ON TRIVIAL TILTING THEORY

TAKUMA AIHARA

Abstract. We explore when an algebra has only trivial tilting module/complex.

Introduction

In mathematics, trivial cases are regularly trivial (insipid and uninteresting). For instance, we study that in group theory, a group with only trivial subgroup is a cyclic group of prime order, and in ring theory, a commutative ring with only trivial ideal is a field. These are first exercises for beginners. Neverthless, we cannot turn away from them.

In this note, we discuss trivial tilting thoery for a finite dimensional algebra Λ over an algebraically closed field. Tilting theory deals with (classical) tilting modules, one-sided tilting complexes, two-sided tilting complexes (derived Picard groups), support τ-tilting modules, Wakamatsu tilting modules and so on. For example, Λ_Λ is a trivial tilting module and the one-sided stalk complexes $\Lambda_\Lambda[m]$ are trivial tilting complexes. We will give answers to the question "when does Λ have only trivial tilting module/complex".

1. Module version

The *right finitistic dimension* $\mathrm{r.fin.dim}\,\Lambda$ of Λ is defined to be the supremum of the projective dimensions of right modules with finite projective dimension. Dually, we define the *left finitistic dimension* $\mathrm{l.fin.dim}\,\Lambda$ of Λ. Note that $\mathrm{r.fin.dim}\,\Lambda$ and $\mathrm{l.fin.dim}\,\Lambda$ do not necessarily coincide. As is well-known, $\mathrm{r.fin.dim}\,\Lambda = 0$ if and only if there is a non-zero homomorphism from every simple module to Λ in $\mathsf{mod}\,\Lambda^{\mathrm{op}}$ [4]. Here, Λ^{op} stands for the opposite algebra of Λ.

A module T is said to be *tilting* if it has finite projective dimension satisfying $\mathrm{Ext}^n_\Lambda(T,T) = 0$ for any positive integer n and there exists an exact sequence $0 \to \Lambda \to T_0 \to T_1 \to \cdots \to T_\ell \to 0$ with $T_i \in \mathsf{add}\,T$; this is also called *Miyashita tilting*. When $\mathrm{proj.dim}\,T \leq 1$, we often call T *classical tilting*.

We state the first observation of this note; it seems to be well-known (or easy to show) for researchers who are familiar with tilting theory.

Theorem 1. *The following are equivalent for an algebra Λ:*

(1) $\mathrm{r.fin.dim}\,\Lambda = 0$*;*
(2) *The module Λ is the only (basic) Miyashita tilting module;*
(3) *It is the only (basic) classical tilting module.*

The detailed version of this paper will be submitted for publication elsewhere.

Remark 2. An affirmative answer to *the finitistic dimension conjecture*, which states that the dimension is always finite, would give us the fact that $\operatorname{r.fin.dim}\Lambda = \operatorname{proj.dim} T$ for some (possibly infinitely generated) tilting module T [3, Theorem 2.6].

2. Complex version

A *tilting* complex T is defined to be a perfect complex satisfying $\operatorname{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)}(T, T[n]) = 0$ for every nonzero integer n and $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda) = \mathsf{thick}\,T$. We denote by $\mathsf{tilt}\,\Lambda$ the set of isomorphism classes of (basic) tilting complexes of Λ. In this section, we explore when Λ has only trivial tilting complex. First, one gives well-known examples; (1) [6], (2) [1] and (3) by Ringel (unpublished paper).

Example 3. The following algebras have only trivial tilting complexes:

(1) local algebras;
(2) selfinjective algebras with cyclic Nakayama permutation;
(3) radical-square-zero algebras satisfying $\operatorname{Ext}^1(S, S') \neq 0$ for all simple modules S, S'.

All algebras above have left and right finitistic dimension zero. However, even if Λ satisfies the property, it does not necessarily hold that Λ admits no nontrivial tilting complex; many selfinjective algebras satisfy both the property and $\mathsf{tilt}\,\Lambda \neq \Lambda[\mathbb{Z}]$, so we should give an example of nonselfinjective algebras.

Example 4. Let Λ be the radical-square-zero algebra presented by the quiver:

$$y \circlearrowright 1 \xrightarrow{x} 2 \circlearrowleft z \ .$$

It is easy to check that $\operatorname{r.fin.dim}\Lambda = 0 = \operatorname{l.fin.dim}\Lambda$.

As is seen in [2, Example 5.10], we have only three indecomposable pretilting complexes up to shift: P_1, P_2 and $X := P_2 \xrightarrow{x} P_1$. This tells us that there precisely exist two types of nontrivial tilting complexes:

$$T := \bigoplus \begin{cases} \qquad\quad P_1 \\ P_2 \xrightarrow[x]{} P_1 \end{cases} , \qquad U := \bigoplus \begin{cases} P_2 \\ P_2 \xrightarrow[x]{} P_1 \end{cases}$$

So, we obtain $\mathsf{tilt}\,\Lambda = \Lambda[\mathbb{Z}] \cup T[\mathbb{Z}] \cup U[\mathbb{Z}]$. Moreover, each component admits the endomorphism algebra isomorphic to Λ, Γ or Γ^{op} (mutually nonisomorphic). Here, Γ is given by the following quiver with relations $\alpha\beta\alpha = \alpha\gamma = \gamma\beta = \gamma^2 = 0$:

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \circlearrowleft \gamma \ .$$

We say that a complex T is *two-term* provided it is of the form $T^{-1} \to T^0$. The subset of $\mathsf{tilt}\,\Lambda$ consisting of two-term tilting complexes is denoted by $\mathsf{2tilt}\,\Lambda$.

A full subcategory of $\mathsf{mod}\,\Lambda$ is said to be a *torsion class* if it is closed under extensions and factors. We say that a torsion class is *ν-stable* provided it is closed under taking the Nakayama functor $\nu := - \otimes_\Lambda D\Lambda$. Here is a useful observation.

Proposition 5. [5, Proposition 5.5] *Let T be a two-term perfect complex of Λ and put $X := H^0(T)$. Then T is tilting if and only if $\mathsf{Fac}\,X$ is a ν-stable functorially finite torsion class of $\mathsf{mod}\,\Lambda$.*

An *Iwanaga–Gorenstein* algebra is defined to have finite left and right selfinjective dimension. We can get a result similar to Theorem 1.

Theorem 6. *The following are equivalent for an Iwanaga–Gorenstein algebra* Λ*:*

(1) Λ *is a selfinjective algebra with cyclic Nakayama permutation;*
(2) $\mathsf{tilt}\,\Lambda = \{\Lambda[m] \mid m \in \mathbb{Z}\}$*;*
(3) $\mathsf{2tilt}\,\Lambda = \{\Lambda, \Lambda[1]\}$.

References

[1] H. Abe and M. Hoshino, On derived equivalences for selfinjective algebras. *Comm. Algebra* **34** (2006), no. 12, 4441–4452.
[2] L. Angeleri-Hugel, S. Koenig, Q. Liu and D. Yang, Ladders and simplicity of derived module categories. *J. Algebra* **472** (2017), 15–66.
[3] L. Angeleri-Hugel and J. Trlifaj, Tilting theory and the finitistic dimension conjectures. *Trans. Amer. Math. Soc.* **354** (2002), no. 11, 4345–4358.
[4] H. Bass, Finitistic dimension and a homological generalization of simple-primary rings. *Trans. Amer. Math. Soc.* **95** (1960), 466–488.
[5] M. Hoshino, Y. Kato and J. Miyachi, On t-structures and torsion theories induced by compact objects. *J. Pure Appl. Algebra* **167** (2002), no. 1, 15–35.
[6] R. Rouquier and A. Zimmermann, Picard groups for derived module categories. *Proc. London Math. Soc. (3)* **87** (2003), no. 1, 197–225.

Department of Mathematics
Tokyo Gakugei University
4-1-1 Nukuikita-machi, Koganei, Tokyo 184-8501, JAPAN
Email address: aihara@u-gakugei.ac.jp