K_0 OF WEAK WALDHAUSEN EXTRIANGULATED CATEGORIES

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Abstract. We modify the axiom of the Waldhausen structure so that it matches better with extriangulated categories. It enables us to define an abelian group $K_0(\mathcal{C})$ of a weak Waldhausen category $\mathcal{C}$ which generalizes that of an extriangulated category. As one might expect, it behaves nicely in the context of Quillen's localization and resolution theorems. We obtain two applications: the first one generalizes exact sequences of the Grothendieck groups associated with the Serre/Verdier localization to some types of "one-sided" exact localizations; the second one reveals close relations between Quillen's theorems and Palu's index.

1. Introduction

The higher algebraic K-theory for an exact category $\mathcal{C}$ was introduced by Quillen, which is now called Quillen's Q-construction [18]. Such a construction makes $\mathcal{C}$ to be the simplicial category $B\mathcal{C}$ by inverting certain morphisms and the K-theory is defined via its geometric realization $|B\mathcal{C}|$. The first foundational result in [18] is the *localization theorem* which extracts a long exact sequence of K-groups from the Serre quotient. The second one is the *resolution theorem* which shows that if we can identify a suitable subcategory $\mathcal{X}$ of an exact category $\mathcal{C}$, then $K(\mathcal{C}) \cong K(\mathcal{X})$. However, not all K-groups can be recovered as those of some abelian/exact categories. It turned out that Quillen's K-theory for exact categories does not possess satisfactory generality that K-theorists had in mind, where triangulated categories come in. To tackle this problem, Waldhausen introduced a generalization of exact categories, now called the Waldhausen category, in which K-theory still exists [21]. As applications of his abstract localization theorem, Thomason-Trobaugh established a K-theory of the derived categories [20] and Schlichting generalized it to any algebraic triangulated category [19].

On one hand, the notion of extriangulated category was introduced by Nakaoka-Palu [13] as a simultaneous generalization of exact categories and triangulated categories. A localization theory of them was also developed in [12] which contains many quotient processes in algebraic contexts as well as the Serre/Verdier quotient. In this article, focusing only on the Grothendieck groups, we generalize a part of the Waldhausen theory on exact categories to the extriangulated case, more specifically, we define the weak Waldhausen extriangulated category $(\mathcal{C}, \mathsf{C}, \mathsf{W})$ together with its Grothendieck group $K_0(\mathcal{C}, \mathsf{C}, \mathsf{W})$.

First, as a benefit of introducing the weak Waldhausen structure, we obtain an exact sequence of Grothendieck groups associated with some localizations such as the Serre/Vedier

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quotient (Theorem 12), which contains an extriangulated counter part of Quillen's localization theorem. The above assertion for the Serre/Verdier quotient goes back to Heller and Grothendieck, respectively. Furthermore, it can apply to abelian localizations of triangulated categories which can be traced back to hearts of t-structures in the sense of [2]. Since then, abelian localizations have been found using cluster tilting subcategories [10]. These constructions were unified in [1] and placed in an extriangulated context in [11]. A generalization from cluster tilting to rigid subcategories was initiated in [3, 4], and has been further developed in the literature.

Our second aim is to reveal a close relation between the resolution theorem and abelian localization. To this end, we establish the extriangulated version (Theorem 14) and it provides a slight generalization and a better understanding for Palu's index which was introduced in connection with the Caldero-Chapoton map [16]. Let triangulated category $\mathcal{C}$ and a 2-cluster tilting subcategory $\mathcal{X} \subseteq \mathcal{C}$ be given. For each object $C \in \mathcal{C}$, Palu's *index* $\mathsf{ind}_{\mathcal{X}}(C)$ *of* C *with respect to* $\mathcal{X}$ is defined as an element of the split Grothendieck group $K_0^{\mathrm{sp}}(\mathcal{X})$. Recently, it is interpreted and generalized via a certain relative extriangulated structure of $\mathcal{C}$ naturally defined by a given subcategory $\mathcal{X}$ [17, 9]. We prove that such results indeed come from the resolution theorem.

Notation and convention. All categories and functors in this article are always assumed to be additive, and subcategories will always be full. For a category $\mathcal{C}$, we denote the class of all morphisms in $\mathcal{C}$ by $\mathsf{Mor}\,\mathcal{C}$, and $\mathsf{mod}\,\mathcal{C}$ is the category of finitely presented contravariant functors from $\mathcal{C}$ to the abelian category Ab of abelian groups.

2. Localization of extriangulated categories

This section is devoted to recall the localization theory of extriangulated category by a suitable thick subcategory, which was introduced in the pursuit of unifying the Serre/Verdier quotient [12]. We also recall a specific case, namely, a localization of triangulated category by an extension-closed subcategory [14].

Nakaoka-Palu's *extriangulated category* is defined to be an additive category $\mathcal{C}$ equipped with

- a biadditive functor $\mathbb{E}\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Ab}$, where Ab is the category of abelian groups, and
- a correspondence $\mathfrak{s}$ that associates an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{g} C]$ of a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$ to each element $\delta \in \mathbb{E}(C, A)$ for any $A, C \in \mathcal{C}$,

where the triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies some axioms. We refer the reader to [13] for an in-depth treatment, see also [15, §2,3]. It turns out that an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is equipped with the class of sequences of the form $A \xrightarrow{f} B \xrightarrow{g} C$ which is called an $\mathfrak{s}$-*conflation*. The pair of an $\mathfrak{s}$-conflation and the corresponding element $\delta \in \mathbb{E}(C, A)$ is called an $\mathfrak{s}$-*triangle* and denoted by $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$. In contrast to triangulated/exact categories, if we state the axiom for extriangulated category, the *realization* $\mathfrak{s}$ is indispensable.

Let us introduce an exact sequence of extriangulated categories as a generalization of the Serre/Verdier quotient. We denote by ET the category of extriangulated categories and exact functors.

Definition 1. A sequence $(\mathcal{N},\mathbb{E}',\mathfrak{s}') \xrightarrow{(F,\phi)} (\mathcal{C},\mathbb{E},\mathfrak{s}) \xrightarrow{(Q,\mu)} (\mathcal{D},\mathbb{F},\mathfrak{t})$ in ET is called an *exact sequence* of extriangulated categories, if the following conditions are fulfilled.

(1) F is fully faithful.
(2) $\operatorname{Im} F = \operatorname{Ker} Q$ holds.
(3) For any map $(G,\psi)\colon (\mathcal{C},\mathbb{E},\mathfrak{s}) \to (\mathcal{D}',\mathbb{F}',\mathfrak{t}')$ in ET with $G\circ F = 0$, there uniquely exists an exact functor $(G',\psi')\colon (\mathcal{D},\mathbb{F},\mathfrak{t}) \to (\mathcal{D}',\mathbb{F}',\mathfrak{t}')$ such that $(G,\psi) = (G',\psi')\circ (Q,\mu)$.

Let us remind a construction of the Verdier quotient: given a triangulated category $\mathcal{C}$ and a thick subcategory $\mathcal{N}\subseteq\mathcal{C}$, we associate the class $\mathcal{S}_{\mathcal{N}}$ of morphisms in $\mathcal{C}$ to $\mathcal{N}$, namely, $\mathcal{S}_{\mathcal{N}} := \{s\in \mathsf{Mor}\,\mathcal{C} \mid {}^{\exists}A \xrightarrow{s} B \to N \to A[1] \text{ with } \mathcal{N}\in\mathcal{N}\}$. Then the Verdier quotient $\mathcal{C}/\mathcal{N}$ is defined to be the Gabriel-Zisman localization $\mathcal{C}[\mathcal{S}_{\mathcal{N}}^{-1}]$ and it gives rise to an exact sequence $\mathcal{N}\to\mathcal{C}\to\mathcal{C}/\mathcal{N}$ in the category of triangulated categories and exact functors.

Similarly to the case of the Verdier quotient, we associate the class $\mathcal{S}_{\mathcal{N}}$ to the pair $(\mathcal{C},\mathcal{N})$ of an extriangulated category $\mathcal{C}$ and a thick subcategory $\mathcal{N}\subseteq\mathcal{C}$. The following is a basic machinery to establish an exact sequence in ET, see [12, Thm. 3.5] for a detailed setup.

Theorem 2. *Let $(\mathcal{C},\mathbb{E},\mathfrak{s})$ be an extriangulated category with a thick subcategory $\mathcal{N}$. Suppose $\mathcal{S}_{\mathcal{N}}$ satisfies conditions* (MR1)–(MR4) *in* [12, Thm. 3.5]. *Then there is an extriangulated category $(\mathcal{C}/\mathcal{N},\widetilde{\mathbb{E}},\widetilde{\mathfrak{s}})$ together with an exact functor $(Q,\mu)\colon(\mathcal{C},\mathbb{E},\mathfrak{s})\to(\mathcal{C}/\mathcal{N},\widetilde{\mathbb{E}},\widetilde{\mathfrak{s}})$. Furthermore, the following natural sequence forms an exact sequence in* ET.

$$(\mathcal{N},\mathbb{E}|_{\mathcal{N}},\mathfrak{s}|_{\mathcal{N}}) \xrightarrow{\ \mathsf{inc}\ } (\mathcal{C},\mathbb{E},\mathfrak{s}) \xrightarrow{(Q,\mu)} (\mathcal{C}/\mathcal{N},\widetilde{\mathbb{E}},\widetilde{\mathfrak{s}}) \tag{2.1}$$

Unfortunately, it is not easy to check the conditions (MR1)–(MR4). Except for the Verdier/Serre quotient, just a few examples of subcategories which yields (2.1) are know, e.g. *biresolving* subcategories [12, §§4.3] and *percolating* subcategories [12, §§4.4].

We now specialize to the case when $(\mathcal{C},\mathbb{E},\mathfrak{s})$ corresponds to a triangulated category and recall the localization theory from [14] that we need.

Setup 3. We fix a triangulated category $\mathcal{C}$ (with suspension [1]) and an extension-closed subcategory $\mathcal{N}\subseteq\mathcal{C}$ that is closed under direct summands. We denote by $(\mathcal{C},\mathbb{E},\mathfrak{s})$ the extriangulated category corresponding to the triangulated category $\mathcal{C}$.

As an application of the relative theory for extriangulated categories [8], we know any extension-closed subcategory $\mathcal{N}$ determines relative structures on $\mathcal{C}$. As pointed out in [5, Prop. A.4], these relative structures are natural from the viewpoint of constructing exact substructures of an exact category.

Proposition 4. [14, Prop. 2.1] *For $A,C\in\mathcal{C}$, define subsets of $\mathbb{E}(C,A)=\mathcal{C}(C,A[1])$ as follows.*

$$\mathbb{E}^L_{\mathcal{N}}(C,A) := \{h\colon C\to A[1] \mid \forall x\colon N\to C \text{ with } N\in\mathcal{N}, \text{ we have } hx\in[\mathcal{N}[1]]\}$$
$$\mathbb{E}^R_{\mathcal{N}}(C,A) := \{h\colon C\to A[1] \mid \forall y\colon A\to N \text{ with } N\in\mathcal{N}, \text{ we have } y\circ h[-1]\in[\mathcal{N}[-1]]\}$$

Then both $\mathbb{E}^L_{\mathcal{N}}$ and $\mathbb{E}^R_{\mathcal{N}}$ give rise to closed subfunctors of $\mathbb{E}$. In particular, putting $\mathbb{E}_{\mathcal{N}} := \mathbb{E}^L_{\mathcal{N}}\cap\mathbb{E}^R_{\mathcal{N}}$, we obtain extriangulated structures

$$\mathcal{C}^L_{\mathcal{N}} := (\mathcal{C},\mathbb{E}^L_{\mathcal{N}},\mathfrak{s}^L_{\mathcal{N}}),\quad \mathcal{C}^R_{\mathcal{N}} := (\mathcal{C},\mathbb{E}^R_{\mathcal{N}},\mathfrak{s}^R_{\mathcal{N}}),\quad \mathcal{C}_{\mathcal{N}} := (\mathcal{C},\mathbb{E}_{\mathcal{N}},\mathfrak{s}_{\mathcal{N}}),$$

all relative to the triangulated structure $(\mathcal{C},\mathbb{E},\mathfrak{s})$.

With respect to the relative structure $\mathcal{C}_{\mathcal{N}}$, the pair $(\mathcal{C},\mathcal{N})$ yields a class $\mathcal{S}_{\mathcal{N}}$ of morphisms in $\mathcal{C}$ satisfying the needed conditions to obtain an exact sequence in ET.

Theorem 5. [14, Thm. A, Lem. 2.4, Cor. 2.11] *We have an exact sequence* $(\mathcal{N},\mathbb{E},\mathfrak{s}) \xrightarrow{\mathsf{inc}} (\mathcal{C},\mathbb{E}_{\mathcal{N}},\mathfrak{s}_{\mathcal{N}}) \xrightarrow{(Q,\mu)} (\mathcal{C}/\mathcal{N},\widetilde{\mathbb{E}}_{\mathcal{N}},\widetilde{\mathfrak{s}}_{\mathcal{N}})$ *in* ET. *Furthermore, if* $\mathsf{Cone}(\mathcal{N},\mathcal{N})=\mathcal{C}$ *holds in the triangulated category* $(\mathcal{C},\mathbb{E},\mathfrak{s})$, *the following are true.*

(1) *The quotient category* $\mathcal{C}/\mathcal{N} := (\mathcal{C}/\mathcal{N},\widetilde{\mathbb{E}}_{\mathcal{N}},\widetilde{\mathfrak{s}}_{\mathcal{N}})$ *is abelian.*
(2) *The quotient functor* (Q,μ) *induces a right exact functor* $Q\colon(\mathcal{C},\mathbb{E}^R_{\mathcal{N}},\mathfrak{s}^R_{\mathcal{N}})\to\mathcal{C}/\mathcal{N}$ *and a left exact functor* $Q\colon(\mathcal{C},\mathbb{E}^L_{\mathcal{N}},\mathfrak{s}^L_{\mathcal{N}})\to\mathcal{C}/\mathcal{N}$. *In addition, it induces a cohomological functor* $Q\colon(\mathcal{C},\mathbb{E},\mathfrak{s})\to\mathcal{C}/\mathcal{N}$.

We call the case $\mathsf{Cone}(\mathcal{N},\mathcal{N})=\mathcal{C}$ *in which we have the resulting abelian category* $\mathcal{C}/\mathcal{N}$ *the* abelian localization *of* $\mathcal{C}$ *by* $\mathcal{N}$.

We can think of hearts of t-structures in the sense of [2] as a prototypical example of the abelian localization. Since then, it has been found and generalized via cluster tilting subcategories [10], rigid subcategories [4, 3] and cotorsion pairs [1]. In turn, Theorem 5 can apply to these phenomenon. To clarify our point of focus, we record the following immediate result.

Example 6. Let $(\mathcal{C},\mathbb{E},\mathfrak{s})$ be a triangulated category and $\mathcal{X}\subseteq\mathcal{C}$ be a contravariantly finite rigid subcategory. We consider an extension closed subcategory $\mathcal{N}:=\mathcal{X}^{\perp_0}=\{C\in\mathcal{C}\mid(\mathcal{X},C)=0\}$. Since $\mathsf{Cone}(\mathcal{N},\mathcal{N})=\mathcal{C}$ is true, Theorem 5 provides a right exact functor $Q\colon\mathcal{C}^R_{\mathcal{N}}\to\mathcal{C}/\mathcal{N}$. Furthermore, we can verify that there exists a natural exact equivalence $\mathcal{C}/\mathcal{N}\cong\mathsf{mod}\,\mathcal{X}$. Thus we have a right exact functor $Q\cong(\mathcal{X},-)$ with the kernel $\mathcal{N}$ as below.

$$(\mathcal{N},\mathbb{E}|_{\mathcal{N}},\mathfrak{s}|_{\mathcal{N}}) \xrightarrow{\mathsf{inc}} (\mathcal{C},\mathbb{E}^R_{\mathcal{N}},\mathfrak{s}^R_{\mathcal{N}}) \xrightarrow{Q} \mathsf{mod}\,\mathcal{X} \tag{2.2}$$

Note that this sequence does not sit in ET any more.

3. Weak Waldhausen categories

We introduce the notion of weak Waldhausen category. This is a simultaneous generalization of the (classical) Waldhausen category and extriangulated category. Also, we define its Grothendieck group.

Definition 7. Let $\mathcal{C}$ be an additive category equipped with a class Seq of *distinguished sequences* of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{3.1}$$

in $\mathcal{C}$, and a class W of morphisms in $\mathcal{C}$. Denote by C (resp. F) the class of morphisms f (resp. g) appearing in a distinguished sequence (3.1). The morphisms in C (resp. F) are called *cofibrations* (resp. *fibrations*) and denoted by $\rightarrowtail$ (resp. $\twoheadrightarrow$). The morphisms in W are called *weak equivalences* and are denoted by $\xrightarrow{\sim}$.

(1) The triplet $(\mathcal{C},\mathsf{Seq},\mathsf{W})$ is called a *weak Waldhausen (additive) category* if the following axioms are satisfied.

(WC0) The class C is closed under composition and contains all isomorphisms.

(WC1) Seq contains all split exact sequences and is closed under isomorphism. Any distinguished sequence (3.1) is a weak cokernel sequence.

(WC2) Any pair (f, c) of a cofibration $A \overset{f}{\rightarrowtail} B$ and a morphism $A \xrightarrow{c} C$ yields a cofibration $A \xrightarrow{\binom{f}{-c}} B \oplus C$. Furthermore, the associated distinguished sequences of the form $A \xrightarrow{\binom{f}{-c}} B \oplus C \xrightarrow{(c'\ f')} D$ satisfy that f' belongs to C.

(WW0) The class W is closed under composition and contains all isomorphisms.

(WW1) (Gluing axiom) Consider a commutative diagram of the form

$$\begin{array}{ccccc} C & \xleftarrow{c} & A & \overset{f}{\rightarrowtail} & B \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ C' & \xleftarrow{c'} & A' & \overset{f'}{\rightarrowtail} & B' \end{array} \tag{3.2}$$

in which all vertical arrows are weak equivalences and the feathered arrows are cofibrations. Then from a distinguished weak cokernel of $\binom{f}{-c}$ to a distinguished weak cokernel of $\binom{f'}{-c'}$, there is an induced morphism that is also a weak equivalence.

(2) The triplet $(\mathcal{C}, \mathsf{Seq}, \mathsf{W})$ is called a *weak coWaldhausen category* if the triplet $(\mathcal{C}^{\mathsf{op}}, \mathsf{Seq}^{\mathsf{op}}, \mathsf{W}^{\mathsf{op}})$ is a weak Waldhausen additive category.

(3) The triplet $(\mathcal{C}, \mathsf{Seq}, \mathsf{W})$ is called a *weak biWaldhausen category* if $(\mathcal{C}, \mathsf{Seq}, \mathsf{W})$ is both weak Waldhausen and weak coWaldhausen.

Example 8. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Define $\mathsf{Seq}_{\mathfrak{s}}$ to be the class of all $\mathfrak{s}$-conflations, and $\mathsf{W}_{\mathfrak{s}}$ to be the class of all isomorphisms in $\mathcal{C}$. Then $(\mathcal{C}, \mathsf{Seq}_{\mathfrak{s}}, \mathsf{W}_{\mathfrak{s}})$ is a weak biWaldhausen category.

We introduce some concepts for weak Waldhausen categories by analogy to the classical theory.

Definition 9. Let $(\mathcal{C}, \mathsf{Seq}, \mathsf{W})$ and $(\mathcal{C}', \mathsf{Seq}', \mathsf{W}')$ be weak Waldhausen categories.

(1) An additive functor $F : \mathcal{C} \to \mathcal{C}'$ is called an *exact functor* if it preserves distinguished sequences and weak equivalences, namely, $F(\mathsf{Seq}) \subseteq \mathsf{Seq}'$ and $F(\mathsf{W}) \subseteq \mathsf{W}'$ hold.

(2) Suppose $(\mathcal{C}, \mathsf{Seq}, \mathsf{V})$ is a weak Waldhausen category with $\mathsf{V} \subseteq \mathsf{W}$. Then the identity functor $\mathsf{id}_{\mathcal{C}} \colon (\mathcal{C}, \mathsf{Seq}, \mathsf{V}) \to (\mathcal{C}, \mathsf{Seq}, \mathsf{W})$ is exact. An object $C \in \mathcal{C}$ is W-*acyclic* if the zero map $0 \rightarrowtail C$ belongs to W. We denote by $\mathcal{N}^{\mathsf{W}}$ the full subcategory of all W-acyclic objects in $(\mathcal{C}, \mathsf{Seq}, \mathsf{V})$. In this case, the subcategory admits a natural weak Waldhausen structure $(\mathcal{N}^{\mathsf{W}}, \mathsf{Seq}', \mathsf{V}')$ which is a restriction of $(\mathcal{C}, \mathsf{Seq}, \mathsf{V})$.

We denote by wWald the category of weak Waldhausen categories and exact functors. Analogously to the case of extriangulated category, we introduce their exact sequence.

Definition 10. The natural sequence

$$(\mathcal{N}^{\mathsf{W}}, \mathsf{Seq}', \mathsf{V}') \xrightarrow{\mathsf{inc}} (\mathcal{C}, \mathsf{Seq}, \mathsf{V}) \xrightarrow{\mathsf{id}} (\mathcal{C}, \mathsf{Seq}, \mathsf{W}) \tag{3.3}$$

in Definition 9(2) is called a *localization sequence*. Moreover it is called an *exact sequence* in wWald if the functor $(\mathcal{C}, \mathsf{Seq}, \mathsf{V}) \xrightarrow{\mathsf{id}_{\mathcal{C}}} (\mathcal{C}, \mathsf{Seq}, \mathsf{W})$ is universal among exact functors $F\colon (\mathcal{C}, \mathsf{Seq}, \mathsf{V}) \to (\mathcal{D}, \mathsf{Seq}', \mathsf{W}')$ with $F|_{\mathcal{N}^{\mathsf{W}}} = 0$, where $(\mathcal{D}, \mathsf{Seq}', \mathsf{W}')$ is a weak Waldhausen category satisfying the *saturation* and *extension axioms* (see [21, p. 327]).

The Grothendieck group for weak Waldhausen categories is defined as follows.

Definition 11. Assume that $(\mathcal{C}, \mathsf{C}, \mathsf{W})$ is a weak Waldhausen category. The *Grothendieck group* $K_0(\mathcal{C}) := K_0(\mathcal{C}, \mathsf{C}, \mathsf{W})$ is defined to be the abelian group freely generated by the set of isomorphism classes $[C]$ of each object $C \in \mathcal{C}$, modulo to the relations:

- $[C] = [C']$ for each weak equivalence $C \xrightarrow{\sim} C'$; and
- $[B] = [A] + [C]$ for each distinguished sequence $A \rightarrowtail B \twoheadrightarrow C$.

To state our abstract localization theorem we define subclasses of $\mathsf{Mor}\,\mathcal{C}$:

- $\mathcal{L}^{\mathsf{ac}} := \mathsf{C} \cap \mathsf{W}$; $\mathcal{R}^{\mathsf{ac}} := \mathsf{F} \cap \mathsf{W}$; and
- $\mathcal{R}^{\mathsf{ac}}_{\mathsf{ret}} := \{g \in \mathsf{Mor}\,\mathcal{C} | g \text{ is a retraction and } \operatorname{Ker} g \in \mathcal{N}\}$.

The first result can be regarded as a version of Shclichting's theorem [19, Thm. 11].

Theorem 12 (Localization Theorem)**.** *Consider a localization sequence of weak Waldhausen categories as* (3.3)*. If we assume that*

(1) W *consists of finite compositions of morphisms from* $\mathcal{L}^{\mathsf{ac}} \cup \mathcal{R}^{\mathsf{ac}}_{\mathsf{ret}} \cup \mathsf{V}$*; or*
(2) W *consists of finite compositions of morphisms from* $\mathcal{L}^{\mathsf{ac}} \cup \mathcal{R}^{\mathsf{ac}} \cup \mathsf{V}$ *and* $\mathcal{C}$ *is a weak biWaldhausen,*

then it becomes an exact sequence in wWald *which induces a right exact sequence in* Ab *as follows.*

$$K_0(\mathcal{N}^{\mathsf{W}}, \mathsf{Seq}', \mathsf{V}') \xrightarrow{K_0(\mathsf{inc})} K_0(\mathcal{C}, \mathsf{Seq}, \mathsf{V}) \xrightarrow{K_0(\mathsf{id})} K_0(\mathcal{C}, \mathsf{Seq}, \mathsf{W}) \longrightarrow 0 \tag{3.4}$$

The second one is an extriangulated version of Quillen's resolution theorem at the level of K_0, see [15, Thm. 4.5] for more details.

Definition 13. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, let $\mathcal{X} \subseteq \mathcal{C}$ be a subcategory and fix an object $C \in \mathcal{C}$. A *finite* $\mathcal{X}$*-resolution (in* $\mathcal{C}$*)* of C is defined to be a complex

$$X_n \xrightarrow{f_{n-1}} \cdots \xrightarrow{g_2 f_1} X_1 \xrightarrow{g_1 f_0} X_0 \xrightarrow{g_0} C, \tag{3.5}$$

where $X_i \in \mathcal{X}$ for each $0 \leq i \leq n$, and $C_{i+1} \xrightarrow{f_i} X_i \xrightarrow{g_i} C_i$ is an $\mathfrak{s}$-conflation for each $0 \leq i \leq n-1$ with $(C_0, C_n) := (C, X_n)$. In this case, we say that the $\mathcal{X}$-resolution is of *length* n.

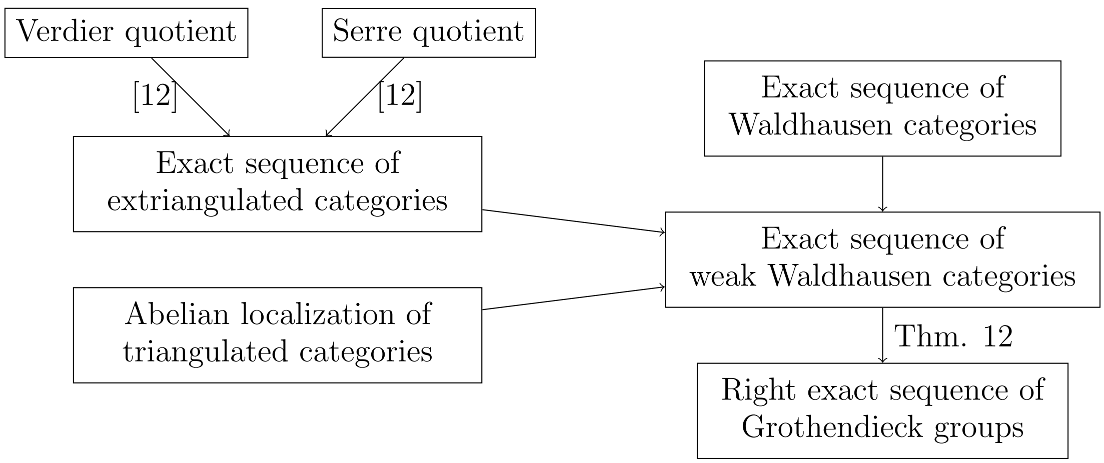
Theorem 14 (Resolution Theorem)**.** *Let* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *be an extriangulated category. Suppose* $\mathcal{X}$ *is an extension-closed subcategory of* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$*, such that* $\mathcal{X}$ *is closed under taking cocones of* $\mathfrak{s}$*-deflations in* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$*. If any object* $C \in \mathcal{C}$ *admits a finite* $\mathcal{X}$*-resolution, then we have an isomorphism*

$$\begin{aligned} K_0(\mathcal{C}, \mathbb{E}, \mathfrak{s}) &\xrightarrow{\cong} K_0(\mathcal{X}, \mathbb{E}|_{\mathcal{X}}, \mathfrak{s}|_{\mathcal{X}}) \\ [C] &\longmapsto \sum_{i=0}^{n} (-1)^i [X_i] \end{aligned}$$

where we consider an $\mathcal{X}$-resolution (3.5) *of $C \in \mathcal{C}$.*

4. Applications

Lastly we demonstrate some usages of our localization and resolution theorem. As expected, an exact sequence in ET induces an exact sequence in wWald. In such a case, we may apply the localization theorem to get a right exact sequence of the Grothendieck groups in Ab, recovering Enomoto-Saito's extriangulated localization theorem [6, Cor. 4.32]. A benefit of weak Waldhausen structures sits in the fact that such a construction still holds for the abelian localization in the sense of Theorem 5. Exact sequences appearing in this article are related to each other as summarized below.



Thus, although the "right exact" sequence (2.2) does not exsist in ET, it induces a natural exact sequences $(\mathcal{N}^{\mathsf{W}}, \mathsf{Seq}', \mathsf{V}') \xrightarrow{\mathsf{inc}} (\mathcal{C}, \mathsf{Seq}, \mathsf{V}) \xrightarrow{\mathsf{id}} (\mathcal{C}, \mathsf{Seq}, \mathsf{W})$ in wWald to which Theorem 12 can apply. Thus, like the case of Enomoto-Saito's theorem, it also induces a right exact sequence in Ab as below.

$$K_0(\mathcal{N}, \mathbb{E}|_{\mathcal{N}}, \mathfrak{s}|_{\mathcal{N}}) \xrightarrow{K_0(\mathsf{inc})} K_0(\mathcal{C}, \mathbb{E}^R_{\mathcal{N}}, \mathfrak{s}^R_{\mathcal{N}}) \xrightarrow{K_0(\mathsf{id})} K_0(\mathcal{C}, \mathsf{Seq}, \mathsf{W}) \longrightarrow 0$$

Furthermore, thanks to the assumption $\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$ in Theorem 5, (the dual of) the resolution theorem applies to the inclusion $\mathcal{N} \subseteq (\mathcal{C}, \mathbb{E}^R_{\mathcal{N}}, \mathfrak{s}^R_{\mathcal{N}})$. It shows the leftmost arrow is an isomorphism $K_0(\mathcal{N}, \mathbb{E}|_{\mathcal{N}}, \mathfrak{s}|_{\mathcal{N}}) \xrightarrow{\cong} K_0(\mathcal{C}, \mathbb{E}^R_{\mathcal{N}}, \mathfrak{s}^R_{\mathcal{N}})$. This isomorphism has been already appeared in the literature, which we now describe.

Example 15. (cf. Example 6) Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a triangulated category and $\mathcal{X} \subseteq \mathcal{C}$ a 2-cluster tilting subcategory. Put $\mathcal{N} := \mathcal{X}[1] = \mathcal{X}^{\perp_0}$. Then the aforementioned isomorphism can be described as follows,

$$\begin{array}{rcl} K_0(\mathcal{C}, \mathbb{E}^R_{\mathcal{N}}, \mathfrak{s}^R_{\mathcal{N}}) & \xrightarrow{\cong} & K_0^{\mathsf{sp}}(\mathcal{X}) \\ {[C]} & \longmapsto & [X_0] - [X_1] \end{array}$$

where we consider a triangle $X_1 \to X_0 \to C \to X_1[1]$ comming from the defining cotorsion pair $(\mathcal{X}, \mathcal{X})$. This isomorphism is known as the *index isomorphism* [17]. In the case of $\mathcal{X} = \mathcal{X}[1]$, by a closer look at this isomorphism, Fedele interpreted the Grothendieck group $K_0(\mathcal{C})$ of the triangulated category as that of the 4-angulated category $\mathcal{X}$ [7, Thm. C].

Due to the very generality of our abstract theorems, we expand their results to wider setup containing the n-cluster tilting subcategory case.

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