WALL-AND-CHAMBER STRUCTURES OF STABILITY PARAMETERS FOR SOME DIMER QUIVERS

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Abstract. It is known that any projective crepant resolution of a three-dimensional Gorenstein toric singularity can be described as the moduli space of representations of a quiver associated to a dimer model for some stability parameter. The space of stability parameters has the wall-and-chamber structure and we can track the variations of projective crepant resolutions by observing such a structure. In this article, we consider dimer models giving rise to projective crepant resolutions of a toric compound Du Val singularity. We show that sequences of zigzag paths, which are special paths on a dimer model, determine the wall-and-chamber structure of the space of stability parameters.

1. Introduction

The moduli space of representations of a quiver, introduced in [10], is defined as the GIT quotient associated to a stability parameter. For some nice singularities, resolutions of singularities can be described as moduli spaces of representations of a quiver. For example, any projective crepant resolution of a three-dimensional Gorenstein quotient singularity $\mathbb{C}^3/G$ defined by the action of a finite subgroup $G \subset \mathrm{SL}(3,\mathbb{C})$ on $\mathbb{C}^3$ can be described as the moduli space of representations of the McKay quiver of G (see [2, 14]). Also, any projective crepant resolution of a three-dimensional Gorenstein toric singularity can be described as the moduli space of representations of the quiver associated to a dimer model (see [9]). It is known that the space of stability parameters associated to a quiver has the *wall-and-chamber structure*, that is, it is decomposed into chambers separated by walls. The moduli spaces associated to stability parameters contained in the same chamber are isomorphic, but a stability parameter contained in another chamber would give a different moduli space. Thus, it is important to detect the wall-and-chamber structure of the space of stability parameters to understand the relationships among projective crepant resolutions of the above singularities. The purpose of this article is to detect the wall-and-chamber structure for a particular class of three-dimensional Gorenstein toric singularities called toric compound Du Val (cDV) singularities. In particular, we will see that the combinatorics of a dimer model associated to a toric cDV singularity control the wall-and-chamber structure.

2. Preliminaries on dimer models and associated quivers

2.1. **Dimer models.** We first introduce dimer models and related notions which are originally derived from theoretical physics (e.g., [4, 6]).

The detailed version of this paper will be submitted for publication elsewhere.

A *dimer model* Γ on the real two-torus $\mathbb{T} := \mathbb{R}^2/\mathbb{Z}^2$ is a finite bipartite graph on $\mathbb{T}$ inducing a polygonal cell decomposition of $\mathbb{T}$. Since Γ is a bipartite graph, the set Γ_0 of nodes of Γ is divided into two subsets Γ_0^+, Γ_0^-, and edges of Γ connect nodes in Γ_0^+ with those in Γ_0^-. We denote by Γ_1 the set of edges. We color the nodes in Γ_0^+ white, and those in Γ_0^- black throughout this article. A *face* of Γ is a connected component of $\mathbb{T}\backslash\Gamma_1$. We denote by Γ_2 the set of faces. In the rest of this article, we assume that any dimer model satisfies a certain nice condition called the *consistency condition*, see e.g., [8, Section 6] for more details. For example, Figure 1 is a consistent dimer model on $\mathbb{T}$, where the outer frame is a fundamental domain of $\mathbb{T}$.

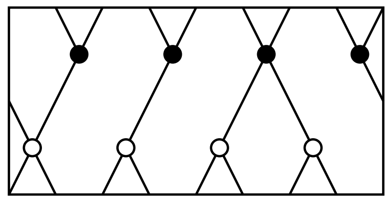

FIGURE 1. An example of a dimer model

We say that a path on a dimer model is a *zigzag path* if it makes a maximum turn to the right on a black node and a maximum turn to the left on a white node. For example, the paths (displayed in thick lines) in Figure 2 are all zigzag paths on the dimer model given in Figure 1.

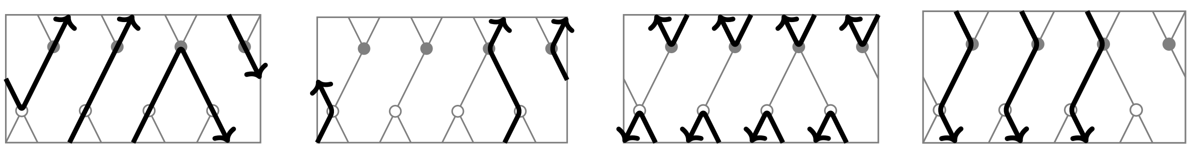

FIGURE 2. Zigzag paths on the dimer model given in Figure 1

We fix two 1-cycles on $\mathbb{T}$ generating the homology group $\mathrm{H}_1(\mathbb{T})$, and take a fundamental domain of $\mathbb{T}$ along such two cycles. Since we can consider a zigzag path z on Γ as a 1-cycle on $\mathbb{T}$, we have the homology class $[z] \in \mathrm{H}_1(\mathbb{T}) \cong \mathbb{Z}^2$, which is called the *slope* of z. Note that for a consistent dimer model Γ, any edge of Γ is contained in exactly two zigzag paths and any slope is a primitive element. Then, for a consistent dimer model Γ, we assign the lattice polygon called the zigzag polygon (cf. [8, Section 12]). Let $[z]$ be the slope of a zigzag path z on Γ. By normalizing $[z] \in \mathbb{Z}^2$, we consider it as an element of the unit circle S^1. Then, the set of slopes has a natural cyclic order along S^1. We consider the sequence $\big([z_i]\big)_{i=1}^k$ of slopes of zigzag paths on Γ such that they are cyclically ordered starting from $[z_1]$, where k is the number of zigzag paths. We note that some slopes may coincide in general. We set another sequence $(w_i)_{i=1}^k$ in $\mathbb{Z}^2$ defined as $w_0 = (0,0)$ and

$$w_{i+1} = w_i + [z_{i+1}]' \quad (i = 0, 1, \ldots, k-1).$$

Here, $[z_{i+1}]' \in \mathbb{Z}^2$ is the element obtained from $[z_{i+1}]$ by rotating 90 degrees in the anti-clockwise direction. One can see that $w_k = (0,0)$ since the sum of all slopes is equal to zero. We call the convex hull of $\{w_i\}_{i=1}^k$ the *zigzag polygon* of Γ and denote it by Δ_Γ. Note that there are several choices of an initial zigzag path z_1, but the zigzag polygon is determined uniquely up to unimodular transformations. By definition, we see that the slope of a zigzag path is an outer normal vector of some side of Δ_Γ, and the number of zigzag paths having the same slope $v \in \mathbb{Z}^2$ coincides with the number of primitive segments of the side of Δ_Γ whose outer normal vector is v.

Example 1. We consider the dimer model in Figure 1 and its zigzag paths as in Figure 2. Then, we have the cyclically ordered sequence of slopes

$$\big((0,-1),\ (0,-1),\ (0,-1),\ (1,1),\ (0,1),\ (0,1),\ (-1,0)\big),$$

where we take a $\mathbb{Z}$-basis of $\mathrm{H}_1(\mathbb{T}) \cong \mathbb{Z}^2$ along the vertical and horizontal lines of the fundamental domain of $\mathbb{T}$. Thus, we have the zigzag polygon as in Figure 3.

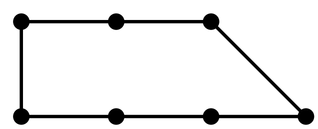

FIGURE 3. The zigzag polygon of the dimer model given in Figure 1

On the other hand, any lattice polygon can be described as the zigzag polygon of a consistent dimer model as follows.

Theorem 2 (see e.g., [5, 8])**.** *For any lattice polygon Δ, there exists a consistent dimer model Γ such that $\Delta = \Delta_\Gamma$.*

2.2. **Toric rings associated to dimer models.** Let Γ be a consistent dimer model. We next consider the cone σ_Γ over the zigzag polygon Δ_Γ, that is, σ_Γ is the cone whose section on the hyperplane at height one is Δ_Γ.

Let $\mathsf{N} := \mathbb{Z}^3$ be a lattice and $\mathsf{M} := \mathrm{Hom}_{\mathbb{Z}}(\mathsf{N}, \mathbb{Z})$ be the dual lattice of N. We set $\mathsf{N}_\mathbb{R} := \mathsf{N} \otimes_\mathbb{Z} \mathbb{R}$ and $\mathsf{M}_\mathbb{R} := \mathsf{M} \otimes_\mathbb{Z} \mathbb{R}$. We denote the standard inner product by $\langle\ ,\ \rangle : \mathsf{M}_\mathbb{R} \times \mathsf{N}_\mathbb{R} \to \mathbb{R}$. For the vertices $\widetilde{v}_1, \dots, \widetilde{v}_n \in \mathbb{Z}^2$ of Δ_Γ, we let $v_i := (\widetilde{v}_i, 1) \in \mathsf{N}$ $(i = 1, \dots, n)$. The cone σ_Γ over Δ_Γ is defined as

$$\sigma_\Gamma := \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_n \subset \mathsf{N}_\mathbb{R}.$$

Then, we consider the dual cone

$$\sigma_\Gamma^\vee := \{x \in \mathsf{M}_\mathbb{R} \mid \langle x, v_i\rangle \geq 0 \text{ for any } i = 1, \dots, n\}.$$

Using this cone, we can define the *toric ring* (*toric singularity*) R_Γ *associated to* Γ as

$$R_\Gamma := \mathbb{C}[\sigma_\Gamma^\vee \cap \mathsf{M}] = \mathbb{C}[t_1^{a_1} t_2^{a_2} t_3^{a_3} \mid (a_1, a_2, a_3) \in \sigma_\Gamma^\vee \cap \mathsf{M}],$$

which is Gorenstein in dimension three. We note that any three-dimensional Gorenstein toric ring can be described with this form. Precisely, let σ be a strongly convex rational polyhedral cone in $\mathsf{N}_\mathbb{R}$ which defines a three-dimensional Gorenstein toric ring R. Then, it is known that, after applying an appropriate unimodular transformation (which does not change the associated toric ring up to isomorphism) to σ, the cone σ can be described as the cone over a certain lattice polygon Δ_R. We call the lattice polygon Δ_R the *toric* *diagram* of R. By Theorem 2, there exists a consistent dimer model Γ such that $\Delta_\Gamma = \Delta_R$ for any three-dimensional Gorenstein toric ring R, in which case we have $R = R_\Gamma$.

2.3. **Quivers associated to dimer models.** Let Γ be a dimer model. As the dual of Γ, we obtain the quiver Q_Γ associated to Γ, which is embedded in $\mathbb{T}$, as follows. We assign a vertex dual to each face in Γ_2 and an arrow dual to each edge in Γ_1. We fix the orientation of any arrow so that the white node is on the right of the arrow. For example, Figure 4 is the quiver associated to the dimer model in Figure 1. We simply denote the quiver Q_Γ by Q unless it causes any confusion. Let $Q = (Q_0, Q_1)$ be the quiver associated to a dimer model, where Q_0 is the set of vertices and Q_1 is the set of arrows. Let $\mathsf{hd}(a), \mathsf{tl}(a) \in Q_0$ be respectively the head and tail of an arrow $a \in Q_1$. A *path* of length $r \geq 1$ is a finite sequence of arrows $\gamma = a_1 \cdots a_r$ with $\mathsf{hd}(a_i) = \mathsf{tl}(a_{i+1})$ for $i = 1, \ldots, r-1$. We define $\mathsf{tl}(a) = \mathsf{tl}(a_1), \mathsf{hd}(a) = \mathsf{hd}(a_r)$ for a path $\gamma = a_1 \cdots a_r$. A *relation* in Q is a $\mathbb{C}$-linear combination of paths of length at least two having the same head and tail. We especially consider relations in Q defined as follows. For each arrow $a \in Q_1$, there exist two paths γ_a^+, γ_a^- such that $\mathsf{hd}(\gamma_a^\pm) = \mathsf{tl}(a)$, $\mathsf{tl}(\gamma_a^\pm) = \mathsf{hd}(a)$ and γ_a^+ (resp. γ_a^-) goes around the white (resp. black) node incident to the edge dual to a clockwise (resp. counterclockwise), see e.g., [12, Figure 6]. We define the set of relations $\mathcal{J}_Q := \{\gamma_a^+ - \gamma_a^- \mid a \in Q_1\}$ and call the pair $(Q, \mathcal{J}_Q)$ the *quiver with relations* associated to Γ.

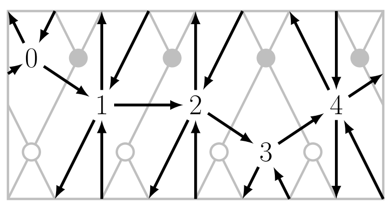

FIGURE 4. The quiver associated to the dimer model given in Figure 1

We then introduce representations of a quiver with relations. A *representation* of $(Q, \mathcal{J}_Q)$ consists of a set of $\mathbb{C}$-vector spaces $\{M_v \mid v \in Q_0\}$ together with $\mathbb{C}$-linear maps $\varphi_a : M_{\mathsf{tl}(a)} \to M_{\mathsf{hd}(a)}$ satisfying the relations $\mathcal{J}_Q$, that is, $\varphi_{\gamma_a^+} = \varphi_{\gamma_a^-}$ for any $a \in Q_1$. Here, for a path $\gamma = a_1 \cdots a_r$, the map φ_γ is defined as the composite $\varphi_{a_1} \cdots \varphi_{a_r}$ of $\mathbb{C}$-linear maps. (Note that in this article, a composite fg of morphisms means we first apply f then g.) In the rest of this article, we assume that the *dimension vector* of any representation $M = ((M_v)_{v \in Q_0}, (\varphi_a)_{a \in Q_1})$ of $(Q, \mathcal{J}_Q)$ is $\underline{1} := (1, \ldots, 1)$, that is, $\underline{1} = (\dim_\mathbb{C} M_v)_{v \in Q_0}$. For representations M, M' of $(Q, \mathcal{J}_Q)$, a morphism from M to M' is a family of $\mathbb{C}$-linear maps $\{f_v : M_v \to M'_v\}_{v \in Q_0}$ such that $\varphi_a f_{\mathsf{hd}(a)} = f_{\mathsf{tl}(a)} \varphi'_a$ for any arrow $a \in Q_1$. We say that representations M and M' are *isomorphic*, if f_v is an isomorphism of vector spaces for all $v \in Q_0$. A representation N of $(Q, \mathcal{J}_Q)$ is called a *subrepresentation* of M if there is an injective morphism $N \to M$.

Next, we introduce moduli spaces parametrizing quiver representations satisfying a certain stability condition. We consider the weight space

$$\Theta(Q) := \left\{ \theta = (\theta_v)_{v \in Q_0} \in \mathbb{Z}^{Q_0} \mid \sum_{v \in Q_0} \theta_v = 0 \right\}$$

and let $\Theta(Q)_{\mathbb{R}} := \Theta(Q) \otimes_{\mathbb{Z}} \mathbb{R}$. We call an element $\theta \in \Theta(Q)_{\mathbb{R}}$ a *stability parameter*.

Let M be a representation of $(Q, \mathcal{J}_Q)$ of dimension vector $\underline{1}$. For a subrepresentation N of M, we define $\theta(N) := \sum_{v \in Q_0} \theta_v(\dim_{\mathbb{C}} N_v)$, and hence $\theta(M) = 0$ in particular. For a stability parameter $\theta \in \Theta(Q)_{\mathbb{R}}$, we introduce θ-stable representations as follows.

Definition 3 (see [10])**.** Let $\theta \in \Theta(Q)_{\mathbb{R}}$. We say that a representation M is *θ-semistable* (resp. *θ-stable*) if $\theta(N) \geq 0$ (resp. $\theta(N) > 0$) for any non-zero proper subrepresentation N of M. We say that θ is *generic* if every θ-semistable representation is θ-stable.

By [10, Proposition 5.3], for a generic parameter $\theta \in \Theta(Q)_{\mathbb{R}}$, one can construct the fine moduli space $\mathcal{M}_\theta(Q, \mathcal{J}_Q, \underline{1})$ parametrizing isomorphism classes of θ-stable representations of $(Q, \mathcal{J}_Q)$ with dimension vector $\underline{1}$ as the GIT (geometric invariant theory) quotient. In the following, we let $\mathcal{M}_\theta = \mathcal{M}_\theta(Q, \mathcal{J}_Q, \underline{1})$ for simplicity. This moduli space gives a projective crepant resolution of a three-dimensional Gorenstein toric singlarity as follows.

Theorem 4 (see [7, Theorem 6.3 and 6.4], [9, Corollary 1.2])**.** *Let Γ be a consistent dimer model, and Q be the associated quiver. Let R_Γ be the three-dimensional Gorenstein toric ring associated to Γ. Then, for a generic parameter $\theta \in \Theta(Q)_{\mathbb{R}}$, the moduli space $\mathcal{M}_\theta$ is a projective crepant resolution of* $\operatorname{Spec} R_\Gamma$. *Moreover, any projective crepant resolution of* $\operatorname{Spec} R_\Gamma$ *can be obtained as the moduli space $\mathcal{M}_\theta$ for some generic parameter $\theta \in \Theta(Q)_{\mathbb{R}}$.*

It is known that the space $\Theta(Q)_{\mathbb{R}}$ of stability parameters has a *wall-and-chamber structure*. Namely, we define an equivalence relation on the set of generic parameters so that $\theta \sim \theta'$ if and only if any θ-stable representation of $(Q, \mathcal{J}_Q)$ is also θ'-stable and vice versa, and this relation gives rise to the decomposition of stability parameters into finitely many chambers which are separated by walls. Here, a *chamber* is an open cone in $\Theta(Q)_{\mathbb{R}}$ consisting of equivalent generic parameters and a *wall* is a codimension one face of the closure of a chamber. Note that any generic parameter lies on some chamber (see [9, Lemma 6.1]), and $\mathcal{M}_\theta$ is unchanged unless a parameter θ moves in the same chamber of $\Theta(Q)_{\mathbb{R}}$.

3. Wall-and-chamber structures for toric cDV singularities

In the following, we detect the wall-and-chamber structure of $\Theta(Q)_{\mathbb{R}}$ for the quiver Q associated to a dimer model giving rise to projective crepant resolutions of a toric compound Du Val singularity. Compound Du Val (cDV) singularities, which are fundamental pieces in the minimal model program, are singularities giving rise to Du Val (or Kleinian, ADE) singularities as hyperplane sections. It is known that toric cDV singularities can be classified into the following two types (e.g., see [3, footnote (18)]):

$$(cA_{a+b-1}) : \mathbb{C}[x, y, z, w]/(xy - z^a w^b),$$
$$(cD_4) : \mathbb{C}[x, y, z, w]/(xyz - w^2),$$

where a, b are integers with $a \geq 1$ and $a \geq b \geq 0$. Note that the former one is a cDV singularity of type cA_{a+b-1} and the latter one is of type cD_4. Since these are three dimensional Gorenstein toric rings, they can also be described as the form explained in Subsection 2.2. In particular, we can take the toric diagram of the toric cDV singularities of type cA_{a+b-1} as the trapezoid, which will be denoted by $\Delta(a, b)$, whose vertices are $(0, 0)$, $(a, 0)$, $(b, 1)$, and $(0, 1)$. For example, Figure 3 shows $\Delta(3, 2)$. By Theorem 2, there exists a consistent dimer model whose zigzag polygon is $\Delta(a,b)$, see [11, Subsection 1.2], [12, Section 5] for the precise construction. In general, such a dimer model is not unique, thus we choose one of them and denote the chosen one by $\Gamma_{a,b}$. By construction, the dimer model $\Gamma_{a,b}$ has $n := a+b$ faces. We label one of the faces with 0, and label the face right next to k with $k+1 \pmod n$ for $k = 0, 1, \ldots, n-1$. Also, we will use these labels as the names of vertices of the associated quiver Q.

We here focus on the toric cA_{n-1} singularity $R_{a,b} := \mathbb{C}[x,y,z,w]/(xy - z^a w^b)$ where $n := a+b$, and the associated dimer model $\Gamma_{a,b}$. Let Q be the quiver obtained as the dual graph of $\Gamma_{a,b}$. By Theorem 4, the quiver Q gives rise to projective crepant resolutions of $\operatorname{Spec} R_{a,b}$ as moduli spaces. By the definition of the zigzag polygon, we have the set $\{u_1, \ldots, u_n\}$ of zigzag paths on $\Gamma_{a,b}$ such that $[u_k]$ is either $(0,-1)$ or $(0,1)$ for $k = 1, \ldots, n$, and $a = \#\{k \mid [u_k] = (0,-1)\}$, $b = \#\{k \mid [u_k] = (0,1)\}$. We rearrange $u_1, \ldots, u_n$ if necessary, and construct the sequence $(u_1, \ldots, u_n)$ of the zigzag paths so that u_k consists of the edges shared by the faces $k-1$ and $k \pmod n$ for any $k = 1, \ldots, n$. Also, we define a total order $<$ on $\{u_1, \ldots, u_n\}$ as $u_n < u_{n-1} < \cdots < u_2 < u_1$.

By [12, Lemma 5.2], we see that any pair of zigzag paths (u_i, u_j) on $\Gamma_{a,b}$ divide the two-torus $\mathbb{T}$ into two parts (see Figure 5). We denote the region containing the face 0 by $\mathcal{R}^-(u_i,u_j)$, and the other region by $\mathcal{R}^+(u_i,u_j)$. By abuse of notation, we also use the notation $\mathcal{R}^\pm(u_i,u_j)$ for the set of vertices of Q contained in $\mathcal{R}^\pm(u_i,u_j)$. Since we essentially use one of $\mathcal{R}^\pm(u_i,u_j)$, we let $\mathcal{R}(u_i,u_j) := \mathcal{R}^+(u_i,u_j)$.

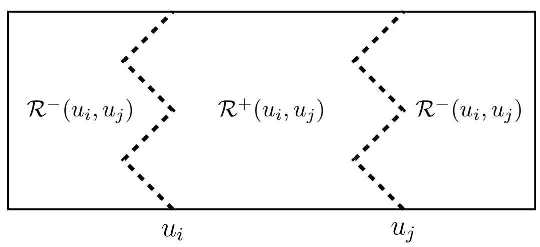

Figure 5

For the quiver Q associated to $\Gamma_{a,b}$, any $\theta \in \Theta(Q)_\mathbb{R}$ satisfies $\theta_0 = -\sum_{v\neq 0} \theta_v$. When we consider $\Theta(Q)_\mathbb{R}$, we employ the coordinates θ_v with $v \neq 0$. Then, the wall-and-chamber structure of $\Theta(Q)_\mathbb{R}$ can be determined by zigzag paths of the dimer model $\Gamma_{a,b}$ as follows.

Theorem 5 (see [12, Theorems 6.11, 6.12, and Corollary 6.13])**.** *Let the notation be the same as above. Then, there exists a one-to-one correspondence between the following sets:*

(a) *the set of chambers in* $\Theta(Q)_\mathbb{R}$*,*

(b) *the set* $\left\{\mathcal{Z}_\omega = (u_{\omega(1)}, \ldots, u_{\omega(n)}) \mid \omega \in \mathfrak{S}_n\right\}$ *of sequences of zigzag paths,*

such that under this correspondence, if a chamber $C \subset \Theta(Q)_\mathbb{R}$ *corresponds to a sequence* $\mathcal{Z}_\omega$*, then for any* $k = 1, \ldots, n-1$*, we have the following:*

(1) *We see that* $W_k := \{\theta \in \Theta(Q)_\mathbb{R} \mid \sum_{v\in\mathcal{R}_k} \theta_v = 0\}$ *is a wall of* C*, where* $\mathcal{R}_k := \mathcal{R}(u_{\omega(k)}, u_{\omega(k+1)})$ *is the region determined by the zigzag paths* $u_{\omega(k)}, u_{\omega(k+1)}$ (*see* Figure 5)*.*

(2) *Any parameter* $\theta \in C$ *satisfies* $\sum_{v\in\mathcal{R}_k}\theta_v > 0$ *(resp.* $\sum_{v\in\mathcal{R}_k}\theta_v < 0$*) if* $u_{\omega(k)} < u_{\omega(k+1)}$ *(resp.* $u_{\omega(k+1)} < u_{\omega(k)}$*).*

(3) *Suppose that* C' *is the chamber separated from* C *by the wall* W_k*. Let* $\theta \in C$ *and* $\theta' \in C'$*. If* $[u_{\omega(k)}] = -[u_{\omega(k+1)}]$*, then* $\mathcal{M}_\theta$ *and* $\mathcal{M}_{\theta'}$ *are related by a flop. If* $[u_{\omega(k)}] = [u_{\omega(k+1)}]$*, then we have* $\mathcal{M}_\theta \cong \mathcal{M}_{\theta'}$*.*

(4) *The action of the adjacent transposition* $s_k \in \mathfrak{S}_n$ *swapping* k *and* $k+1$ *on* $\mathcal{Z}_\omega$ *induces a crossing of the wall* W_k *in* $\Theta(Q)_\mathbb{R}$*. In particular, the chambers in* $\Theta_\mathbb{R}(Q)$ *can be identified with the Weyl chambers of type* A_{n-1}*.*

For the case cD_4, we have similar results as shown in [12, Theorem 8.1], although some modifications are required. Note that the homological minimal model program [13] also detects the wall-and-chamber structure of $\Theta(Q)_\mathbb{R}$, whereas our method provides a more combinatorial way to observe it.

In addition, it is known that the projective crepant resolution $\mathcal{M}_\theta$ can also be described as the toric variety associated to the toric fan induced from a triangulation of $\Delta(a,b)$ (see e.g., [1, Chapter 11]). For the sequence $\mathcal{Z}_\omega$ corresponding to a chamber $C \subset \Theta(Q)_\mathbb{R}$, there is a certain way to obtain the triangulation of $\Delta(a,b)$ giving rise to the projective crepant resolution $\mathcal{M}_\theta$ with $\theta \in C$, see [12, Subsection 6.1] for more details.

Example 6 (The suspended pinch point (cf. [9, Example 12.5]))**.** We consider the dimer model Γ shown in the left of Figure 6. We can see that the zigzag polygon of Γ is $\Delta(2,1)$. We also consider the zigzag paths u_1, u_2, u_3 shown in the right of Figure 6. In particular, the slopes of these zigzag paths are $[u_1] = [u_2] = (0,-1)$, and $[u_3] = (0,1)$. We fix a total order $u_3 < u_2 < u_1$.

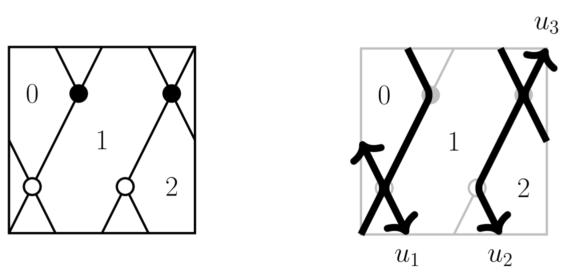

FIGURE 6. The dimer model Γ whose zigzag polygon is $\Delta(2,1)$ (left), the zigzag paths u_1, u_2, u_3 on Γ (right).

Let Q be the quiver associated to Γ. Then the space of stability parameters is

$$\Theta(Q)_\mathbb{R} = \{\theta = (\theta_0, \theta_1, \theta_2) \mid \theta_0 + \theta_1 + \theta_2 = 0\}.$$

By Theorem 5, we have the wall-and-chamber structure of $\Theta(Q)_\mathbb{R}$ as shown in Figure 7. For example, the sequence (u_3, u_2, u_1) corresponds to the chamber C described as

$$C = \{\theta \in \Theta(Q)_\mathbb{R} \mid \theta_1 > 0,\ \theta_2 > 0\}.$$

Indeed, since $\mathcal{R}(u_3, u_2) = \{2\}$ and $u_3 < u_2$, any parameter in C satisfies the inequality $\theta_2 > 0$. Also, since $\mathcal{R}(u_2, u_1) = \{1\}$ and $u_2 < u_1$, any parameter in C also satisfies the inequality $\theta_1 > 0$. A crossing of the wall $\theta_2 = 0$ of C corresponds to a swapping of u_3 and u_2. Also, a crossing of the wall $\theta_1 = 0$ of C corresponds to a swapping of u_2 and u_1.

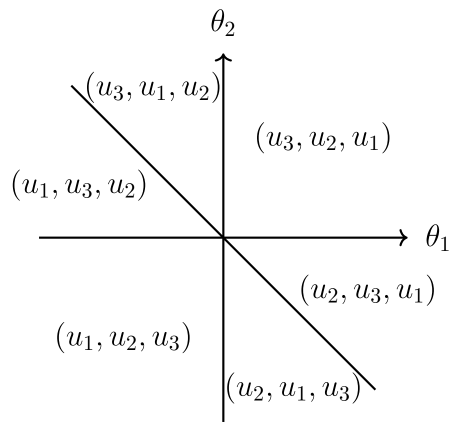

FIGURE 7

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