QUIVER HEISENBERG ALGEBRAS AND THE ALGEBRA $B(Q)$

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Abstract. This is a report on ongoing joint work with Martin Herschend about quiver Heisenberg algebras (QHA) and the algebra ${}^{v}B(Q)$. In this note, we mainly investigate QHA of Dynkin type. The first main result tells that QHA ${}^{v}\Lambda(Q)$ of Dynkin type is finite dimensional if and only if the weight $v \in \mathbf{k}Q_0$ is regular (see Definition 1), and moreover that if this is the case, ${}^{v}\Lambda(Q)$ is a symmetric algebra. In the case $\mathrm{char}\mathbf{k} = 0$, the "if" part of the first statement is proved by Etingof and Rains [10], and the second is verified for a generic weight by Etingof, Latour and Rains [11].

Compare to the preprojective algebras $\Pi(Q)$ which are only Frobenius in general, QHA ${}^{v}\Lambda(Q)$ can be said to be well-behaved, since they are always symmetric. Making use of this, we investigate silting theory of QHA of Dynkin type. We obtain results which are analogous to the results for $\Pi(Q)$ by Aihara-Mizuno [3].

1. Introduction

Throughout this note $\mathbf{k}$ is an algebraically closed field and Q is a finite acyclic quiver. For $\mathbf{k}Q$-module M, the dimension vector $\underline{\dim} M$ is regarded as an element of $\mathbf{k}Q_0 = \mathbf{k} \times \cdots \times \mathbf{k}$ (not of $\mathbb{Z}Q_0$).

For an element $v \in \mathbf{k}Q_0$, which we call *weight*, we define the *weighted dimension* of M to be

$$ {}^{v}\dim M := \sum_{i \in Q_0} v_i \dim e_i M. $$

Definition 1. A weight $v \in \mathbf{k}Q_0$ is called *regular* if

$$ {}^{v}\dim M \neq 0 \quad (\forall M \in \mathrm{ind}\, Q) $$

Remark 2. In the case Q is Dynkin and char $\mathbf{k} = 0$, the vector space $\mathbf{k}Q_0$ may be identified with the Cartan subalgebra $\mathfrak{h}$ of the semi-simple Lie algebra $\mathfrak{g}$ corresponding to Q. By Gabriel's theorem the dimension vectors of indecomposable $\mathbf{k}Q$-modules are precisely the roots of $\mathfrak{g}$, so the regularity given here coincides with that are used by Etingof-Rains [10].

Example 3. Let Q be a directed A_3-quiver.

$$ Q: \ 1 \xrightarrow{\ \alpha\ } 2 \xrightarrow{\ \beta\ } 3 \,. $$

The dimension vectors of indecomposable modules are

$$ \begin{smallmatrix}1\\0\\0\end{smallmatrix},\ \begin{smallmatrix}0\\1\\0\end{smallmatrix},\ \begin{smallmatrix}0\\0\\1\end{smallmatrix},\ \begin{smallmatrix}1\\1\\0\end{smallmatrix},\ \begin{smallmatrix}0\\1\\1\end{smallmatrix},\ \begin{smallmatrix}1\\1\\1\end{smallmatrix}. $$

The detailed version of this paper will be submitted for publication elsewhere.

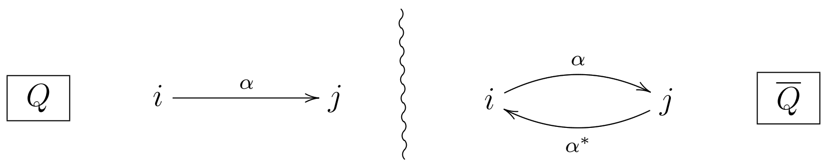
Thus, regularity of a weight $v = (v_1, v_2, v_3)^t$ is

$$v_1 \neq 0,\ v_2 \neq 0,\ v_3 \neq 0,$$
$$v_1 + v_2 \neq 0,\ v_2 + v_3 \neq 0,\ v_1 + v_2 + v_3 \neq 0.$$

Looking the weighted dimension of simple modules S_i $(i \in Q_0)$, we obtain

Lemma 4. *A regular weight v is sincere i.e., $v_i \neq 0$ $(\forall i \in Q_0)$.*

Let $\overline{Q}$ be the double of Q.



For $i \in Q_0$, ρ_i denotes the mesh relation at i

$$\rho_i := \sum_{\alpha \in Q_1 : t(\alpha) = i} \alpha\alpha^* - \sum_{\alpha \in Q_1 : h(\alpha) = i} \alpha^*\alpha.$$

Definition 5. The quiver Heisenberg algebra ${}^v\Lambda(Q)$ with the weight $v \in \mathbf{k}Q_0$ is defined to be

$${}^v\Lambda(Q) := \frac{\mathbf{k}[z]\overline{Q}}{(\rho_i - v_i z e_i \mid i \in Q_0)}.$$

Remark 6. This algebra is a special case of algebras studied in [6, 7, 10].

Remark 7. If v is sincere, then ${}^v\Lambda(Q)$ is isomorphic to the algebra which was given in previous talks of QHA, via the isomorphism

$${}^v\Lambda(Q) \cong \frac{\mathbf{k}\overline{Q}}{([a, {}^v\rho] \mid a \in \overline{Q}_1)}, \quad z \mapsto {}^v\rho$$

where ${}^v\rho := \sum_i v_i^{-1}\rho_i$ the "weighted mesh relation" and $[a, {}^v\rho] = a{}^v\rho - {}^v\rho a$ is the commutator.

We recall an indecomposable decomposition of ${}^v\Lambda(Q)$ as $\mathbf{k}Q$-module.

Theorem 8 ([12])**.** *If v is regular, then as $\mathbf{k}Q$-modules*

$${}^v\Lambda(Q) \cong \bigoplus_M M^{\dim M}.$$

where M runs over representatives of isomorphism class of indecomposable preprojective modules.

In particular, in the case Q is Dynkin, if v is regular, ${}^v\Lambda(Q)$ is finite dimensional. One of our main result asserts that the converse holds and moreover, if this is the case, ${}^v\Lambda(Q)$ is symmetric.

Theorem 9 ((1) [12], (2) [13])**.** *Let Q be a Dynkin quiver. The followings hold.*

(1) *A weight v is regular if and only if ${}^v\Lambda(Q)$ is finite dimensional.*

(2) *If a weight v is regular, then ${}^{v}\Lambda(Q)$ is symmetric.*

Remark 10. In the case $\mathrm{char}\mathbf{k} = 0$, Etingof-Latour-Rains [11] showed that ${}^{v}\Lambda(Q)$ is symmetric for a generic weight v.

In the next section, we explain keys of proofs. In the third section, we discuss silting theory of ${}^{v}\Lambda(Q)$.

2. Proof of Theorem 9

2.1. **Proof of Theorem 9(1).** We only have to prove "if" direction. We do this by proving the contraposition. Namely, we show that if v is not regular, then ${}^{v}\Lambda(Q)$ is infinite dimensional. In the case v is not sincere, using an explicit presentation of ${}^{v}\Lambda(Q)$ by a quiver with relations, we can directly check that $\dim {}^{v}\Lambda(Q) = \infty$. Thus we may assume that v is sincere (and not regular). In that case, we conclude $\dim {}^{v}\Lambda(Q) = \infty$ by the following proposition.

To state the proposition, we recall that ${}^{v}\Lambda(Q)$ acquires a grading that counts the number of extra arrows α^*, which we call the $*$-*grading*. Let ${}^{v}\Lambda(Q)_n$ denote the $*$-degree n-part of ${}^{v}\Lambda(Q)$. It is clear that ${}^{v}\Lambda(Q)_0 = \mathbf{k}Q$ and ${}^{v}\Lambda(Q)_n$ has a canonical structure of $\mathbf{k}Q$-bimodule.

Proposition 11 ([12])**.** *Assume that v is sincere but not regular. Let M be an indecomposable $\mathbf{k}Q$-module such that ${}^{v}\dim M = 0$. Then for any $n \geq 0$, M is a direct summand of ${}^{v}\Lambda(Q)_n \otimes_{\mathbf{k}Q} M$ as $\mathbf{k}Q$-module.*

In particular ${}^{v}\Lambda(Q)_n \neq 0$ for all $n \geq 0$.

The case $n = 0$ is clear. For the case $n = 1$, we recall that there is a canonical exact triangle which is obtained from analysis of QHA and preprojective algebra

$$M \to {}^{v}\widetilde{\Lambda}(Q)_1 \otimes^{\mathbb{L}}_{\mathbf{k}Q} M \to \nu_1^{-1} M \to,$$

in the derived category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}Q\mathrm{mod})$ where ${}^{v}\widetilde{\Lambda}(Q)$ is the derived quiver Heisenberg algebra given in the next section. We can show that ${}^{v}\dim M = 0$ if and only if the above exact triangle splits. We note that in the case ${}^{v}\dim M \neq 0$, the exact triangle is an almost split exact triangle.

The case $n \geq 2$ uses the following exact triangle

$$\widetilde{\Pi}_1 \otimes^{\mathbb{L}} {}^{v}\widetilde{\Lambda}_{n-2} \otimes^{\mathbb{L}} M \to {}^{v}\widetilde{\Lambda}_1 \otimes^{\mathbb{L}} {}^{v}\widetilde{\Lambda}_{n-1} \otimes^{\mathbb{L}} M \to {}^{v}\widetilde{\Lambda}_n \otimes^{\mathbb{L}} M$$

Please see [12] for details.

2.2. **Proof of Theorem 9(2).** Main ingredients of our proof is the followings:

(i) A general result about derived preprojective algebra of d-representation finite algebra.

(ii) The algebra ${}^{v}B(Q)$.

(iii) A direct computation of the cohomology algebra of derived QHA.

2.2.1. Let A be a d-representation finite algebra. Iyama-Oppermann [15] showed that the $d+1$-preprojective algebra $\Pi := \Pi_{d+1}(A)$ is Frobenius. Let ν be the Nakayama automorphism of Π, i.e., $\Pi \cong {}_{\nu}\mathrm{D}(\Pi)$ as Π-bimodules.

Theorem 12 ([13]). *Let $\widetilde{\Pi}$ be the $d+1$-derived preprojective algebra of A. Then, the cohomology algebra $\mathrm{H}(\widetilde{\Pi})$ of the derived $d+1$-preprojective algebra $\widetilde{\Pi}$ is isomorphic to the skew polynomial algebra $\Pi[u;\nu]$*

$$\mathrm{H}(\widetilde{\Pi}) \cong \Pi[u;\nu]$$

as cohomologically graded algebras, where u is a formal variable of cohomological degree $-d$ and

$$au = u\nu(a) \quad (\forall a \in \Pi).$$

This theorem connects the Nakayama automorphism ν to the algebra structure of $\mathrm{H}(\widetilde{\Pi})$.

2.2.2. We introduce a finite dimensional algebra ${}^{v}B(Q)$.

Definition 13. For a quiver Q and a regular weight v, we define

$${}^{v}B(Q) := \begin{pmatrix} \mathbf{k}Q & {}^{v}\Lambda(Q)_1 \\ 0 & \mathbf{k}Q \end{pmatrix}$$

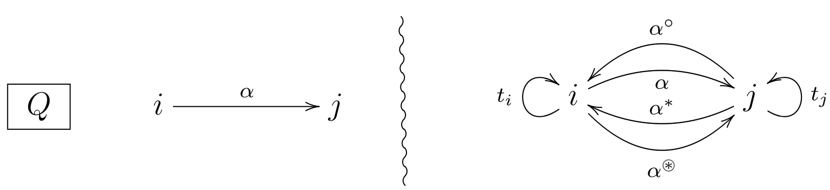
the bypath algebra (a.k.a., 2-path algebra) of Q.

The algebra ${}^{v}B(Q)$ has various properties that are 1-dimension higher version of that of the path algebra $\mathbf{k}Q$. Among other things, we have a 1-dimension higher version of Gabriel's dichotomy of representation types.

Theorem 14 ([13]). *The followings hold.*

(1) *${}^{v}B(Q)$ is 2-representation finite if and only if Q is Dynkin.*
(2) *${}^{v}B(Q)$ is 2-representation infinite if and only if Q is non-Dynkin.*

Recall that the derived QHA ${}^{v}\widetilde{\Lambda}(Q)$ is a DGA explicitly defined by the quiver



the differential is defined by

$$d(\alpha) := 0, d(\alpha^*) := 0, d(\alpha^\circ) := -[\alpha^*, {}^{v}\rho], d(\alpha^\circledast) := [\alpha, {}^{v}\rho],$$
$$d(t_i) := \sum_{\alpha\in Q_1} e_i[\alpha, \alpha^\circ]e_i + \sum_{\alpha\in Q_1} e_i[\alpha^*, \alpha^\circledast]e_i.$$

If $\mathrm{char}\mathbf{k} \neq 2$, ${}^{v}\widetilde{\Lambda}(Q)$ is the Ginzburg dg-algebra $\mathcal{G}(\overline{Q}, W)$ where

$$W := -\frac{1}{2}{}^{v}\rho\rho = -\frac{1}{2}\sum_{i\in Q_0} v_i^{-1}\rho_i^2.$$

Lemma 15 ([13]). *The 3-derived preprojective algebra of ${}^{v}B(Q)$ and the 2-ed quasi-Veronese algebra of ${}^{v}\widetilde{\Lambda}(Q)$ are isomorphic*

$$\widetilde{\Pi}_3({}^{v}B(Q)) \cong {}^{v}\widetilde{\Lambda}(Q)^{[2]}.$$

2.2.3. By (more or less) direct computation we have

Theorem 16 ([12]). *Assume that Q is Dynkin and v is regular. Then,*

$$\mathrm{H}({}^{v}\widetilde{\Lambda}(Q)) \cong {}^{v}\Lambda(Q)[u]$$

where u is a formal variable of cohomological degree -2.

Comparing the right hand sides of the isomorphisms given in Theorem 12 for ${}^{v}B(Q)$ and Theorem 16 via Lemma 15, we conclude that $\nu_\Lambda = \mathrm{id}_\Lambda$ up to inner automorphisms.

3. Silting theory of QHA of Dynkin type

Compare to the preprojective algebras $\Pi(Q)$ which are only Frobenius in general, QHA ${}^{v}\Lambda(Q)$ can be said to be well-behaved, since they are always symmetric. Making use of this, we investigate silting theory of QHA of Dynkin type. Before doing this, first we introduce a general construction of a tilting complex.

3.1. In this subsection, Q denote a quiver which is not necessarily Dynkin.

Let $i \in Q_0$. We define a complex $T^{(i)}$ over ${}^{v}\Lambda$ to be

$$T^{(i)} := {}^{v}\Lambda(1-e_i) \oplus \left[{}^{v}\Lambda e_i \xrightarrow{(\pm a^*)_{a\in h^{-1}(i)}} \bigoplus_{a\in h^{-1}(i)} {}^{v}\Lambda e_{t(a)} \right]$$

where the right factor is a complex placed in $-1, 0$-th cohomological. degree.

This complex is a “family version” of the tilting complex of Crawley-Boevey-Kimura [8]. The reduction $\Pi \otimes_{{}^{v}\Lambda} T^{(i)}$ is the tilting complex introduced by Baumann-Kamniter [4] and Buan-Iyama-Reiten-Scott [5].

Let $r : W_Q \curvearrowright \mathbf{k}Q_0$ be the dual action. Let r_i be the action of the Coxeter generator s_i.

Theorem 17 ([13]). *The complex $T^{(i)}$ is a tilting complex and*

$$\mathrm{End}_{{}^{v}\Lambda}(T^{(i)})^{\mathrm{op}} \cong {}^{r_i(v)}\Lambda.$$

3.2. From now we assume that Q is Dynkin. We note $\mathrm{silt}{}^{v}\Lambda = \mathrm{tilt}{}^{v}\Lambda$ by Theorem 5.

Then, it is straightforward to check that $T^{(i)}$ is the left silting mutation of ${}^{v}\Lambda$:

$$T^{(i)} = \mu_i^-({}^{v}\Lambda).$$

Thus, taking iterated mutations

$${}^{w(v)}\Lambda \cong \mathrm{End}_{{}^{v}\Lambda}(\mu_{i_n}^- \cdots \mu_{i_1}^-({}^{v}\Lambda))^{\mathrm{op}}$$

where $w = s_{i_n} \cdots s_{i_1}$.

There are following bijections,

$$W_Q \xrightarrow{1:1} \mathrm{sttilt}\Pi(Q) \xrightarrow{1:1} 2\mathrm{silt}\Pi(Q).$$

the first is established by Mizuno [14], the second is a consequence of a general result due to Adachi-Iyama-Reiten [1]

The weighted mesh relation ${}^v\rho$ is central in ${}^v\Lambda(Q)$ and we have a canonical isomorphism ${}^v\Lambda(Q)/({}^v\rho) \cong \Pi(Q)$. Applying a general result by Eisele-Janssens-Raedschelders [9], we obtain bijections

$$W_Q \xrightarrow{1:1} \mathrm{sttilt}{}^v\Lambda(Q) \xrightarrow{1:1} 2\mathrm{silt}{}^v\Lambda(Q)$$

which is given by

$$w = s_{i_n} \cdots s_{i_1} \mapsto \mu^+_{i_n} \cdots \mu^+_{i_1}({}^v\Lambda)$$

By general criteria due to Aihara-Mizuno [3], we conclude that ${}^v\Lambda(Q)$ is silting discrete.

As a consequence of the preceding consideration, we obtain the following results which are analogous to the results for $\Pi(Q)$ by Aihara-Mizuno [3].

Theorem 18 ([13]). *Assume that Q is Dynkin and v is regular.*

(1) *The algebra ${}^v\Lambda(Q)$ is silting discrete.*
(2) *A silting complex T is a tilting complex and*

$$\mathrm{End}_{{}^v\Lambda(Q)}(T)^{\mathrm{op}} \cong {}^{w(v)}\Lambda(Q)$$

for some $w \in W_Q$.

Theorem 19 ([13]). *Let B_Q be the braid group of Q. There is a bijection*

$$B_Q \xrightarrow{1:1} \mathrm{silt}{}^v\Lambda(Q), \ b \mapsto \mu_b({}^v\Lambda(Q)).$$

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