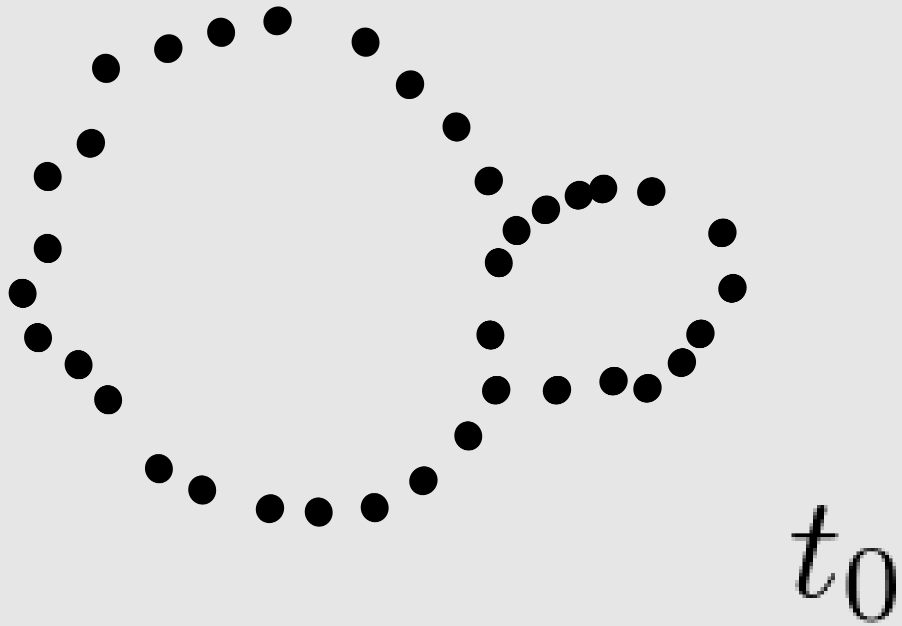
パーシステントホモロジー解析

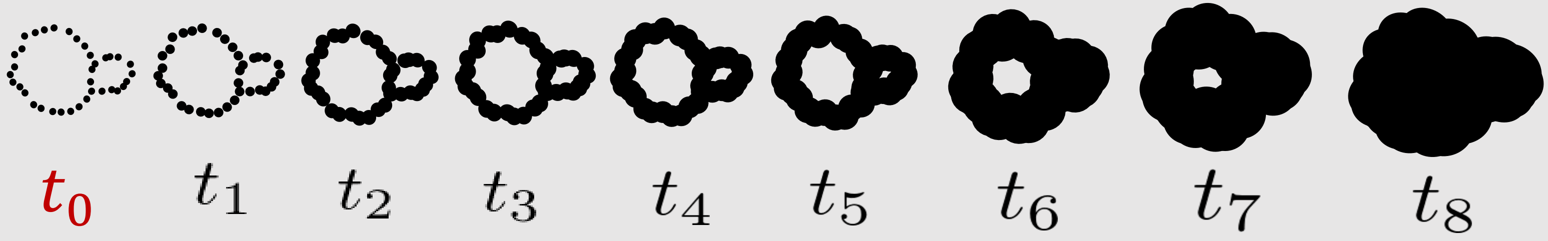
データの形(穴や空洞)の
「パーシステンス」(持続性)
に着目



点群データから半径パラメータtを大きくすることにより

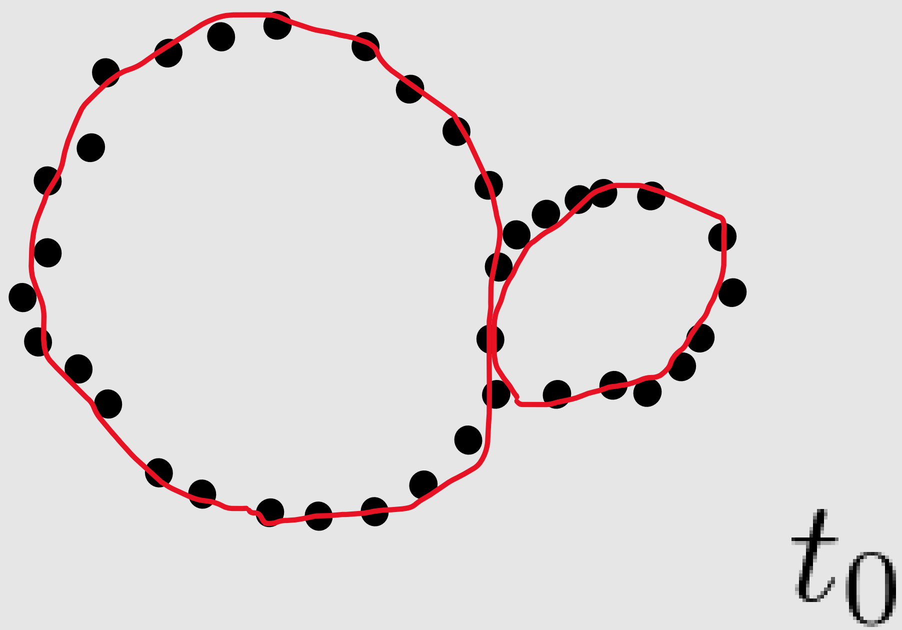
フィルトレーションを構成

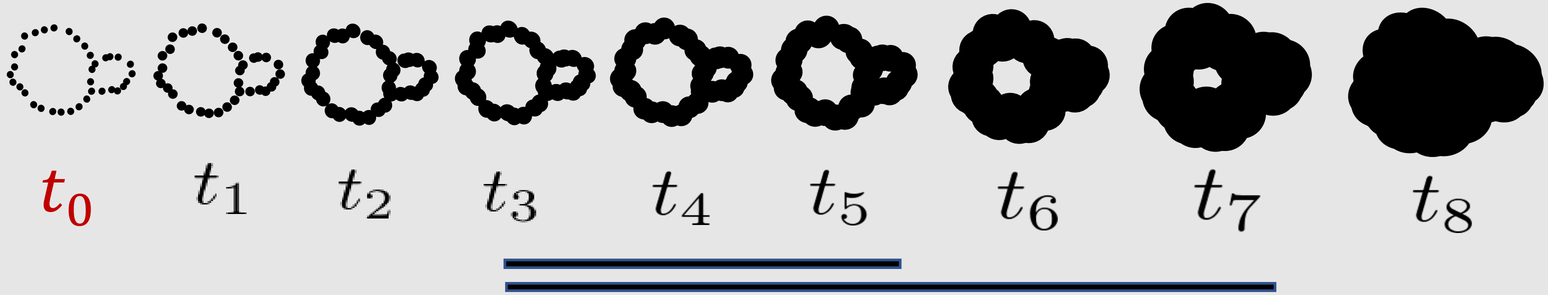




点群データから半径パラメータtを大きくすることにより

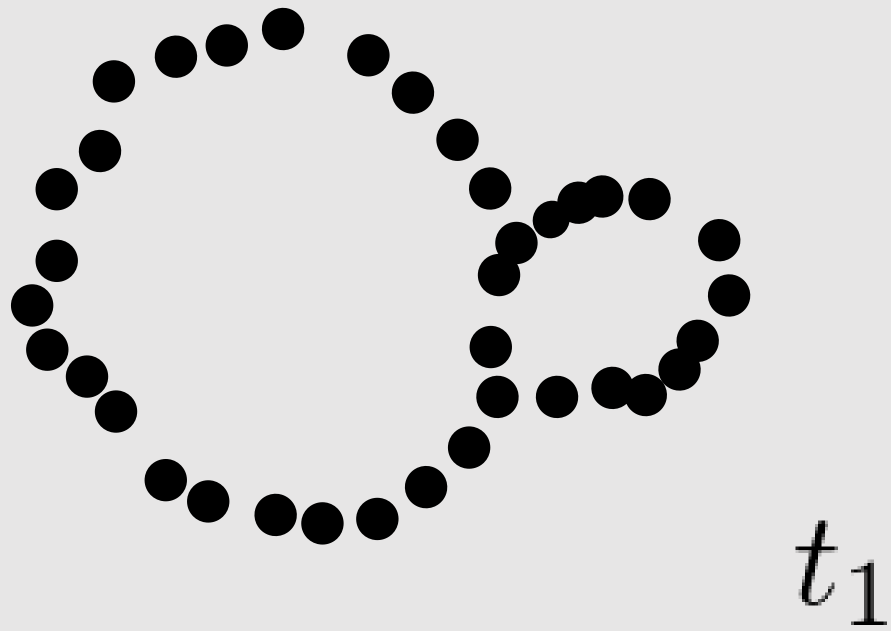
フィルトレーションを構成

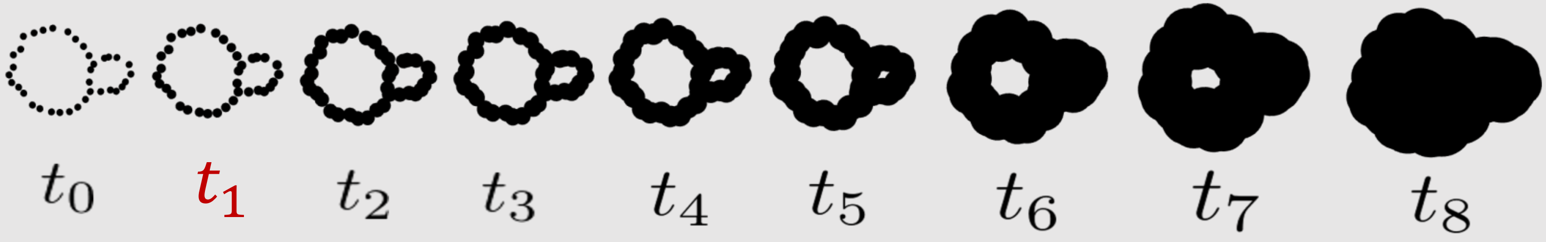




点群データから半径パラメータtを大きくすることにより

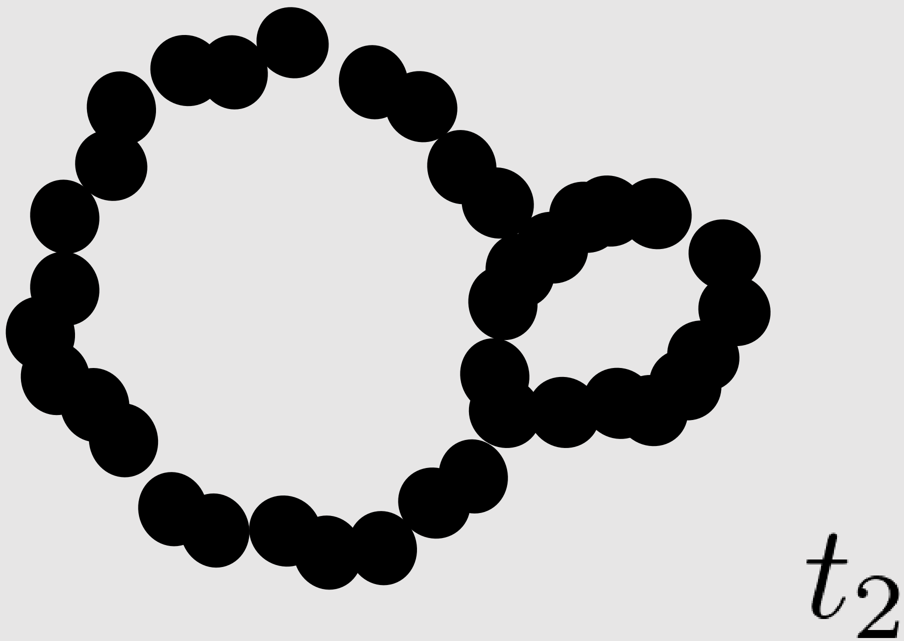
フィルトレーションを構成

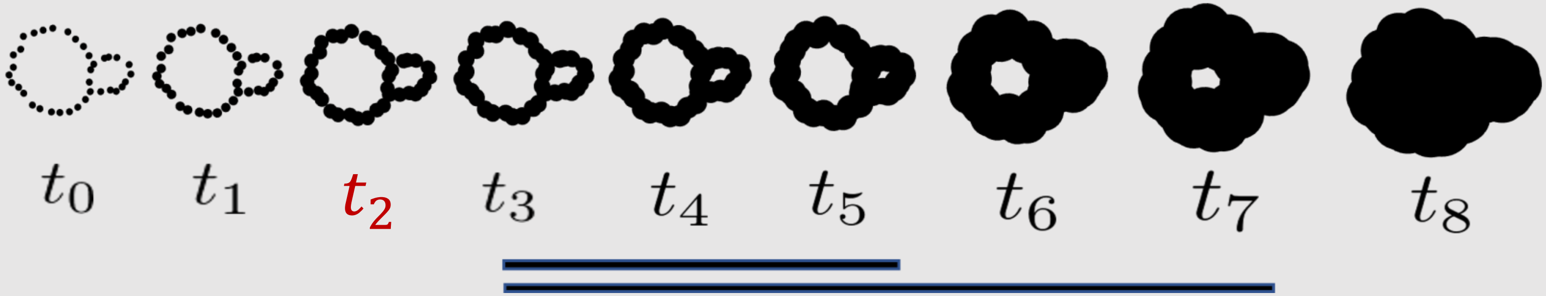




点群データから半径パラメータtを大きくすることにより

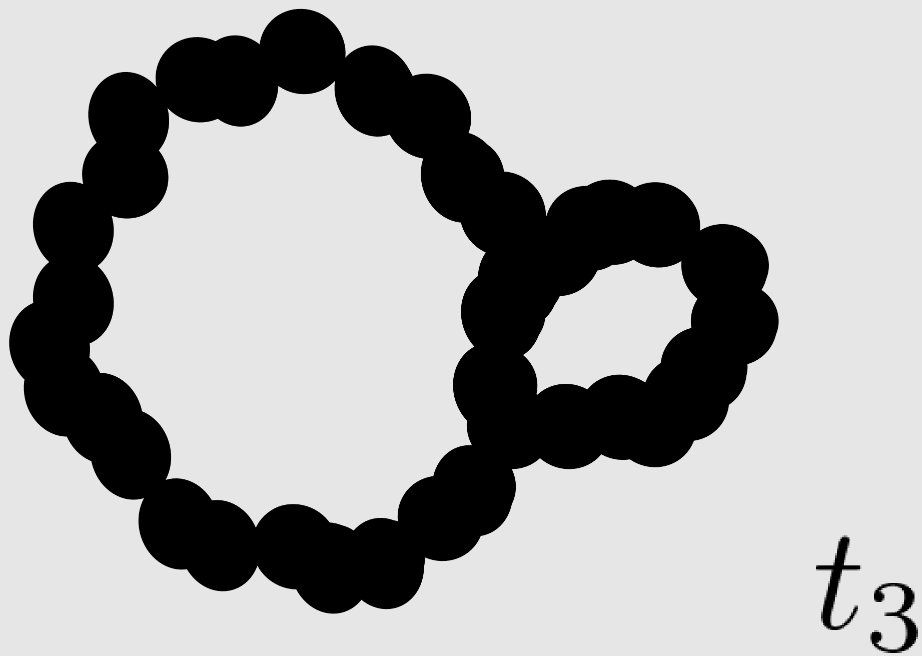
フィルトレーションを構成

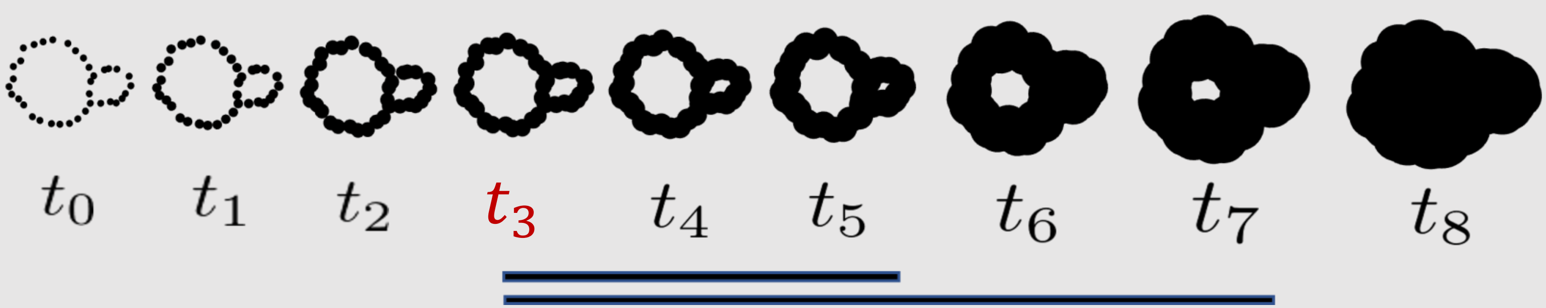




点群データから半径パラメータtを大きくすることにより

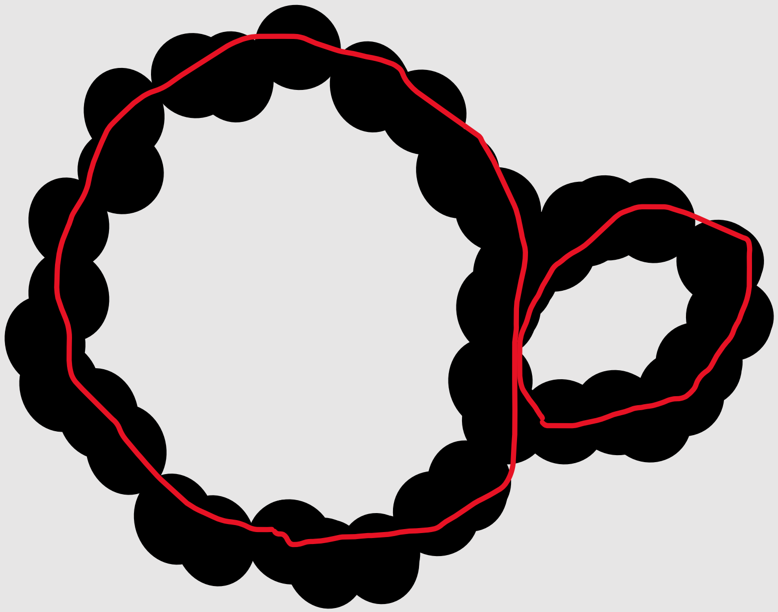
フィルトレーションを構成



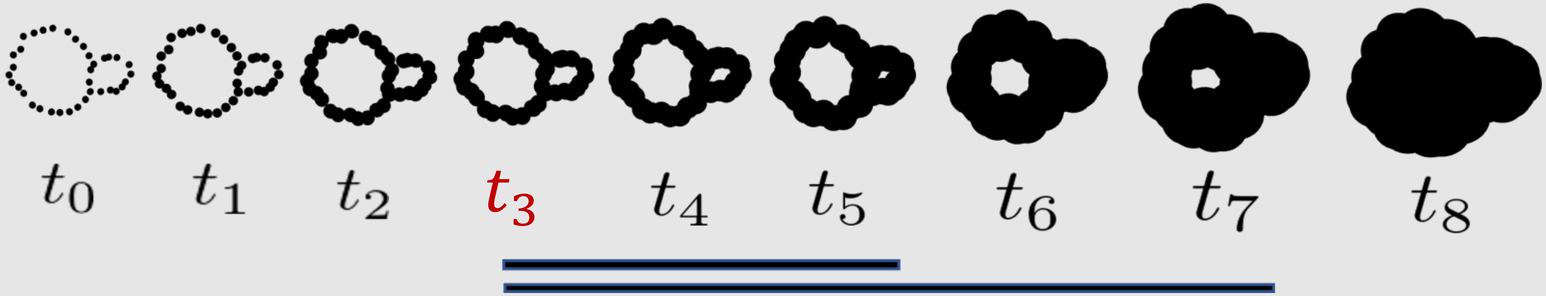


点群データから半径パラメータtを大きくすることにより

フィルトレーションを構成

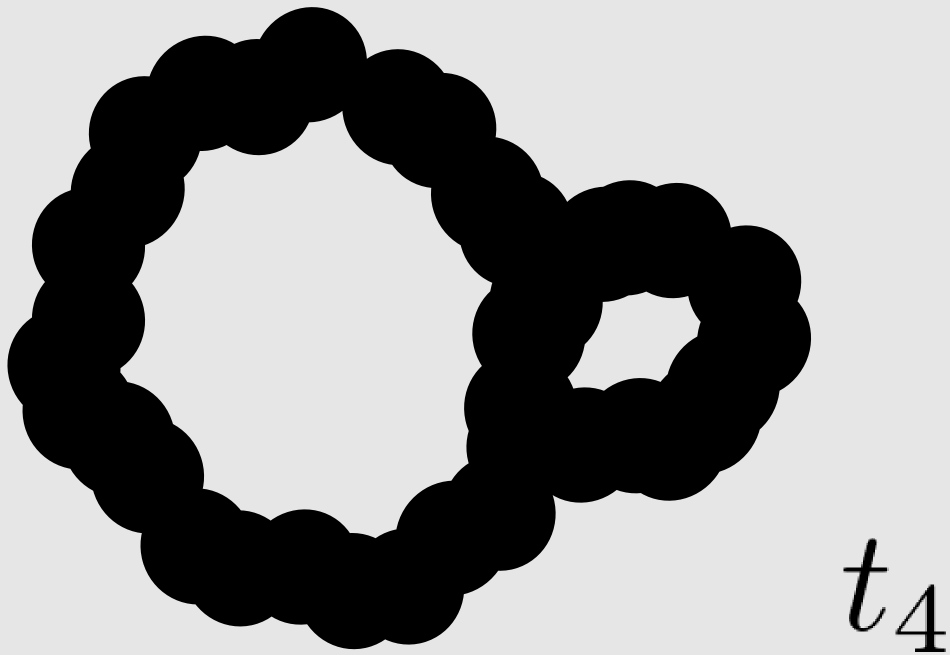


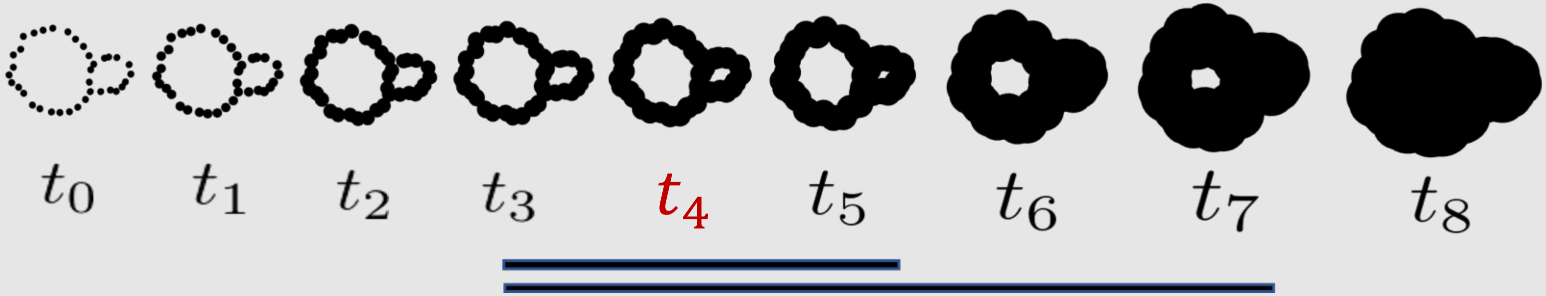
t_3 穴の生成



点群データから半径パラメータtを大きくすることにより

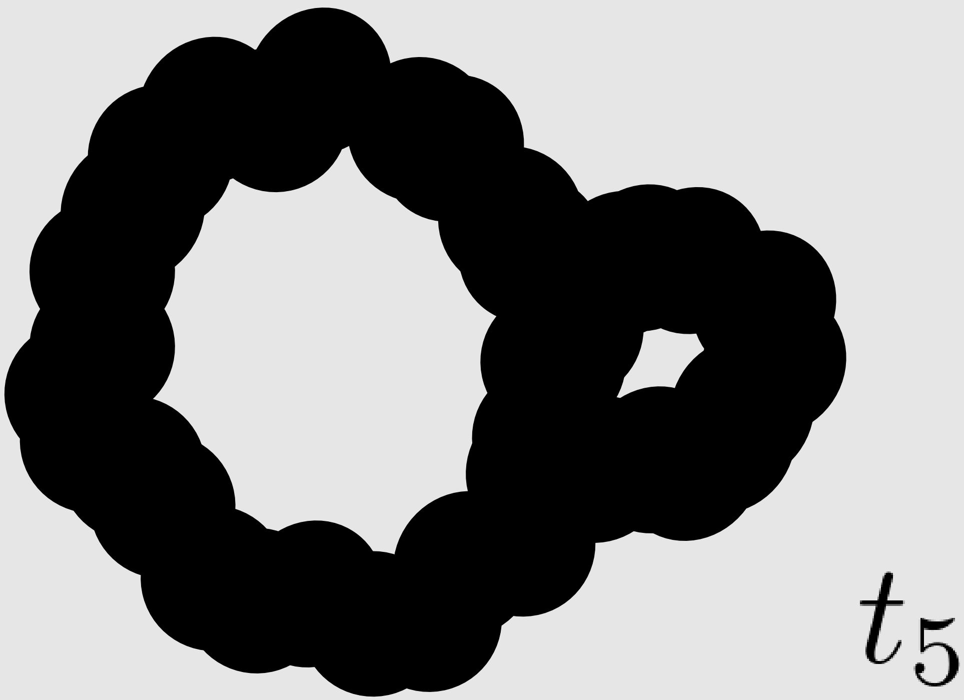
フィルトレーションを構成

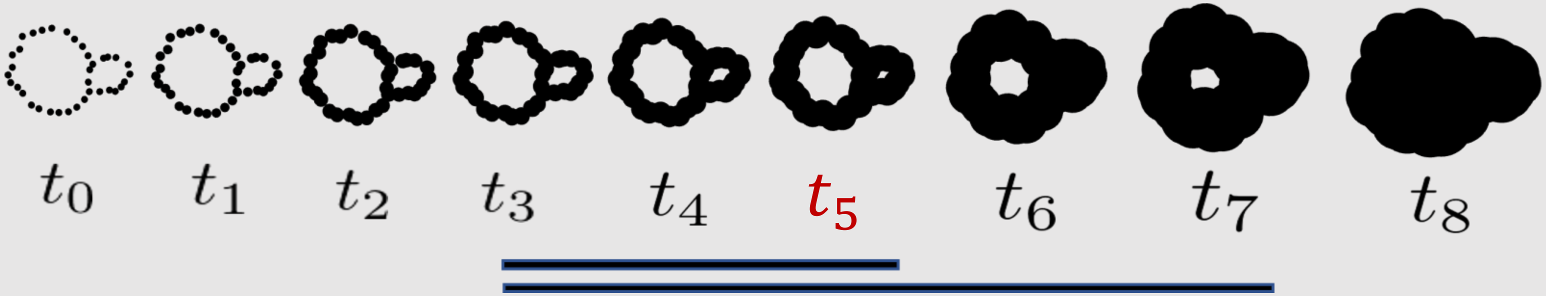




点群データから半径パラメータtを大きくすることにより

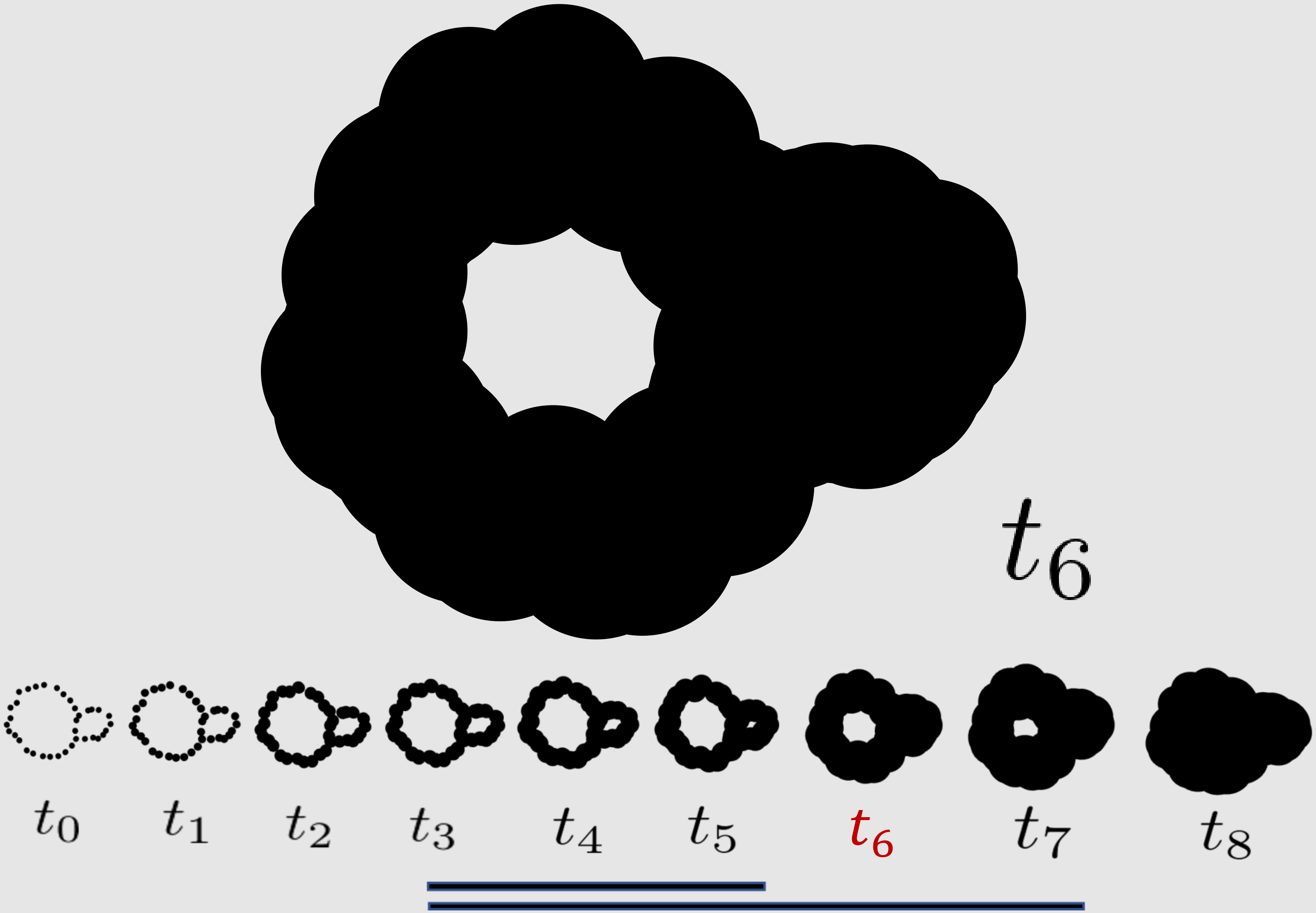
フィルトレーションを構成





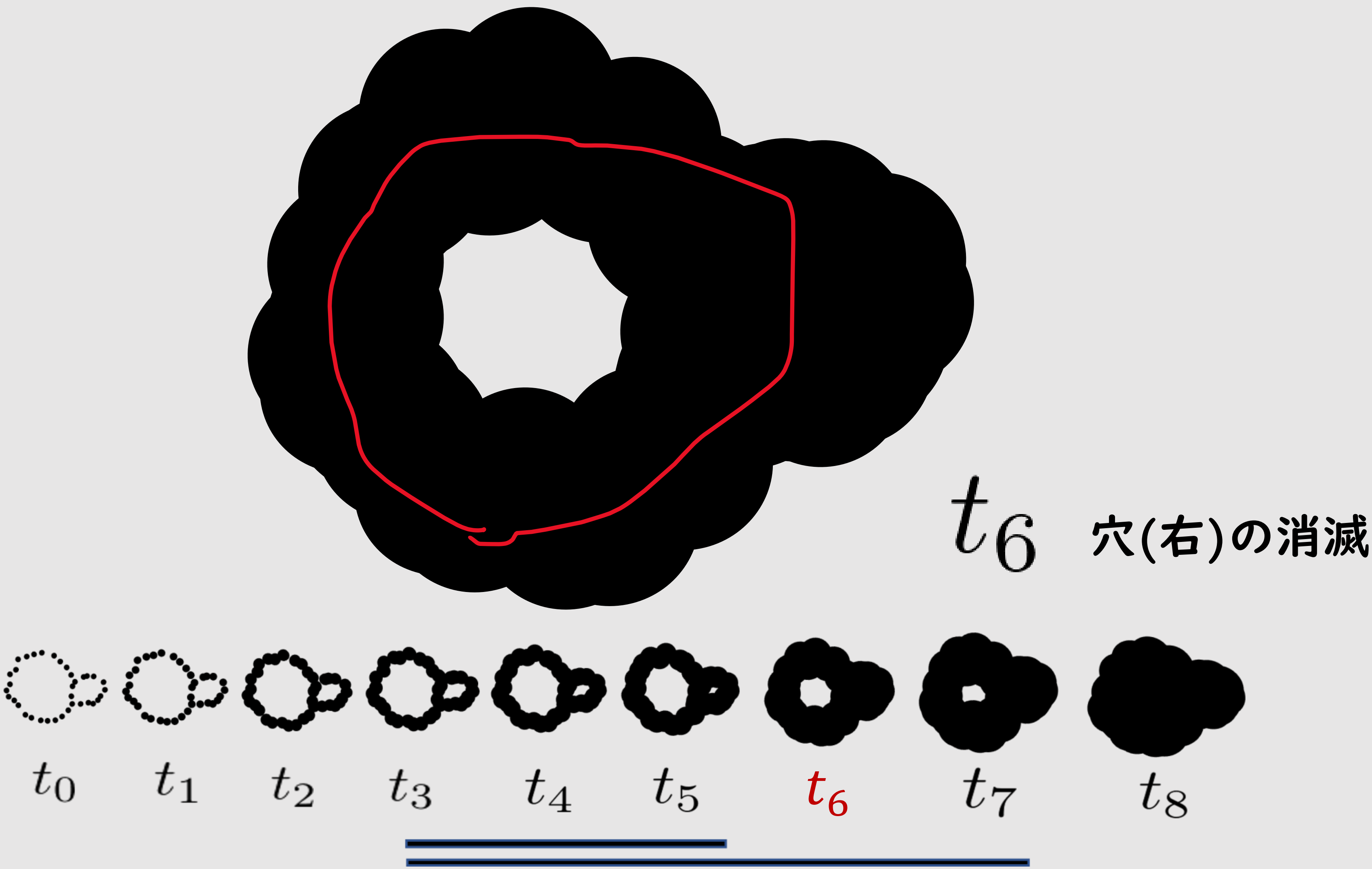
点群データから半径パラメータtを大きくすることにより

フィルトレーション を構成



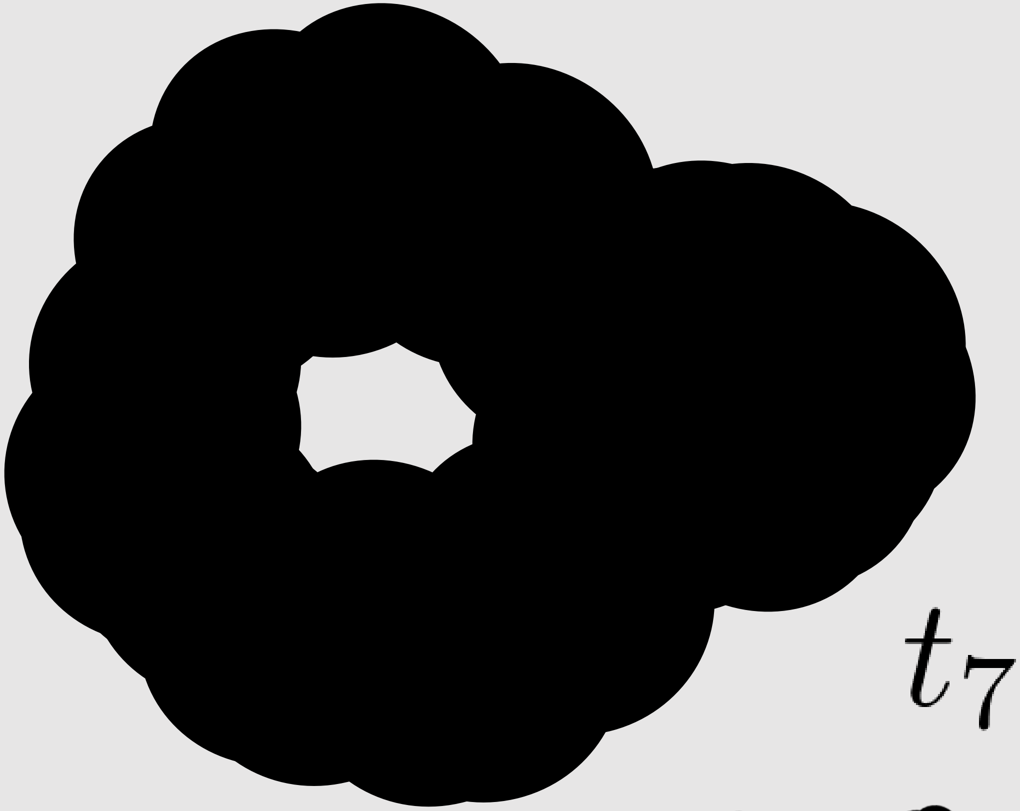
点群データから半径パラメータtを大きくすることにより

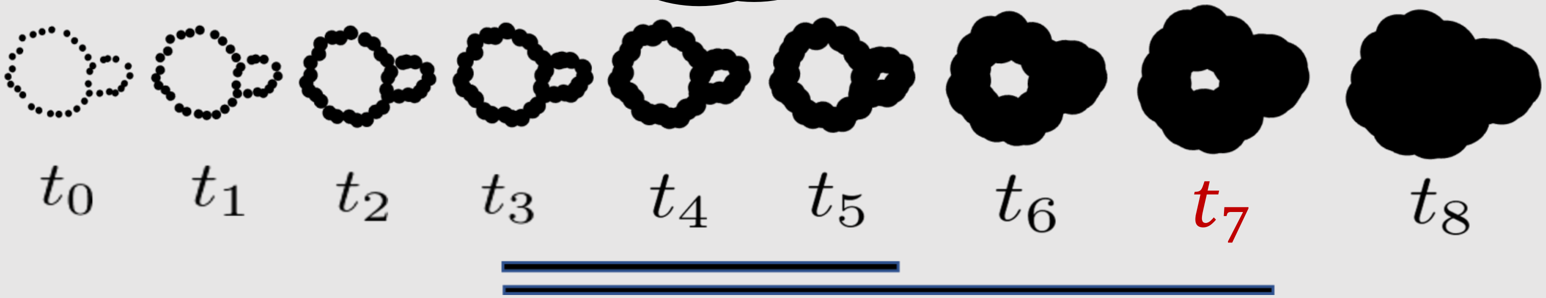
フィルトレーションを構成



点群データから半径パラメータtを大きくすることにより

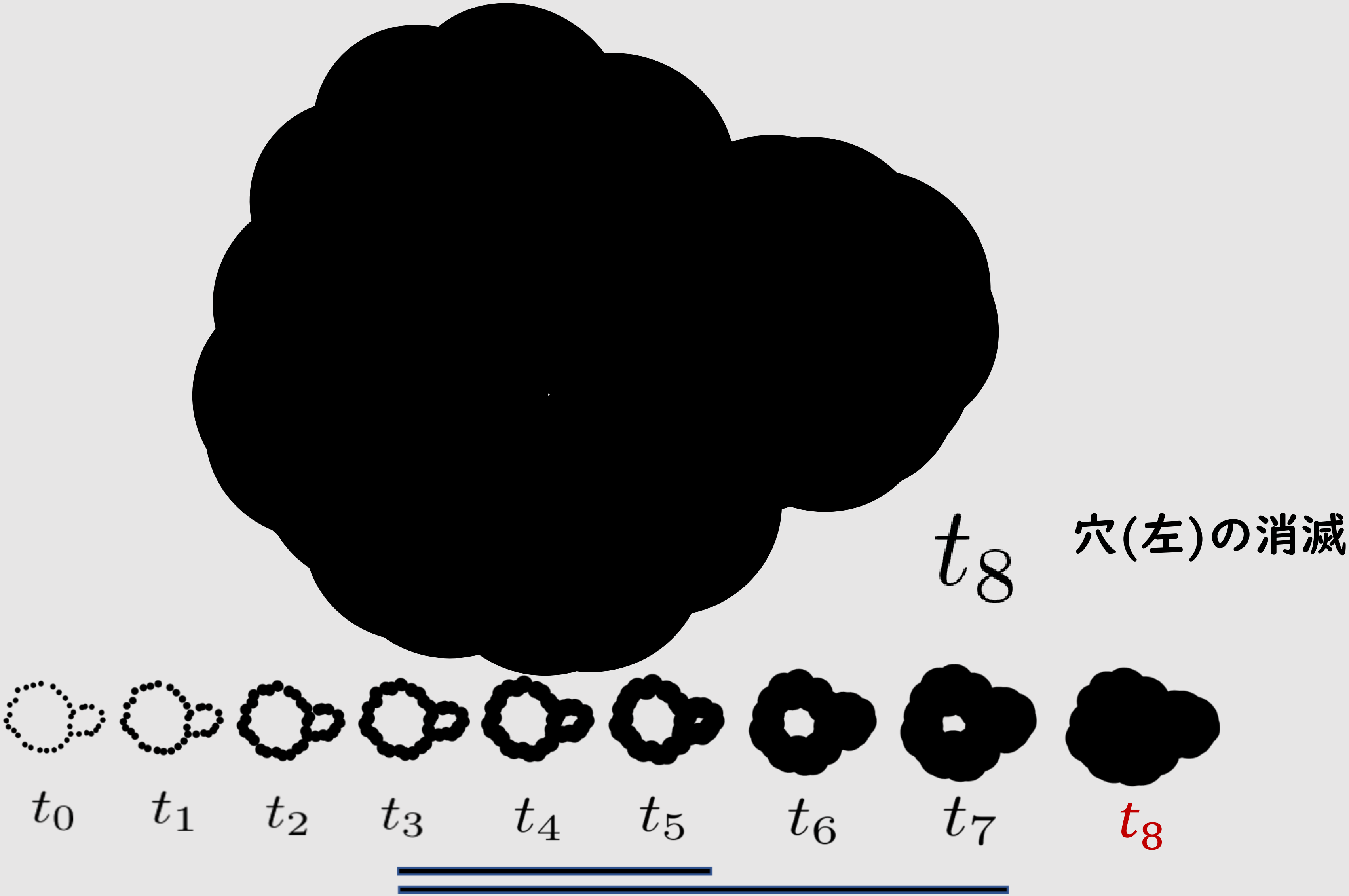
フィルトレーションを構成





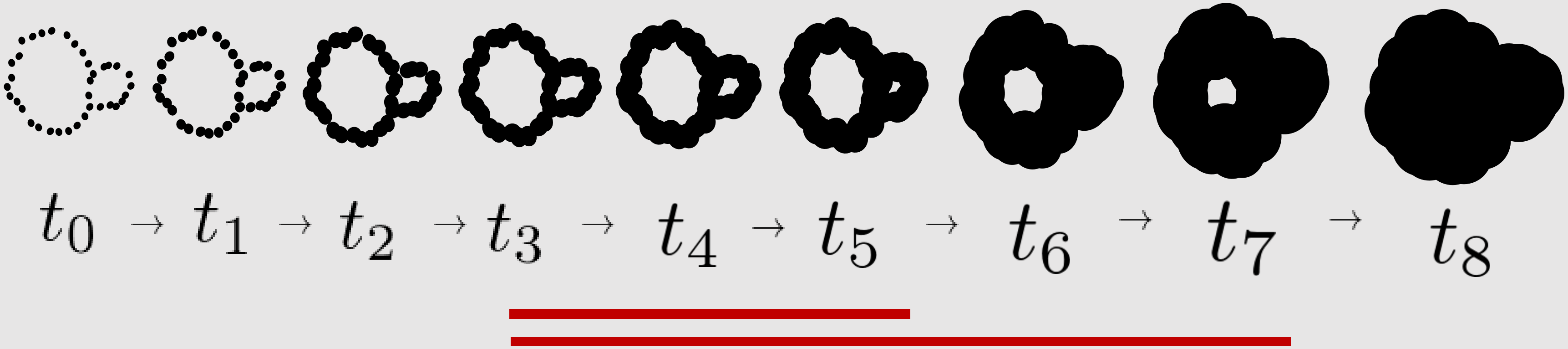
点群データから半径パラメータtを大きくすることにより

フィルトレーションを構成



点群データから半径パラメータtを大きくすることにより

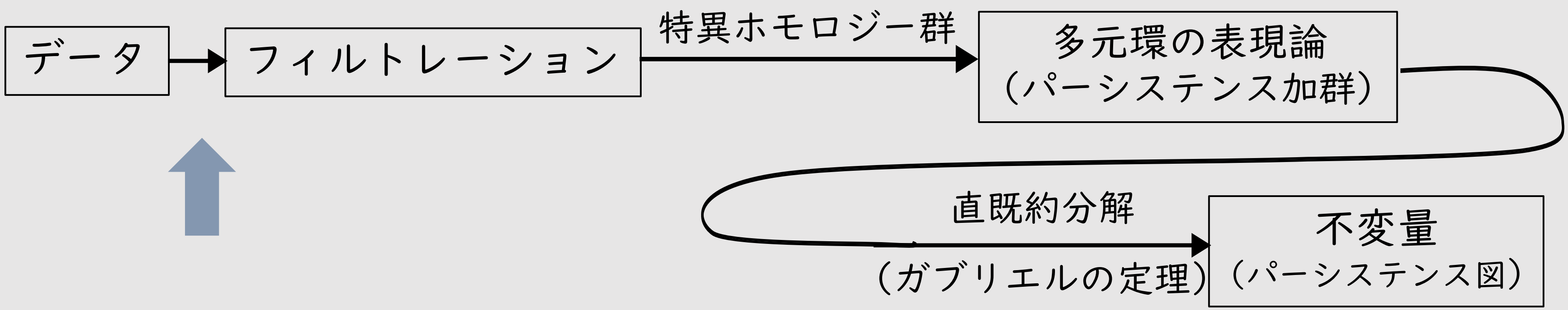
フィルトレーションを構成



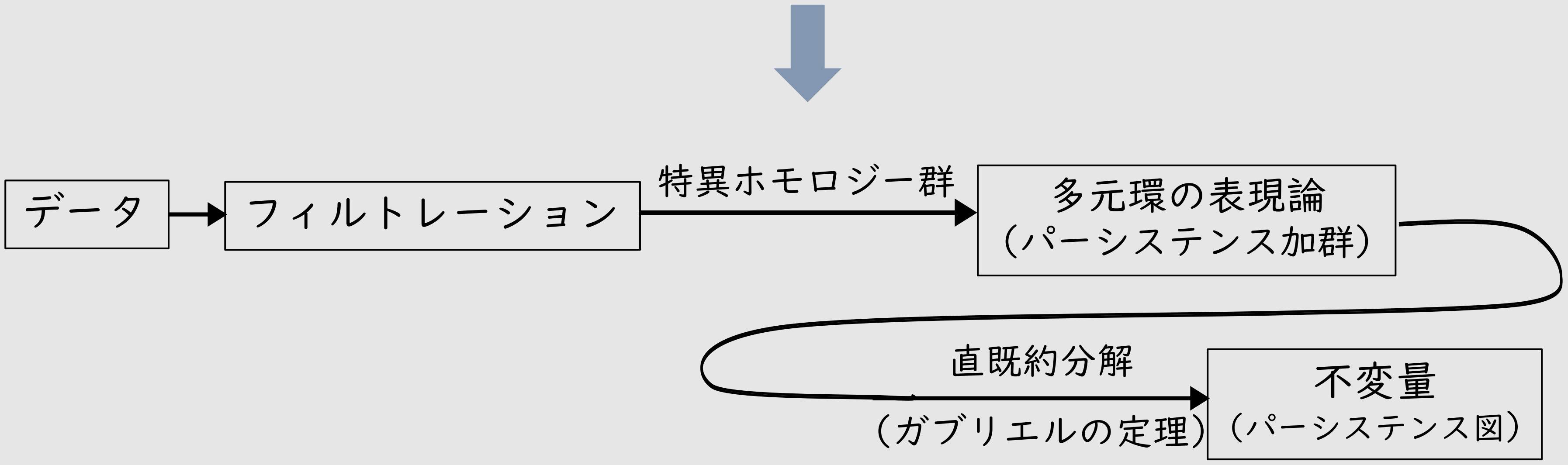
区間 [3,5], 区間 [3,7] (持続性=life-time)

によって穴の生成(birth)と消滅(death)を記述.

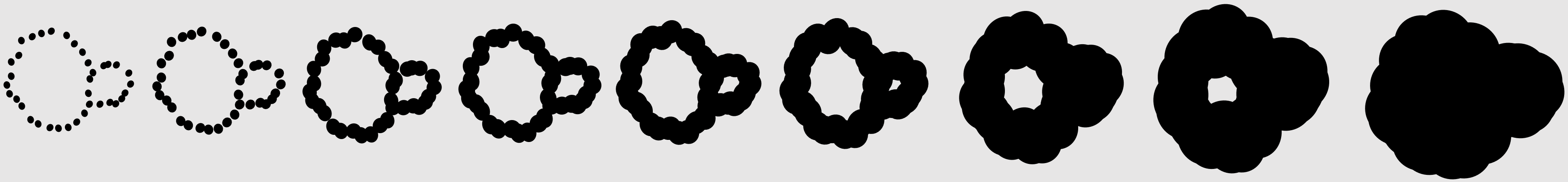
点群データから半径パラメータtを大きくすることにより
フィルトレーションを構成



フィルトレーション→多元環の表現論



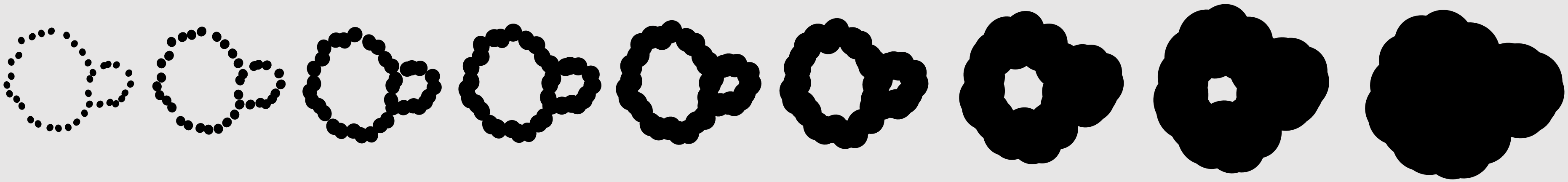
フィルトレーション→多元環の表現論



$$t_0 \to t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_8$$

$H_1(-)$

フィルトレーション→多元環の表現論

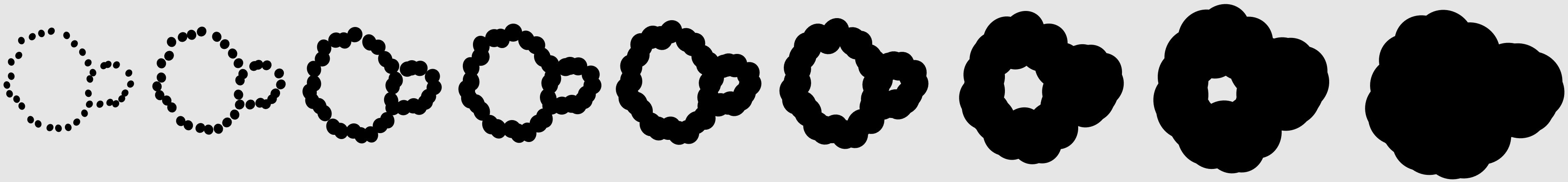


$$t_0 \to t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_8$$

$H_1(-)$

$$H_1(t_0) \to H_1(t_1) \to H_1(t_2) \to H_1(t_3) \to H_1(t_4) \to H_1(t_5) \to H_1(t_6) \to H_1(t_7) \to H_1(t_8)$$

フィルトレーション→多元環の表現論



$$t_0 \to t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_8$$

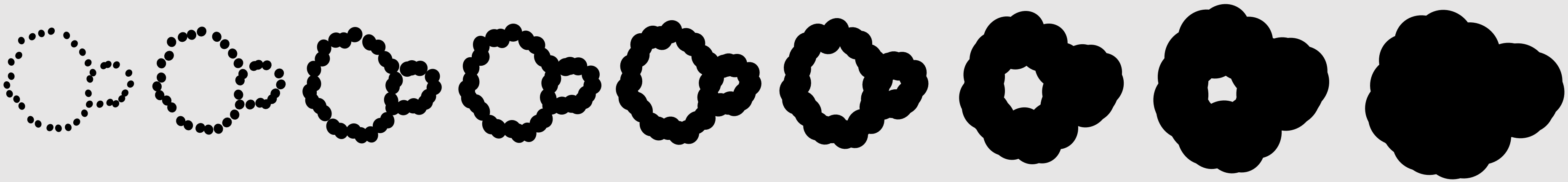
$H_1(-)$

$$H_1(t_0) \to H_1(t_1) \to H_1(t_2) \to H_1(t_3) \to H_1(t_4) \to H_1(t_5) \to H_1(t_6) \to H_1(t_7) \to H_1(t_8)$$

$$\wr|| $$

$$0 \to 0 \to 0 \to k^2 \xrightarrow{\mathrm{id}} k^2 \xrightarrow{\mathrm{id}} k^2 \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} k \xrightarrow{\mathrm{id}} k \to 0$$

フィルトレーション→多元環の表現論



$$t_0 \to t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_8$$

$H_1(-)$

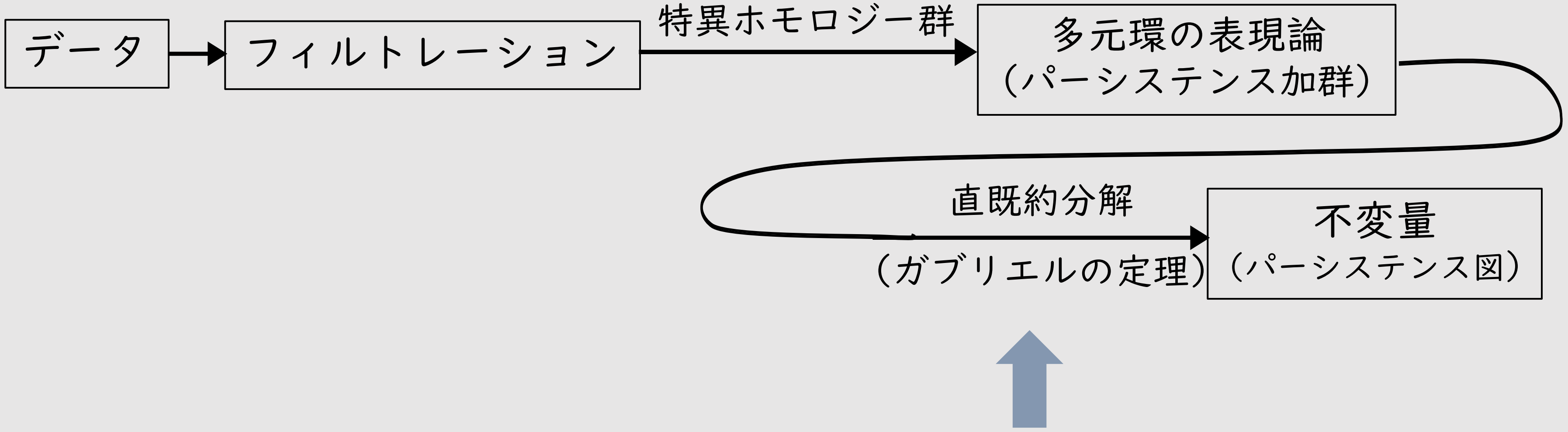
$$H_1(t_0)\to H_1(t_1)\to H_1(t_2)\to H_1(t_3)\to H_1(t_4)\to H_1(t_5)\to H_1(t_6)\to H_1(t_7)\to H_1(t_8)$$

$$\cong$$

$$0 \to 0 \to 0 \to k^2 \xrightarrow{\mathrm{id}} k^2 \xrightarrow{\mathrm{id}} k^2 \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} k \xrightarrow{\mathrm{id}} k \to 0$$

A型箙の表現

パーシステンス加群→パーシステンス図



パーシステンス加群→パーシステンス図

Gabriel's theorem for type A-quivers

For a quiver

$$1 \to \cdots \to n$$

and its representations

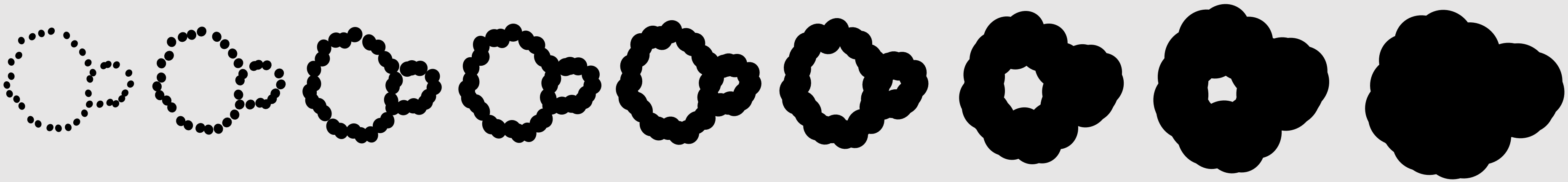
$$V : V_1 \to \cdots \to V_n,$$

we have a unique decomposition of V

$$V \cong \bigoplus_{i=1}^{n} I[b_i, d_i]^{m_{b_i,d_i}},$$

where $I[b_i, d_i] \coloneqq \cdots \to 0 \to \underset{b_i}{k} \overset{\mathrm{id}}{\to} \cdots \overset{\mathrm{id}}{\to} \underset{d_i}{k} \to 0 \to \cdots$.

パーシステンス加群→パーシステンス図



$$t_0 \to t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_8$$

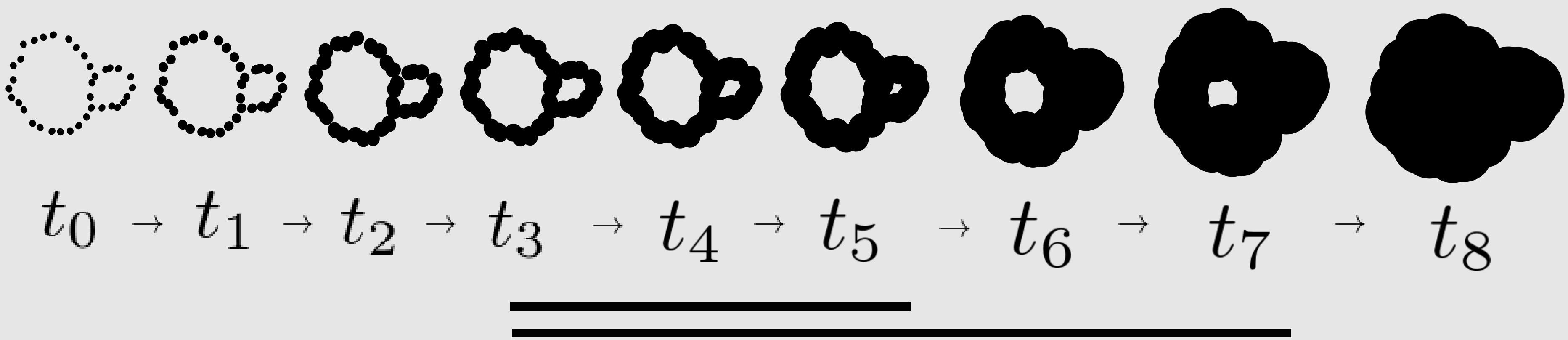
$H_1(-)$

$$H_1(t_0)\to H_1(t_1)\to H_1(t_2)\to H_1(t_3)\to H_1(t_4)\to H_1(t_5)\to H_1(t_6)\to H_1(t_7)\to H_1(t_8)$$

$$\cong$$

$$0 \to 0 \to 0 \to k^2 \xrightarrow{\mathrm{id}} k^2 \xrightarrow{\mathrm{id}} k^2 \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} k \xrightarrow{\mathrm{id}} k \to 0$$

パーシステンス加群→パーシステンス図



$H_1(-)$

$$H_1(t_0) \to H_1(t_1) \to H_1(t_2) \to H_1(t_3) \to H_1(t_4) \to H_1(t_5) \to H_1(t_6) \to H_1(t_7) \to H_1(t_8)$$

$$\cong$$

$$0 \to 0 \to 0 \to k^2 \xrightarrow{\mathrm{id}} k^2 \xrightarrow{\mathrm{id}} k^2 \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} k \xrightarrow{\mathrm{id}} k \to 0$$

$$\cong$$

$I[3,5]$

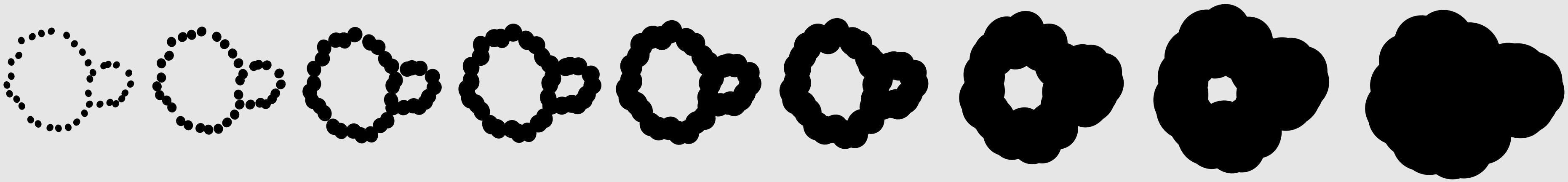
$$0 \to 0 \to 0 \to k \to k \to k \to 0 \to 0 \to 0$$

$$\oplus$$

$I[3,7]$

$$0 \to 0 \to 0 \to k \to k \to k \to k \to k \to 0$$

パーシステンス加群→パーシステンス図



$$t_0 \to t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_8$$

$H_1(-)$

直既約加群 ⟷ 位相的特徴の持続性

$$H_1(t_0)\to H_1(t_1)\to H_1(t_2)\to H_1(t_3)\to H_1(t_4)\to H_1(t_5)\to H_1(t_6)\to H_1(t_7)\to H_1(t_8)$$

$$\cong$$

$$0 \to 0 \to 0 \to k^2 \xrightarrow{\mathrm{id}} k^2 \xrightarrow{\mathrm{id}} k^2 \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} k \xrightarrow{\mathrm{id}} k \to 0$$

$$\cong$$

$I[3,5]$

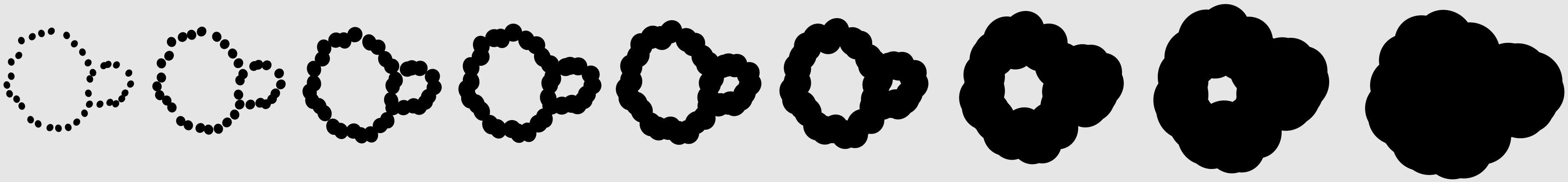
$$0 \to 0 \to 0 \to k \to k \to k \to 0 \to 0 \to 0$$

$$\oplus$$

$I[3,7]$

$$0 \to 0 \to 0 \to k \to k \to k \to k \to k \to 0$$

パーシステンス加群→パーシステンス図



$$t_0 \to t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_8$$

$H_1(-)$

直既約加群 ⟷ 位相的特徴の持続性

$$H_1(t_0) \to H_1(t_1) \to H_1(t_2) \to H_1(t_3) \to H_1(t_4) \to H_1(t_5) \to H_1(t_6) \to H_1(t_7) \to H_1(t_8)$$

$$\cong$$

$$0 \to 0 \to 0 \to k^2 \xrightarrow{\mathrm{id}} k^2 \xrightarrow{\mathrm{id}} k^2 \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} k \xrightarrow{\mathrm{id}} k \to 0$$

$$\cong$$

$I[3,5]$

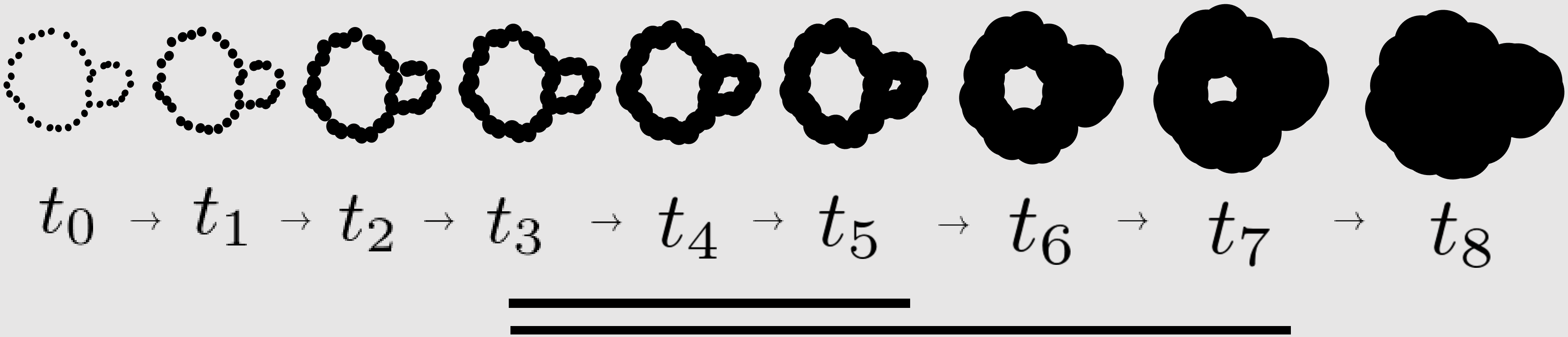
$$0 \to 0 \to 0 \to k \to k \to k \to 0 \to 0 \to 0$$

$$\oplus$$

$I[3,7]$

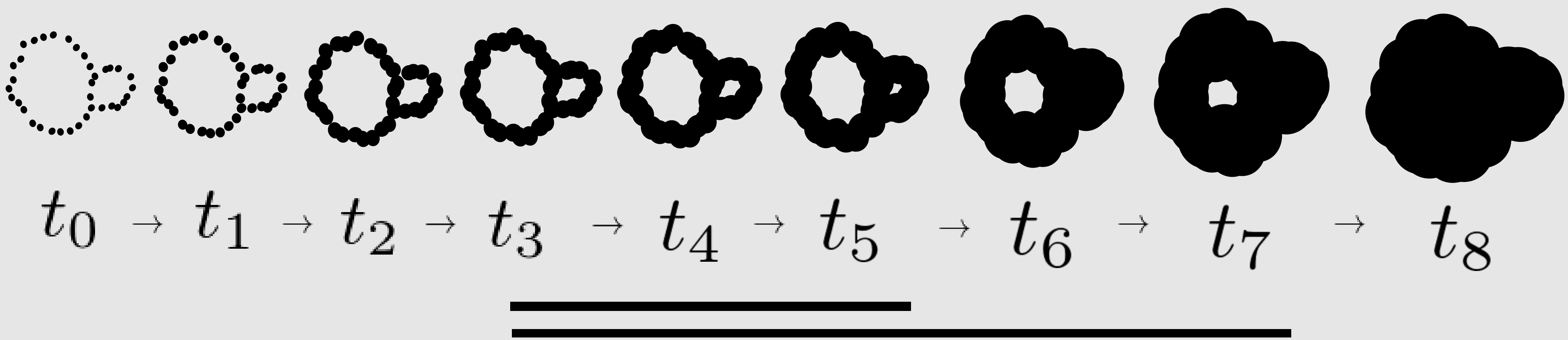
$$0 \to 0 \to 0 \to k \to k \to k \to k \to k \to 0$$

パーシステンス加群→パーシステンス図

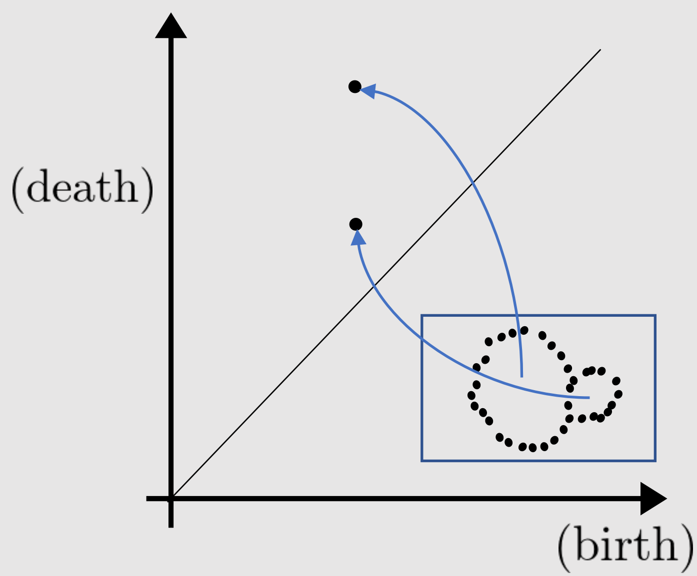


$\Longleftrightarrow \; I[3,5] \oplus I[3,7]$

パーシステンス加群→パーシステンス図



$\Longleftrightarrow I[3,5] \oplus I[3,7]$



パーシステンス図

データの形を記述

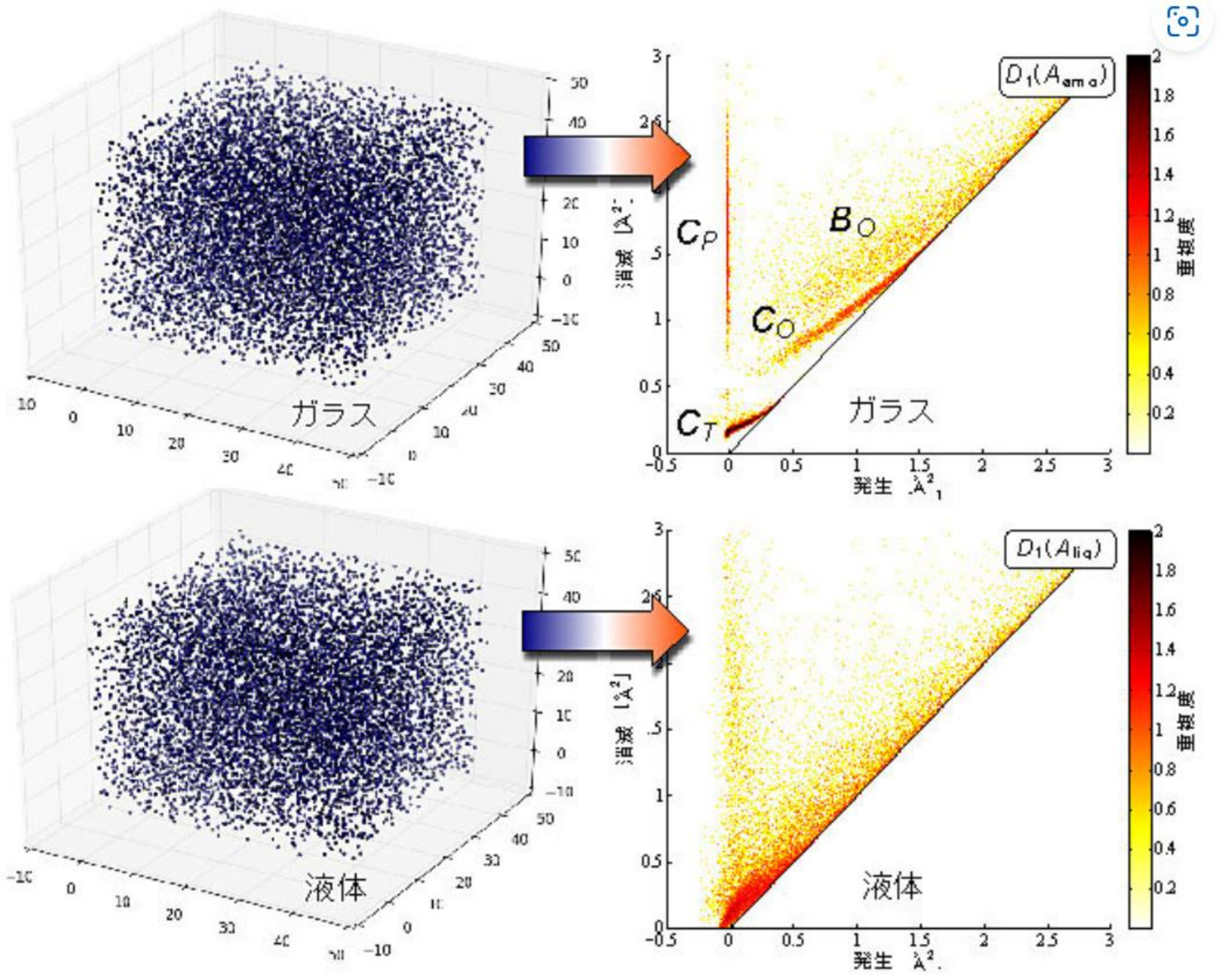
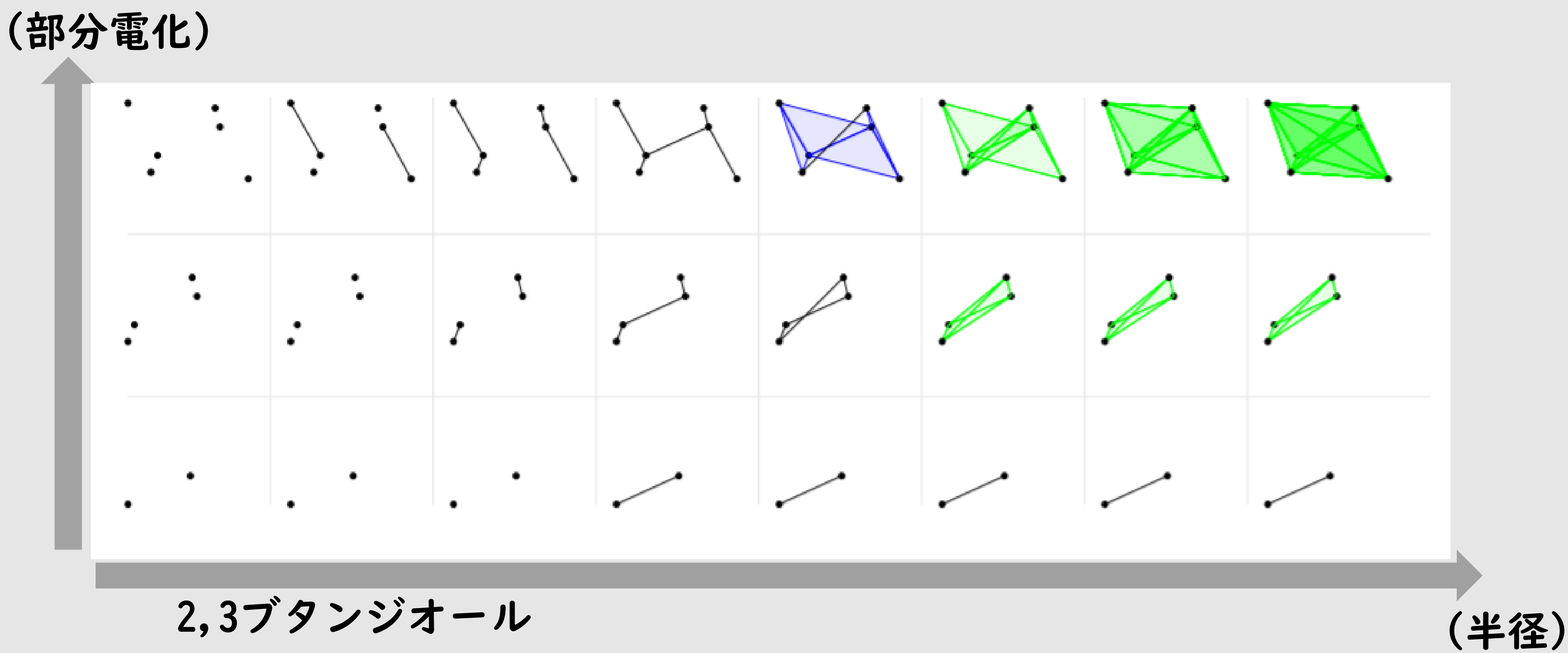


図１　ＳｉＯ２の原子配置（左）とそのパーシステントホモロジー（右）

共同発表：ガラスの「形」を数学的に解明～トポロジーで読み解く無秩序の中の秩序～ (jst.go.jp)から引用

多パラメータのパーシステントホモロジー解析



Keller B, Lesnick M, Willke TL. Persistent Homology for Virtual Screening. ChemRxiv. Cambridge: Cambridge Open Engage; 2018

多パラメータのパーシステントホモロジー解析

なぜ？

例　・データに幾何以外の情報を持たせる(部分電化)

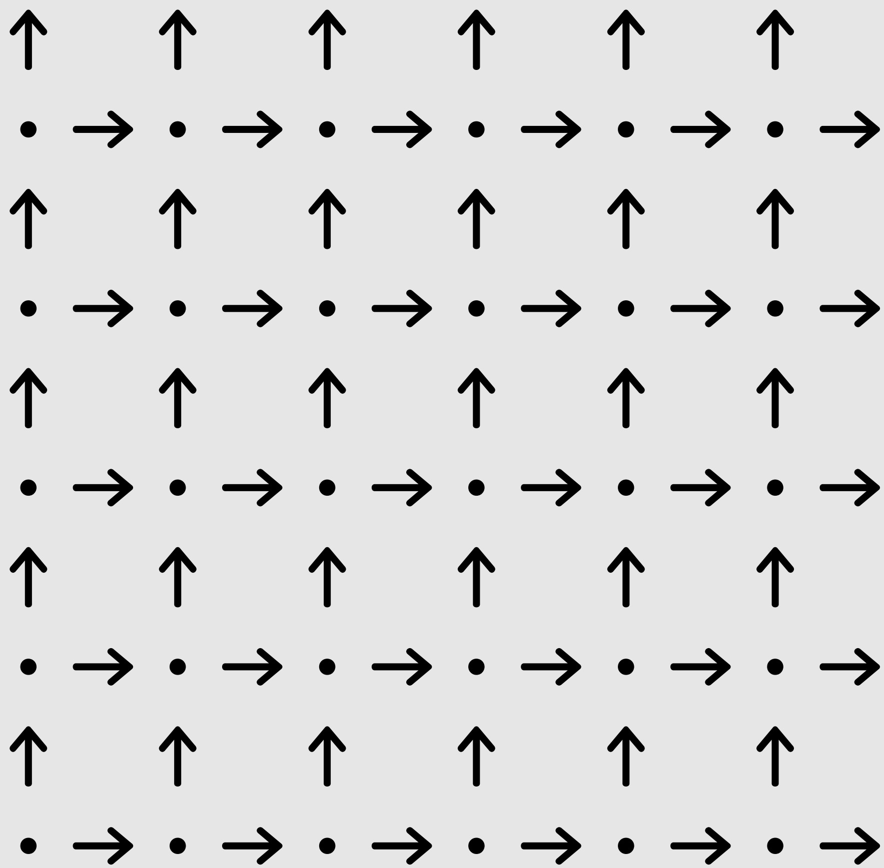
・ノイズ削除、パラメータに点の密度を持たせる

・画像解析　（白黒の濃淡）

1パラメータよりもデータを理解できる？

多パラメータのパーシステントホモロジー解析

グリッドの表現は扱いずらい



Wild表現型

多パラメータのパーシステントホモロジー解析

加群の例

$$
\begin{array}{ccccccc}
K & \longrightarrow & K & \longrightarrow & K & \longrightarrow & K \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K & \longrightarrow & K & \longrightarrow & K & \longrightarrow & K \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K & \longrightarrow & K & \longrightarrow & K & \longrightarrow & K.
\end{array}
$$

$$
\begin{array}{ccccccc}
K & \longrightarrow & K & \longrightarrow & O & \longrightarrow & O \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K & \longrightarrow & K & \longrightarrow & K & \longrightarrow & O \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
O & \longrightarrow & O & \longrightarrow & K & \longrightarrow & K.
\end{array}
$$

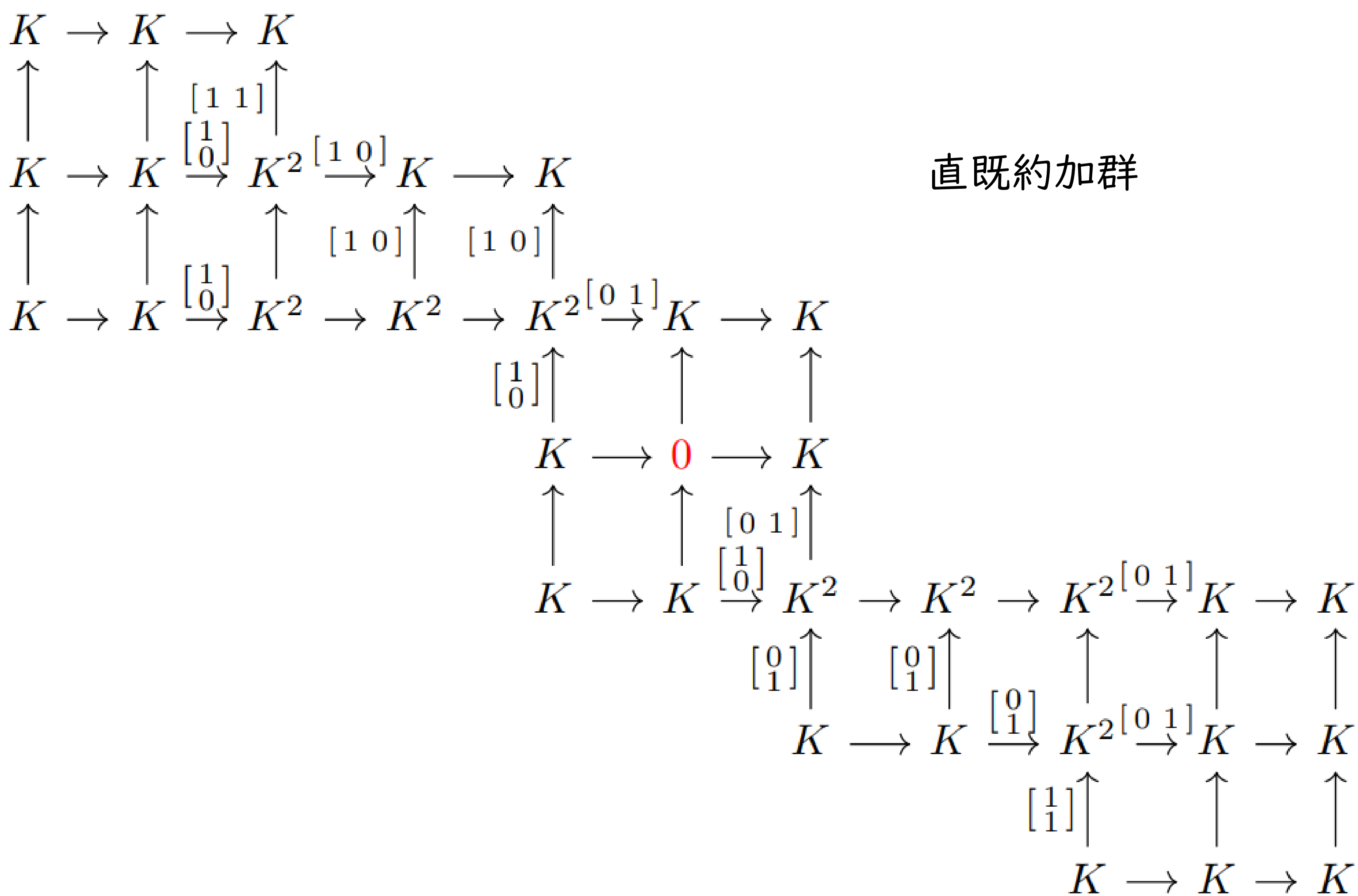
多パラメータのパーシステントホモロジー解析

加群の例

$$
\begin{array}{ccccccc}
K & \longrightarrow & K & \longrightarrow & K & \longrightarrow & K \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K & \longrightarrow & K & \longrightarrow & K & \longrightarrow & K \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K & \longrightarrow & K & \longrightarrow & K & \longrightarrow & K.
\end{array}
$$

$$
\begin{array}{ccccccc}
K & \longrightarrow & K & \longrightarrow & O & \longrightarrow & O \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K & \longrightarrow & K & \longrightarrow & K & \longrightarrow & O \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
O & \longrightarrow & O & \longrightarrow & K & \longrightarrow & K.
\end{array}
$$

多パラメータのパーシステントホモロジー解析



M.Buchet, Emerson G. Escolar “Every 1D Persistence Module is a Restriction of Some Indecomposable 2D Persistence Module” *Journal of Applied and Computational Topology*

半順序集合の表現(homological algebra)

・Magnus Bakke Botnan, Steffen Oppermann, and Steve Oudot. “Signed barcodes for multi-parameter persistence via rank decompositions and rank-exact resolutions.” *In International Symposium on Computational Geometry*, 2021

・Benjamin Blanchette, Thomas Brüstle, and Eric J Hanson. “Homological approximations in persistence theory.” *Canadian Journal of Mathematics*, pages 1–38, 2021.

・Hideto Asashiba, Emerson G Escolar, Ken Nakashima, and Michio Yoshiwaki. “Approximation by interval-decomposables and interval resolutions of persistence modules.” *Journal of Pure and Applied Algebra*, 227(10):107397, 2023.

など

モチベーション

半順序集合上の(データから得られるような)複雑な加群を理解するために

“取り扱いやすい加群” を用いて理解したい.

モチベーション

半順序集合上の(データから得られるような)複雑な加群を理解するために

“取り扱いやすい加群” を用いて理解したい.

区間加群
(Interval module)

$$
\begin{array}{ccccccc}
K & \longrightarrow & K & \longrightarrow & K & \longrightarrow & K \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K & \longrightarrow & K & \longrightarrow & K & \longrightarrow & K \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K & \longrightarrow & K & \longrightarrow & K & \longrightarrow & K.
\end{array}
\qquad
\begin{array}{ccccccc}
K & \longrightarrow & K & \longrightarrow & O & \longrightarrow & O \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K & \longrightarrow & K & \longrightarrow & K & \longrightarrow & O \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
O & \longrightarrow & O & \longrightarrow & K & \longrightarrow & K.
\end{array}
$$

区間加群の例

モチベーション

半順序集合上の(データから得られるような)複雑な加群を理解するために

“取り扱いやすい加群” を用いて理解したい.

区間加群
(Interval module)

区間近似
(Interval approximation)

$$
\begin{array}{ccccccc}
K & \longrightarrow & K & \longrightarrow & K & \longrightarrow & K \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K & \longrightarrow & K & \longrightarrow & K & \longrightarrow & K \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K & \longrightarrow & K & \longrightarrow & K & \longrightarrow & K.
\end{array}
\qquad
\begin{array}{ccccccc}
K & \longrightarrow & K & \longrightarrow & O & \longrightarrow & O \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K & \longrightarrow & K & \longrightarrow & K & \longrightarrow & O \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
O & \longrightarrow & O & \longrightarrow & K & \longrightarrow & K.
\end{array}
$$

区間加群の例

発表の流れ

■位相的データ解析とは？

■パーシステンス加群(隣接代数の加群)

■得られた結果

Persistence module (1/7)

- Let P be a finite partially ordered set (poset).

  (we see it as a category by $a \leqq b \Leftrightarrow \exists ! \, a \to b$)

-

Persistence module (1/7)

- Let P be a finite partially ordered set (poset).

  (we see it as a category by $a \leqq b \Leftrightarrow \exists ! \; a \to b$)

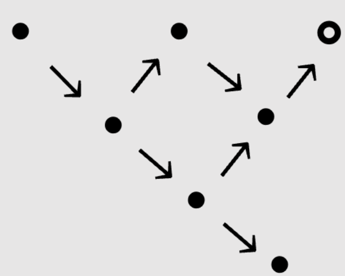
- *Persistence modules over P* are functors from P to k-mod

  (or equivalently modules over incidence algebra $k[P]$).

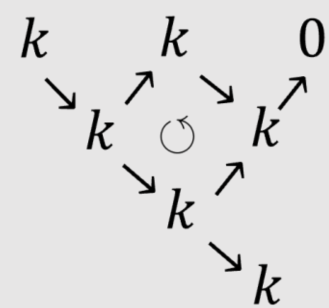
Intervals (2/7)

- A full subposet I of P is called *interval* if I is
  (1) connected (the Hasse diagram of I is connected),
  (2) convex ($x \leq y \leq z$, and $x, z \in I$ imply $y \in I$).

$I \subset P$:

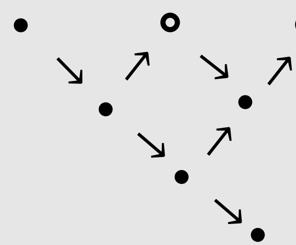


k_I :

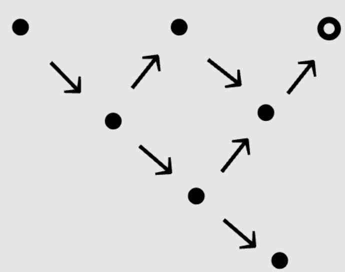


Intervals (2/7)

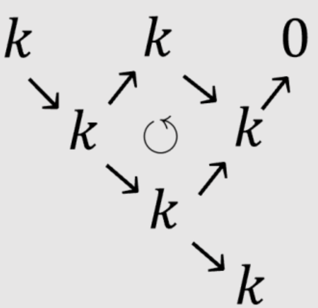
- A full subposet I of P is called *interval* if I is
  (1) connected (the Hasse diagram of I is connected),
  (2) convex ($x \leq y \leq z$, and $x, z \in I$ imply $y \in I$).



$I \subset P$:



k_I :

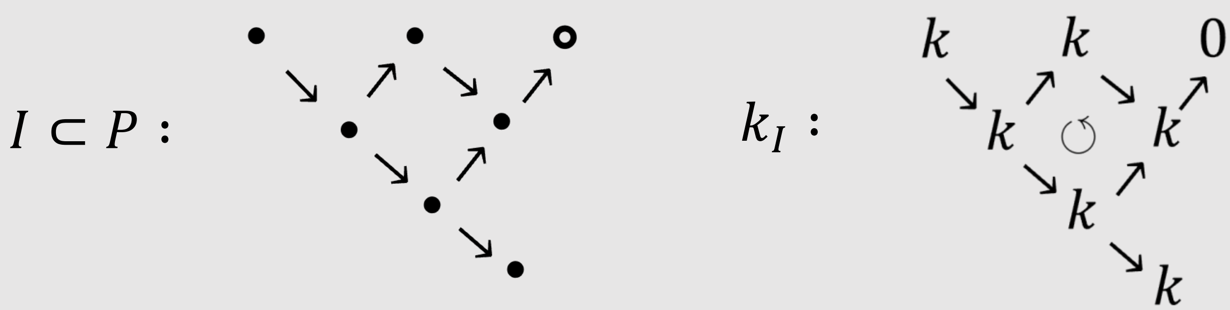


Intervals (2/7)

- A full subposet I of P is called *interval* if I is
  (1) connected (the Hasse diagram of I is connected),
  (2) convex ($x \leq y \leq z$, and $x, z \in I$ imply $y \in I$).

- For an interval I of P, the *interval module* k_I is defined by
  $k_I(p) \coloneqq k$ for $p \in I$, otherwise $k_I(p) \coloneqq 0$,
  $k_I(a \to b) \coloneqq \mathrm{id}_k$ for $a, b \in I$, otherwise 0.

-



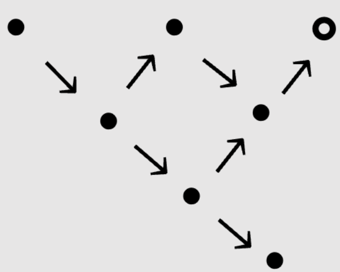
Intervals (2/7)

- A full subposet I of P is called *interval* if I is
  (1) connected (the Hasse diagram of I is connected),
  (2) convex ($x \leq y \leq z$, and $x, z \in I$ imply $y \in I$).

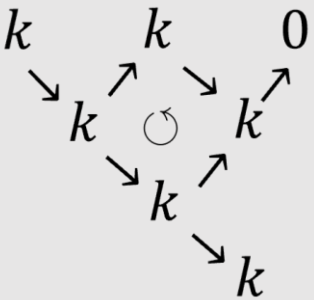
- For an interval I of P, the *interval module k_I* is defined by
  $k_I(p) \coloneqq k$ for $p \in I$, otherwise $k_I(p) \coloneqq 0$,
  $k_I(a \to b) \coloneqq \mathrm{id}_k$ for $a, b \in I$, otherwise 0.

- A module is *interval decomposable* if the module decomposes into interval modules.

$I \subset P$:



k_I :

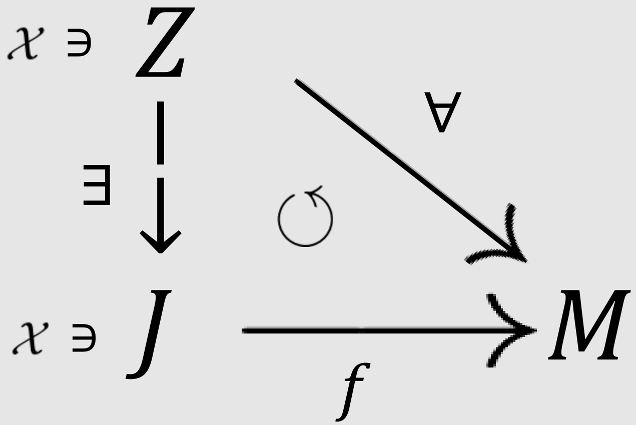


Right $\mathcal{X}$-approximation (3/7)

- A : a finite dimensional k-algbra
- $\mathcal{X}$: a full subcategory of mod A satisfying certain conditions.
  (proj(A) $\subseteq \mathcal{X}$, functorial finite, closed under direct summand, …)
  e.g. $\mathcal{X}$ = the set of all interval decomposable modules.

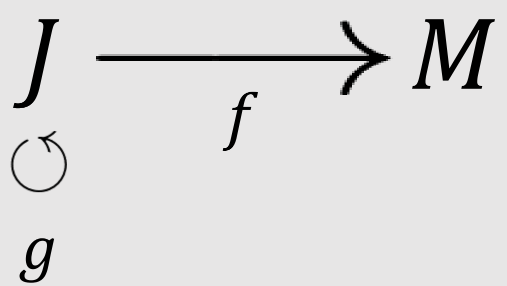
- A *right $\mathcal{X}$-approximation of M* is a morphism $f : J \to M$ with $J \in \mathcal{X}$

s.t. for any $Z \in \mathcal{X}$, $\mathrm{Hom}_A(Z, f) : \mathrm{Hom}_A(Z, J) \to \mathrm{Hom}_A(Z, M)$
is surjective.



Right minimal (4/7)

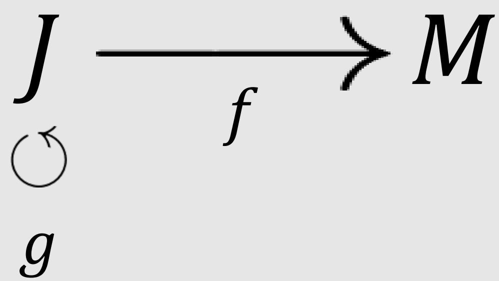
- A morphism $f: J \to M$ is *right minimal* if $fg = f$ implies g is an automorphism.



-

Right minimal (4/7)

- A morphism $f: J \to M$ is *right minimal* if $fg = f$ implies g is an automorphism.



- A morphism $f: J \to M$ is *right minimal $\mathcal{X}$-approximation of M* if f is right minimal and right $\mathcal{X}$-approximation of M.

Right minimal $\mathcal{X}$-resolution(5/7)

- *Right minimal $\mathcal{X}$-resolution of M* is an exact sequence

Right minimal $\mathcal{X}$-resolution(5/7)

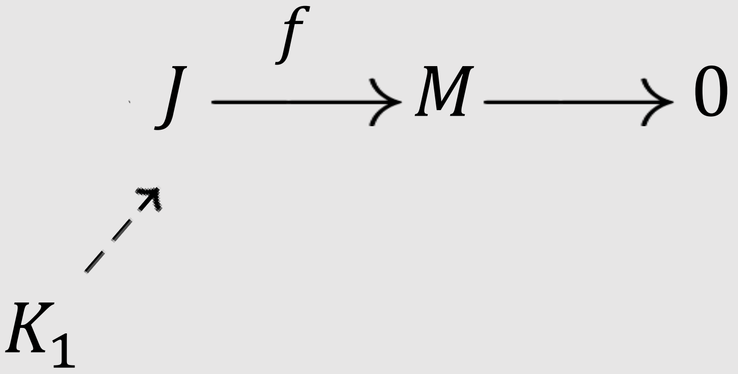
- *Right minimal $\mathcal{X}$-resolution of M* is an exact sequence

$$J \xrightarrow{f} M \longrightarrow 0$$

right minimal $\mathcal{X}$-approximation of M

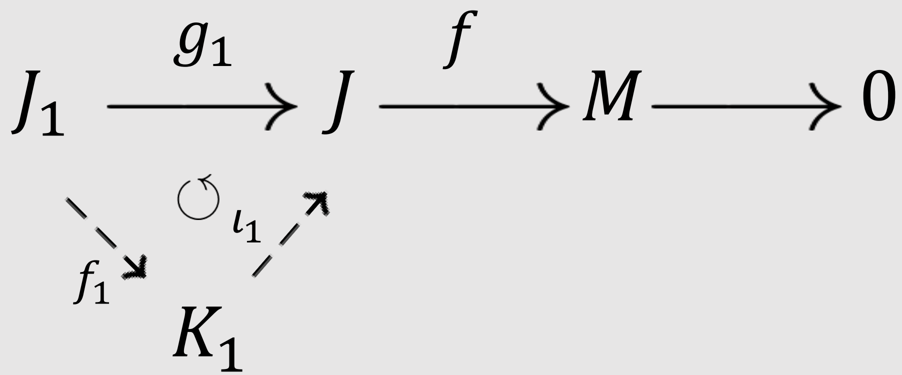
Right minimal $\mathcal{X}$-resolution(5/7)

- *Right minimal $\mathcal{X}$-resolution of M* is an exact sequence



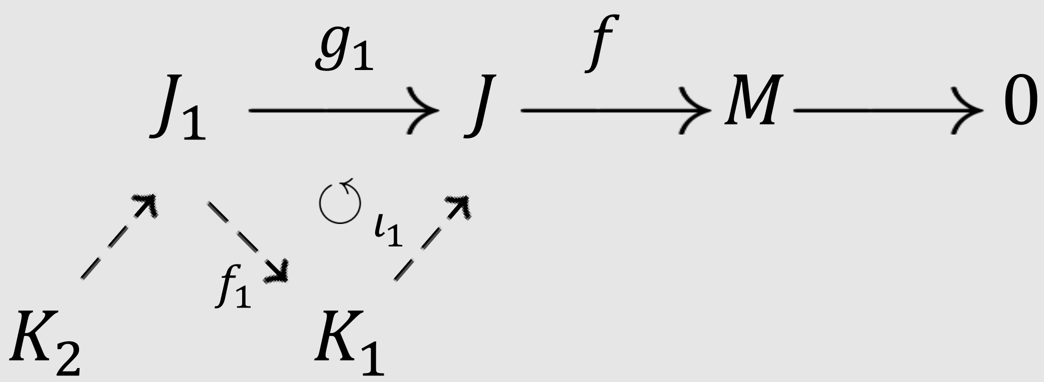
Right minimal $\mathcal{X}$-resolution(5/7)

- *Right minimal $\mathcal{X}$-resolution of M* is an exact sequence



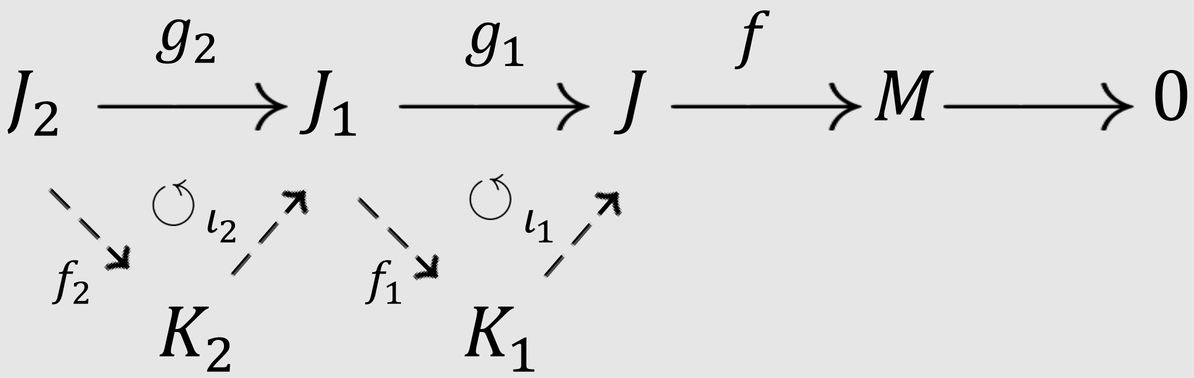
Right minimal $\mathcal{X}$-resolution(5/7)

- *Right minimal $\mathcal{X}$-resolution of M* is an exact sequence



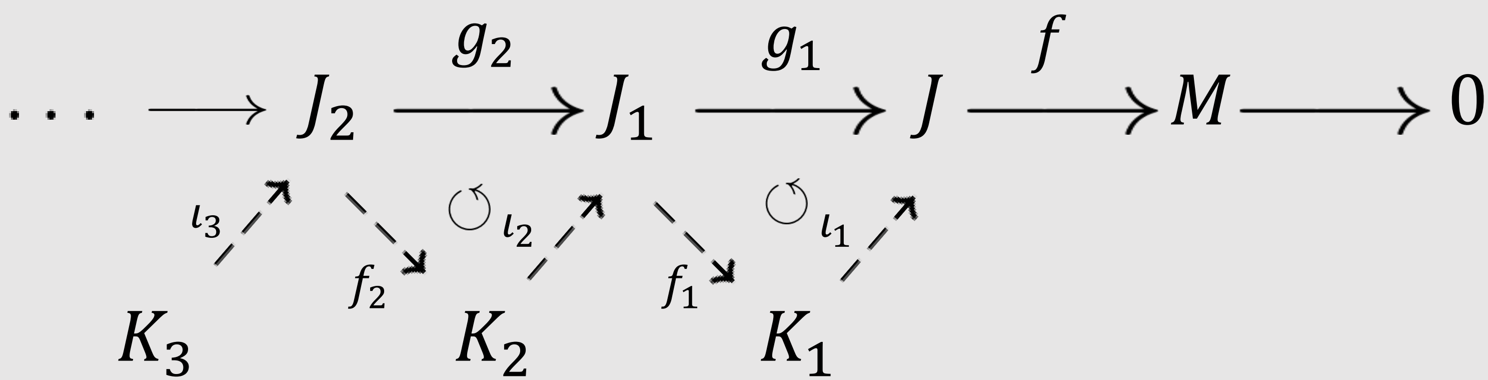
Right minimal $\mathcal{X}$-resolution(5/7)

- *Right minimal $\mathcal{X}$-resolution of M* is an exact sequence



Right minimal $\mathcal{X}$-resolution(5/7)

- *Right minimal $\mathcal{X}$-resolution of M* is an exact sequence



Resolution dimension (6/7)

- If M has a right minimal $\mathcal{X}$-resolution of the form

$$0 \to J_m \xrightarrow{g_m} \cdots \to J_2 \xrightarrow{g_2} J_1 \xrightarrow{g_1} J \xrightarrow{f} M \to 0,$$

then we say that the $\mathcal{X}$-*resolution dimension of* M is m and write $\mathcal{X}$-res-dim $M = m$.

Otherwise, we say that the $\mathcal{X}$-resolution dimension of M is infinity.

Interval resolution global dimension (7/7)

Now, we consider

- $k[P]$: the incidence algebra of a poset P.
- $\mathcal{I}_P$: the set of interval decomposable modules over $k[P]$.
- For a module M, let int-res-dim(M) be the resolution dimension of M with respect to $\mathcal{I}_P$.
-

Interval resolution global dimension (7/7)

Now, we consider

- $k[P]$: the incidence algebra of a poset P.
- $\mathcal{J}_P$: the set of interval decomposable modules over $k[P]$.
- For a module M, let int-res-dim(M) be the resolution dimension of M with respect to $\mathcal{J}_P$.

- *interval resolution global dimension of* $k[P]$ is

$$\text{int-res-gldim}(k[P]) := \sup\{ \text{int-res-dim}(M) \mid M \in \text{mod}\, k[P] \}.$$

Interval resolution global dimension (7/7)

Now, we consider

- $k[P]$: the incidence algebra of a poset P.
- $\mathcal{I}_P$: the set of interval decomposable modules over $k[P]$.
- For a module M, let int-res-dim(M) be the resolution dimension of M with respect to $\mathcal{I}_P$.

- *interval resolution global dimension of* $k[P]$ is

$$\text{int-res-gldim}(k[P]) := \sup\{ \text{int-res-dim}(M) \mid M \in \text{mod}\, k[P] \}.$$

[Asashiba-Escolar-Nakashima-Yoshiwaki, Proposition 4.5, 2023]

int-res-gldim($k[P]$) is finite for any finite poset P.

Interval resolution global dimension (7/7)

(0) Let G be a direct sum of all interval modules k_I over $k[P]$.
Note that all indecomposable projective (resp., injective) are interval modules.

(1)[AENY, 23] shows that any submodule of interval module is an interval decomposable module.

(2) Then, $\Gamma \coloneqq \mathrm{End}(\mathrm{G})$ is a left strongly quasi-hereditary algebra by [Ringel, 09, Theorem 5] (see also [Iyama, 03]). In particular, Γ has the finite global dimension.

(3)We have int-res-gldim$(k[P])$ = gldim $(\Gamma) - 2 < \infty$.([Erdmann-Holm-Iyama-Schröer, 17])

See [AENY, 23] for the detail.

結果 (1)Direct summand injectivity

(2)Monotonicity

(3)Classification of posets with int-res-gldim=0

Theorem 1 [Aoki-Escolar-T]

Let P be a finite poset, $\mathcal{I}_P$ be the set of interval decomposable modules over $k[P]$. For any right minimal $\mathcal{I}_P$-approximation of M

$$f = (f_i) : \bigoplus_{i=1}^{n} k_{I_i} \to M,$$

the following holds.

(1) f is surjective.
(2) Each $f_i : k_{I_i} \to M$ is injective.
(3) supp $M =$ supp $(\bigoplus_{i=1}^{n} k_{I_i})$.

In particular, every k_{I_i} is given by an interval submodule of M.

Remark

Recently, [Asashiba, 2023, Proposition 4.8, arXiv:2307.06559] gave the essentially same result (see also [Blanchette-Brüstle-Hanson, Proposition 6.7, 2021, Canadian Journal of Mathematics, 1-38]).

Example

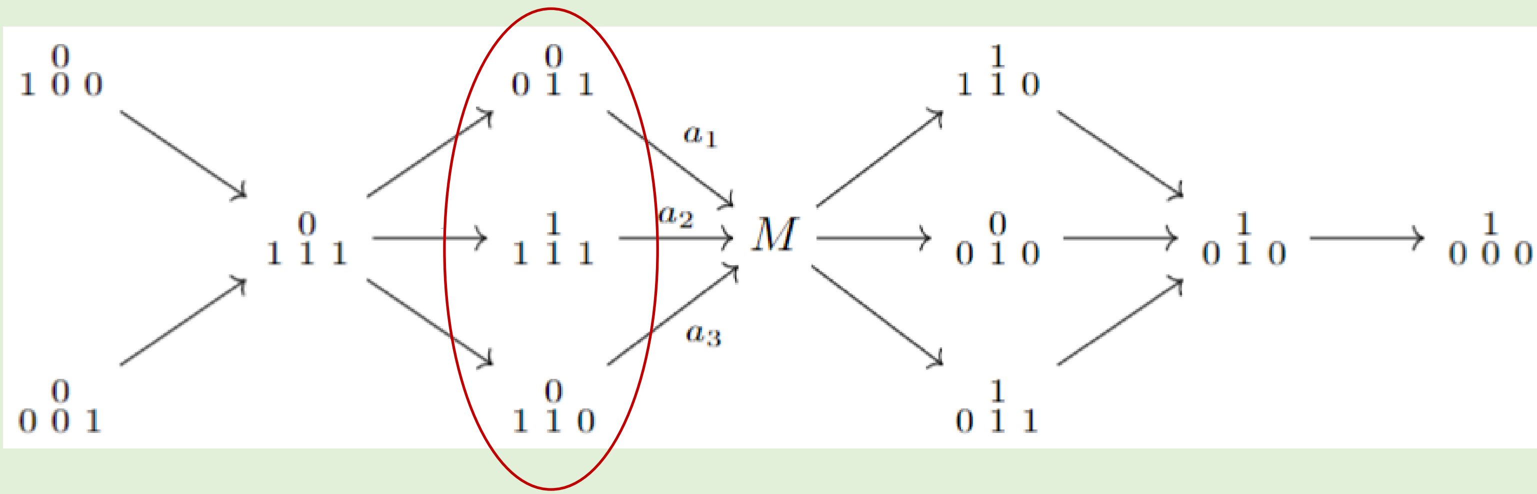
$$P := \begin{array}{ccccc} & & \bullet & & \\ & & \downarrow & & \\ \bullet & \leftarrow & \bullet & \rightarrow & \bullet \end{array},$$

$$M := \begin{array}{ccccc} & & k & & \\ & & \downarrow \begin{bmatrix}1\\1\end{bmatrix} & & \\ k & \underset{[1,0]}{\leftarrow} & k^2 & \underset{[0,1]}{\rightarrow} & k. \end{array}$$

→

Example

$$P := \begin{array}{ccccc} & & \bullet & & \\ & & \downarrow & & \\ \bullet & \leftarrow & \bullet & \rightarrow & \bullet \end{array}, \qquad M := \begin{array}{ccccc} & & k & & \\ & & \downarrow \begin{bmatrix}1\\1\end{bmatrix} & & \\ k & \underset{[1,0]}{\leftarrow} & k^2 & \underset{[0,1]}{\rightarrow} & k. \end{array}$$



An approximation of M is given by

$$\begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \end{smallmatrix} \xrightarrow{[a_1, a_2, a_3]} M$$

結果 (1)Direct summand injectivity

(2)Monotonicity

(3)Classification of posets

with int-res-gldim=0

Theorem 2 [Aoki-Escolar-T]

Let P be a finite poset. For any full subposet Q of P, the following inequality holds.

$$\text{int-res-gldim}\, k[Q] \leq \text{int-res-gldim}\, k[P].$$

Remark

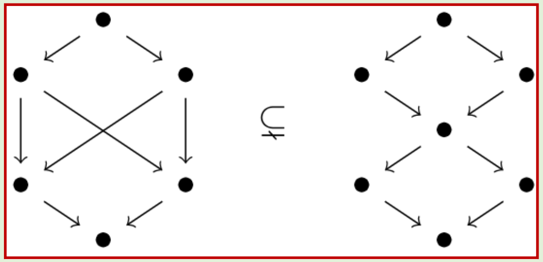
Theorem 2 [Aoki-Escolar-T]

Let P be a finite poset. For any full subposet Q of P, the following inequality holds.

$$\text{int-res-gldim}\, k[Q] \leq \text{int-res-gldim}\, k[P].$$

Remark

The above monotonicity does not hold for (usual) global dimension in general [Igusa-Zacharia, 1990].



Poset	Q	$\subset$	P	
Global dimension	3	>	2	
Interval global dimension	1	<	2	(over a field with two elements)

For a full subposet Q of P, we have an isomorphism

$$k[Q] \cong ek[P]e$$

of k-algebras, where $e \coloneqq \Sigma_{x \in Q} e_x$. It induces adjoint functors

$$\mathrm{T} \coloneqq - \otimes_{k[Q]} ek[P]$$

$$\operatorname{mod} k[P] \xrightarrow{\text{Res}} \operatorname{mod} k[Q]$$

$$\mathrm{L} \coloneqq \operatorname{Hom}_{k[Q]}(ek[P], -)$$

- Res preserves interval decomposability of modules.
- T and L do NOT preserve interval decomposability of modules in general.

We find a functor Θ that sends to interval modules over Q to interval modules over P by using T and L.

The functor Θ

Using adjoint functors, we have

$$\begin{array}{ccc} \mathrm{Hom}_{k[Q]}(M,M) & \cong & \mathrm{Hom}_{k[P]}\big(\mathrm{T}(M),\mathrm{L}(M)\big). \\ \Cup & & \Cup \\ 1_M & \longmapsto & \theta_M \end{array}$$

For a given module $M \in \mathrm{mod}\, k[Q]$, let

$$\Theta(M) \coloneqq \mathrm{Im}(\theta_M).$$

It gives rise to a functor Θ. It is called *intermediate extension* in [Kuhn, 94], and *prolongedment intermédiare* in [Beilison-Bernstein-Deligne, 82].

Proposition For a given interval I of Q, let k_I be the corresponding interval $k[Q]$-module. Then, we have

$$\Theta(k_I) \cong k_{\mathrm{conv}(I)},$$

where $\mathrm{conv}(I)$ is the smallest interval of P containing I.

The functor Θ

We obtain a pair of functors

$$\operatorname{mod} k[P] \underset{\text{Res}}{\overset{\Theta}{\rightleftarrows}} \operatorname{mod} k[Q]$$

satisfying the following properties :

(i) Res preserves interval decomposability of modules.
(ii) Θ sends interval modules to interval modules by Proposition.
(iii) $1_{\operatorname{mod} k[Q]} \cong \text{Res} \circ \Theta$.

Proposition For any $M \in \operatorname{mod} k[Q]$, we have the following inequality

$$\text{int-res-dim}(M) \leq \text{int-res-dim}\big(\Theta(M)\big).$$

Since M is an arbitrary module, we obtain the desired inequality

$$\text{int-res-gldim}(k[Q]) \leq \text{int-res-gldim}(k[P]). \qquad \square$$

結果 (1)Direct summand injectivity

(2)Monotonicity

(3)Classification of posets

with int-res-gldim=0

Theorem 3 [Aoki-Escolar-T]

Let P be a finite poset. The following are equivalent.

(a) Every $k[P]$ modules is interval decomposable
(or equivalently, int-res-gldim $k[P] = 0$).

(b) The Hasse diagram of P is one of the following form:

$$1 \longleftrightarrow \cdots \longleftrightarrow n$$

A_n (a)

$$\begin{array}{ccccc} & & 1 \to 2 \to \cdots \to n & & \\ & \nearrow & & \searrow & \\ \hat{0} & & \circlearrowleft & & \hat{1} \\ & \searrow & & \nearrow & \\ & & 1' \to 2' \to \cdots \to m' & & \end{array}$$

$C_{n,m}$

Theorem 3 [Aoki-Escolar-T]

Let P be a finite poset. The following are equivalent.

(a) Every $k[P]$ modules is interval decomposable
(or equivalently, int-res-gldim $k[P] = 0$).

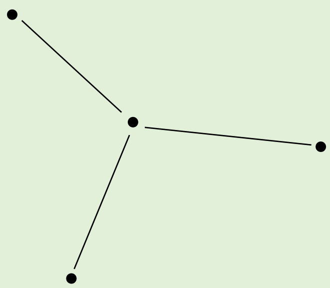
(b) The Hasse diagram of P is $1 \longleftrightarrow \cdots \longleftrightarrow n$ or

$$\begin{array}{ccccc} & 1 \to 2 \to \cdots \to n & \\ \nearrow & & \searrow \\ \hat{0} & \circlearrowleft & \hat{1} \\ \searrow & & \nearrow \\ & 1' \to 2' \to \cdots \to m' & \end{array}$$

$A_n\,(a)$ $\qquad$ $C_{n,m}$

Idea of proof (a ⇒ b)

- P does not have a vertex with degree 3.
-
-



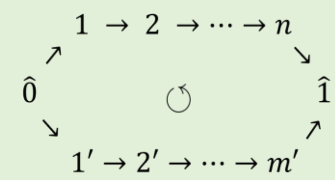
Theorem 3 [Aoki-Escolar-T]

Let P be a finite poset. The following are equivalent.

(a) Every $k[P]$ modules is interval decomposable
(or equivalently, int-res-gldim $k[P] = 0$).

(b) The Hasse diagram of P is $1 \longleftrightarrow \cdots \longleftrightarrow n$ or

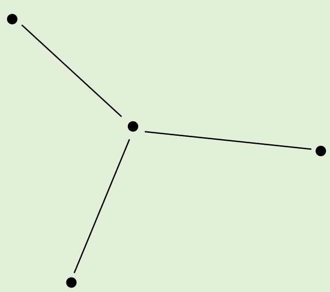
$A_n\ (a)$



$C_{n,m}$

Idea of proof (a ⇒ b)

- P does not have a vertex with degree 3.
- P is either A_n or $\tilde{A}_\ell$ for some n and ℓ.
-



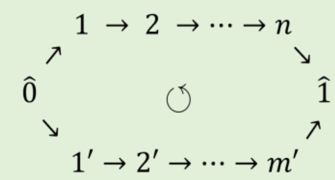
Theorem 3 [Aoki-Escolar-T]

Let P be a finite poset. The following are equivalent.

(a) Every $k[P]$ modules is interval decomposable
(or equivalently, int-res-gldim $k[P] = 0$).

(b) The Hasse diagram of P is $1 \longleftrightarrow \cdots \longleftrightarrow n$ or

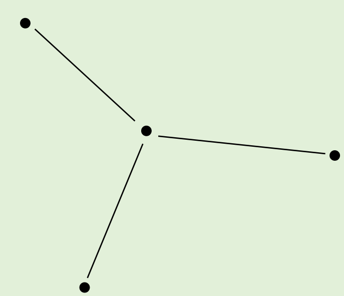
$A_n\ (a)$



$C_{n,m}$

Idea of proof (a ⇒ b)

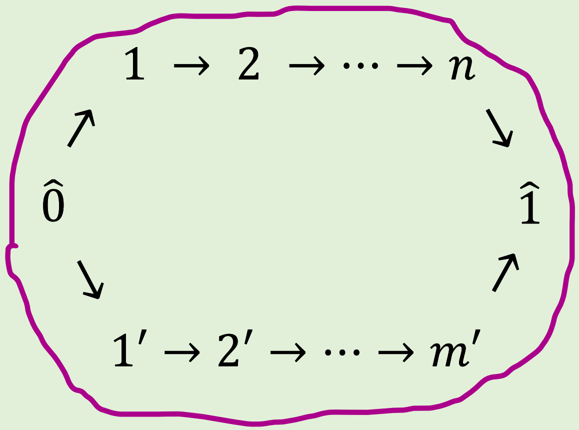
- P does not have a vertex with degree 3.
- P is either A_n or $\tilde{A}_\ell$ for some n and ℓ.
- The only $C_{n,m}$'s representations are always interval decomposable. □



Intervals in $C_{n,m}$

The intervals in $C_{n,m}$ are following forms.

(1)

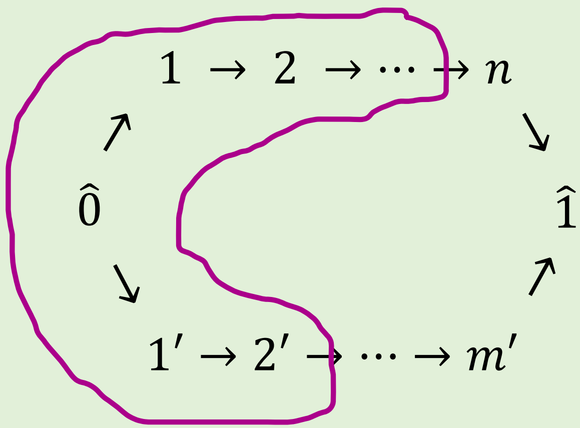


$\hat{0} \in I$, $\hat{1} \in I$ (all)

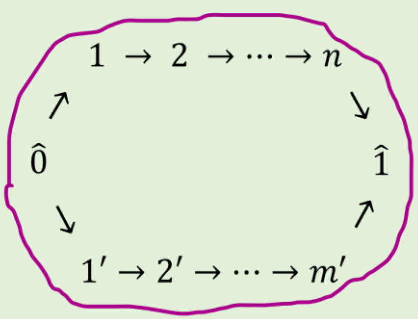
Intervals in $C_{n,m}$

The intervals in $C_{n,m}$ are following forms.

(2)



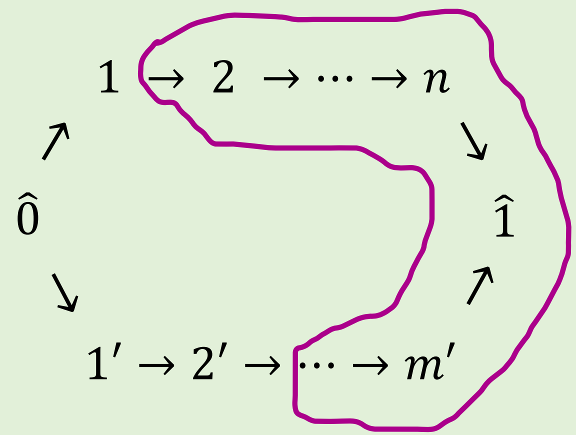
$\hat{0} \in I\ ,\ \ \hat{1} \notin I$



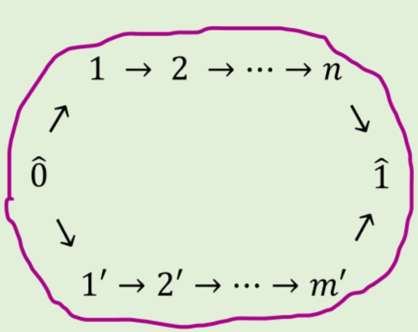
Intervals in $C_{n,m}$

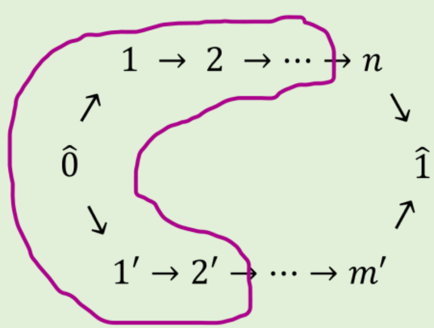
The intervals in $C_{n,m}$ are following forms.

(3)



$\hat{0} \notin I$, $\hat{1} \in I$

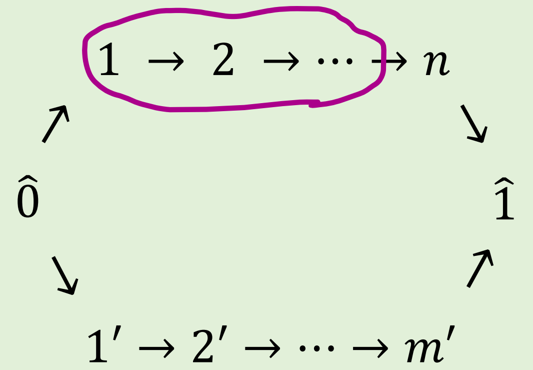




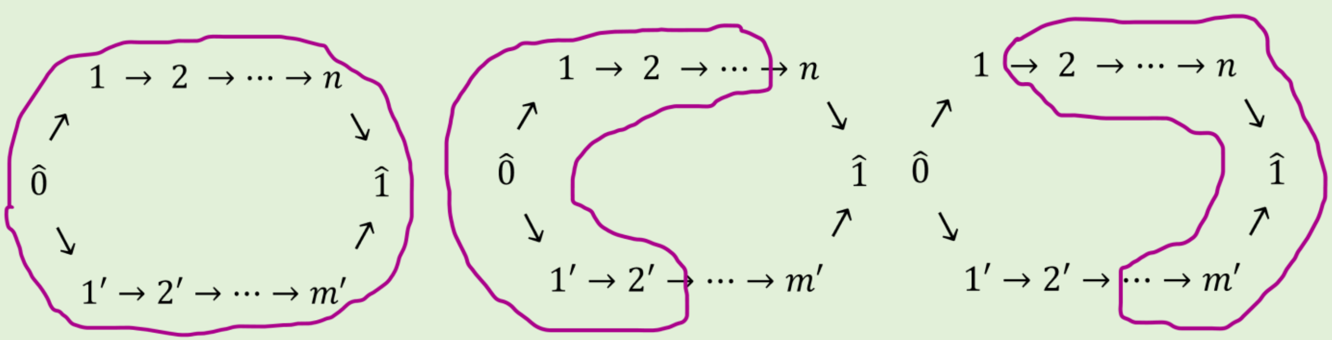
Intervals in $C_{n,m}$

The intervals in $C_{n,m}$ are following forms.

(4)



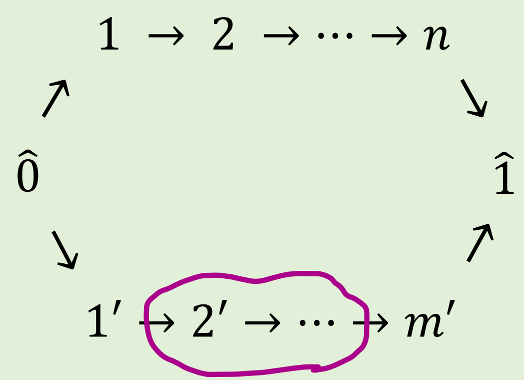
$\hat{0} \notin I$, $\hat{1} \notin I$



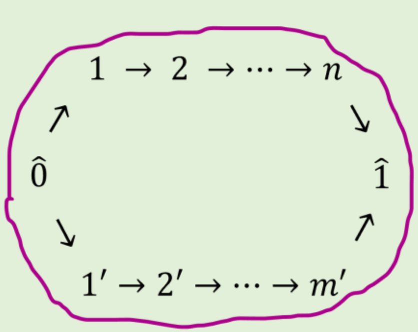
Intervals in $C_{n,m}$

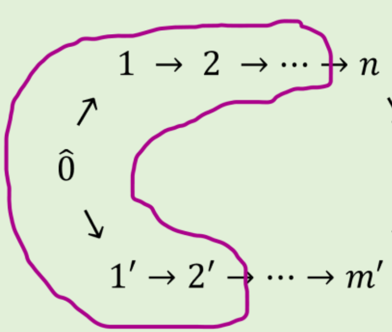
The intervals in $C_{n,m}$ are following forms.

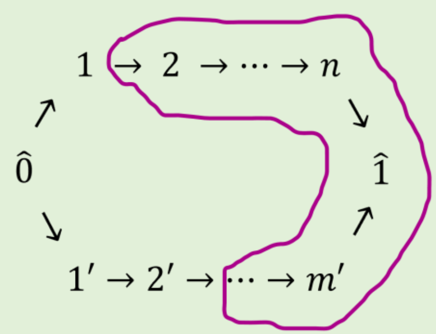
(5)

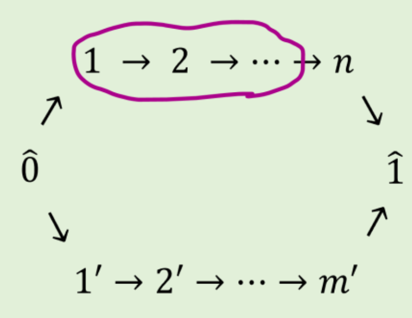


$\hat{0} \notin I\ ,\ \ \hat{1} \notin I$









区間分解可能性に着目

$\bullet \to \bullet \to \bullet \to \bullet \to \bullet$ ⇒ $\bullet \longleftrightarrow \bullet \longleftrightarrow \bullet \longleftrightarrow \bullet \longleftrightarrow \bullet$

一方通行からzigzagへ

・Carlsson, Gunnar, and Vin De Silva. “Zigzag persistence.” *Foundations of computational mathematics* 10 (2010): 367-405.

・Botnan, Magnus, and Michael Lesnick. “Algebraic stability of zigzag persistence modules.” *Algebraic & geometric topology* 18.6 (2018): 3133-3204.

⇓

$$
\begin{array}{ccccc}
 & & 1 \to 2 \to \cdots \to n & & \\
 & \nearrow & & \searrow & \\
\hat{0} & & \circlearrowleft & & \hat{1} \\
 & \searrow & & \nearrow & \\
 & & 1' \to 2' \to \cdots \to m' & &
\end{array}
$$

Discussion

- Can we apply $C_{n,m}$ to topological data analysis? Stability?
- Does int-res-gldim depend on the characteristic of fields?
- Computation using GAP package QPA("Quiver and Path Algebras")

  -pmgap : $(n \times m)$-grid by E. G. Escolar

  -our project: an arbitrary finite poset
  e.g. interval approximations / resolutions of modules

GAP - Groups, Algorithm, Programming -
a System for Computational Discreate Algebra

ご清聴ありがとうございました。

参考文献

- Carlsson, Gunnar, and Vin De Silva. "Zigzag persistence." Foundations of computational mathematics 10 (2010): 367-405.
- Carlsson, Gunnar, Vin De Silva, and Dmitriy Morozov. "Zigzag persistent homology and real-valued functions." Proceedings of the twenty-fifth annual symposium on Computational geometry. 2009.
- Botnan, Magnus, and Michael Lesnick. "Algebraic stability of zigzag persistence modules." Algebraic & geometric topology 18.6 (2018): 3133-3204.
- Asashiba, Hideto. "Relative Koszul coresolutions and relative Betti numbers." arXiv preprint arXiv:2307.06559 (2023).
- Bauer, Ulrich, et al. "Cotorsion torsion triples and the representation theory of filtered hierarchical clustering." Advances in Mathematics 369 (2020): 107171.
- Keller B, Lesnick M, Willke TL. Persistent Homology for Virtual Screening. ChemRxiv. Cambridge: Cambridge Open Engage; 2018.
- Hiraoka, Y., Nakamura, T., Hirata, A., Escolar, E. G., Matsue, K., & Nishiura, Y. “Hierarchical structures of amorphous solids characterized by persistent homology.” Proceedings of the National Academy of Sciences, 113(26), (2016):7035-7040.

参考文献

- Kiyoshi Igusa and Dan Zacharia. “On the cohomology of incidence algebras of partially ordered sets.” Communications in Algebra, 18(3) (1990):873–887.
- M.Buchet, Emerson G. Escolar“Every 1D Persistence Module is a Restriction of Some Indecomposable 2D Persistence Module” Journal of Applied and Computational Topology.
- Blanchette, Benjamin, Thomas Brüstle, and Eric J. Hanson. "Homological approximations in persistence theory." Canadian Journal of Mathematics (2021): 1-38.
- Magnus Bakke Botnan, Steffen Oppermann, and Steve Oudot. “Signed barcodes for multi-parameter persistence via rank decompositions and rank-exact resolutions.” In International Symposium on Computational Geometry (2021).
- Hideto Asashiba, Emerson G Escolar, Ken Nakashima, and Michio Yoshiwaki. “Approximation by interval-decomposables and interval resolutions of persistence modules.” Journal of Pure and Applied Algebra, 227(10):107397, (2023).
- Assem, Ibrahim, Daniel Simson, and Andrzej Skowronski. “Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory.” Cambridge University Press, (2006).
- 池祐一, E. G. エスカラ, 大林一平, 鍛冶静雄 「位相的データ解析から構造発見へ」サイエンス社, 2023.

参考文献

Erdmann, K., Holm, T., Iyama, O., & Schröer, J. “Radical embeddings and representation dimension.” *Advances in mathematics*, *185*(1) (2004):159-177.

Chacholski, W., Guidolin, A., Ren, I., Scolamiero, M., & Tombari, F. (2022). “Effective computation of relative homological invariants for functors over posets.” arXiv preprint arXiv:2209.05923.

Kuhn, Nicholas J. “Generic representations of the finite general linear groups and the Steenrod algebra: II.” *K-theory* 8.4 (1994): 395-428.