GOVOROV–LAZARD TYPE THEOREMS, BIG COHEN–MACAULAY MODULES, AND COHEN–MACAULAY HEARTS

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Abstract. Let R be a Cohen–Macaulay local ring with a canonical module and let A be an R-order. We report that a Govorov–Lazard type theorem holds for the category of weak (balanced) big Cohen–Macaulay modules over A. This theorem, which is a generalization of a result due to Holm for the case $R = A$, enables us to show that every complete pure-injective big Cohen–Macaulay A-module is a direct summand of a direct product of finitely generated CM A-modules, provided that R is complete. This fact is well known if R is artinian. We also study big Cohen–Macaulay modules over a non-Cohen–Macaulay local ring R, using the Cohen–Macaulay heart of R.

1. Introduction

Let A be a ring and denote by $\mathsf{Mod}\,A$ (resp. $\mathsf{mod}\,A$) the category of (right) A-modules (resp. finitely presented A-modules). Let $\mathcal{C}$ be an additive subcategory of $\mathsf{Mod}\,A$ closed under direct limits. It is a delicate problem in general whether every module in $\mathcal{C}$ can be presented as a direct limit of modules in $\mathcal{C} \cap \mathsf{mod}\,A$. If this is possible, we say that a *Govorov–Lazard type theorem* holds for $\mathcal{C}$, and write

$$\varinjlim(\mathcal{C} \cap \mathsf{mod}\,A) = \mathcal{C}.$$

For example, it is well known that $\varinjlim \mathsf{mod}\,A = \mathsf{Mod}\,A$. Govorov [3] and Lazard [4] independently proved that $\varinjlim \mathsf{proj}\,A = \mathsf{Flat}\,A$, where $\mathsf{Flat}\,A$ (resp. $\mathsf{proj}\,A$) denotes the category of flat (resp. finitely generated projective) A-modules. Moreover, for an Iwanaga–Gorenstein ring A, Enochs and Jenda [2] showed that a Govorov–Lazard type theorem holds for the category $\mathsf{GFlat}\,A$ of Gorenstein-flat A-modules, where $(\mathsf{GFlat}\,A) \cap \mathsf{mod}\,A$ coincides with the category of finitely generated Gorenstein-projective A-modules. If A is not Iwanaga–Gorenstein, a Govorov–Lazard type theorem may not hold for $\mathsf{GFlat}\,A$; this is due to Holm and Jørgensen [6].

2. Results

Let R be a commutative noetherian local ring. An R-module M is called *(balanced) big CM* (=Cohen–Macaulay) if every system of parameters of R is an M-regular sequence. We call an R-module M a *weak* big CM if every system of parameters of R is a weak M-regular sequence (cf. [5]). We denote by $\mathsf{WCM}\,R$ the category of weak big CM modules. Then $\mathsf{WCM}\,R \cap \mathsf{mod}\,R = \mathsf{CM}\,R$, where the right-hand side denotes the category of (maximal) CM modules. Holm [5] showed that $\varinjlim \mathsf{CM}\,R = \mathsf{WCM}\,R$ holds for any CM local ring R with a canonical module. Our first result extends this to orders over a CM local ring R

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with a canonical module. Recall that a (possibly noncommutative) R-algebra A is said to be an *R-order* if A is CM as an R-module. We denote by $\mathsf{CM}\,A$ (resp. $\mathsf{WCM}\,A$) the category of A-modules being CM (resp. weak big CM) as R-modules.

Theorem 1. *Let R be a CM local ring with a canonical module and let A be an R-order. Then we have*

$$\varinjlim \mathsf{CM}\,A = \mathsf{WCM}\,A.$$

If R is artinian, then A is an Artin R-algebra, and then It is well known that every pure-injective module over a A is a direct summand of a direct product of finitely generated A-modules. Using the above theorem, we can extend this fact to an order A over a complete CM local ring R. Note that a *CM* (resp. *big CM*) A-module means an A-module which is CM (resp. big CM) as an R-module.

Corollary 2. *Let R be a complete CM local ring and let A be an R-order. Then every pure-injective complete big CM module is a direct summand of a direct product of CM A-modules.*

By André' noble work [1], every commutative noetherian local ring R admits a big Cohen–Macaulay module. On the other hand, it is still an open question if every complete noetherian local ring R admits a (finitely generated) CM R-module. This question is known as the *small CM conjecture*. Then there might be little hope that we could have $\varinjlim \mathsf{CM}\,R = \mathsf{WCM}\,R$ in general. So we would like to give another formulation.

Assume that R is a homomorphic image of a CM local ring. We use the *Cohen–Macaulay heart* $\mathcal{H}_{\mathsf{CM}}$ of R introduced in [7]. This is the heart of some compactly generated generated t-structure in the (unbounded) derived category $\mathsf{D}\,(R)$. There are several remarkable facts: $\mathcal{H}_{\mathsf{CM}}$ is a locally coherent Grothendieck category and derived equivalent to $\mathsf{Mod}\,R$. Furthermore, we have

$$\mathcal{H}_{\mathsf{CM}} \cap \mathsf{Mod}\,R = \mathsf{WCM}\,R.$$

Denote by $\mathsf{fp}(\mathcal{H}_{\mathsf{CM}})$ the subcategory of finitely presented objects in $\mathcal{H}_{\mathsf{CM}}$. The locally coherence of $\mathcal{H}_{\mathsf{CM}}$ implies that a Govorov–Lazard type theorem holds for $\mathcal{H}_{\mathsf{CM}}$, that is, each object in $\mathcal{H}_{\mathsf{CM}}$ is a direct limit of objects in $\mathsf{fp}(\mathcal{H}_{\mathsf{CM}})$:

$$\varinjlim \mathsf{fp}(\mathcal{H}_{\mathsf{CM}}) = \mathcal{H}_{\mathsf{CM}}.$$

Hence we have:

Proposition 3. *Let R be a homomorphic image of a CM local ring. Then every weak big CM module is a direct limit of finitely presented objects in $\mathcal{H}_{\mathsf{CM}}$.*

Remark 4. When R admits a dualizing complex D (such that $\inf\{i \mid H^i(D) \neq 0\} = 0$), there is an equivalence

$$\mathrm{RHom}_R(-, D) : (\mathsf{mod}\,R)^{\mathsf{op}} \xrightarrow{\sim} \mathsf{fp}(\mathcal{H}_{\mathsf{CM}}).$$

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