THE FIRST EULER CHARACTERISTIC AND THE DEPTH OF ASSOCIATED GRADED RINGS

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ABSTRACT. The homological property of the associated graded ring of an ideal is an important problem in commutative algebra. In this talk, we explore the structure of the associated graded ring of $\mathfrak{m}$-primary ideals in the case where the first Euler characteristic attains almost minimal value in a Cohen-Macaulay local ring.

Key Words: commutative ring, Cohen-Macaulay local ring, associated graded ring, first Euler characteristic, Hilbert function, Hilbert coefficient, stretched ideal.

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1. INTRODUCTION

Throughout this report, let A be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. For simplicity, we may assume the residue class field $A/\mathfrak{m}$ is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and let

$$R = \mathrm{R}(I) := A[It] \quad \subseteq A[t] \text{ and } \quad R' = \mathrm{R}'(I) := A[It, t^{-1}] \quad \subseteq A[t, t^{-1}]$$

denote, respectively, the Rees algebra and the extended Rees algebra of I. Let

$$G = \mathrm{G}(I) := R'/t^{-1}R' \cong \bigoplus_{n \geq 0} I^n/I^{n+1}$$

denotes the associated graded ring of I. Let $M = \mathfrak{m}G + G_+$ be the graded maximal ideal in G. Let $\ell_A(N)$ denote, for an A-module N, the length of N.

Let $Q = (a_1, a_2, \cdots, a_d) \subseteq I$ be a parameter ideal in A which forms a reduction of I. Then, we set

$$\chi_1(a_1t, a_2t, \ldots, a_dt; G) := \ell(G/(a_1t, a_2t, \ldots, a_dt)G) - \mathrm{e}(a_1t, a_2t, \ldots, a_dt; G_M)$$

and call it the *first Euler characteristic* of G relative to $a_1t, a_2t, \ldots, a_dt$ (c.f. [1, 2, 11]), where $\mathrm{e}(a_1t, a_2t, \ldots, a_dt; G_M)$ denotes the multiplicity of G_M with respect to $a_1t, a_2t, \ldots, a_dt$.

It is well-known that $\chi_1(a_1t, a_2t, \ldots, a_dt; G) \geq 0$ holds true, and the equality

$$\chi_1(a_1t, a_2t, \ldots, a_dt : G) = 0$$

holds true if and only if the associated graded ring G is Cohen-Macaulay. The aim of this talk is to explore the structure of the associated graded ring G with $\chi_1(a_1t, a_2t, \ldots, a_dt; G) = 1$ and, in particular, we prove that $\operatorname{depth} G = d - 1$.

The detailed version of this paper will be submitted for publication elsewhere.

In this report we will also study the Hilbert series and coefficients of $\mathfrak{m}$-primary ideals. We set the power series

$$HS_I(z) = \sum_{n=0}^{\infty} \ell_A(I^n/I^{n+1})z^n$$

and call it the Hilbert series of I. It is also well known that this series is rational and that there exists a polynomial $h_I(z)$ with integer coefficients such that $h_I(1) \neq 0$ and

$$HS_I(z) = \frac{h_I(z)}{(1-z)^d}.$$

As is well known, for a given $\mathfrak{m}$-primary ideal I, there exist integers $\{\mathrm{e}_k(I)\}_{0\leq k\leq d}$ such that the equality

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d\mathrm{e}_d(I)$$

holds true for all integers $n \gg 0$. For each $0 \leq k \leq d$, $\mathrm{e}_k(I)$ is called the k-th *Hilbert coefficient* of I.

The main result of this report is the following.

Theorem 1. *The following conditions are equivalent to each other.*

(1) $\chi_1(a_1t, a_2t, \cdots, a_dt; G) = 1$,
(2) $\mathrm{e}_0(I) = \ell_A(A/I) + \sum_{n\geq 1} \ell_A(I^n/QI^{n-1} + I^{n+1}) - 1$,
(3) *the Hilbert series $HS_I(z)$ of I is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \sum_{n=1}^{r_I} \ell_A(I^n/QI^{n-1} + I^{n+1})z^n - z^s}{(1-z)^d}$$

for some $s > 0$.

When this is the case we have the following.

(*i*) $s = \min\{n \geq 1 \mid QI^{n-1} \cap I^{n+1} \neq QI^n\}$,
(*ii*) $\mathrm{e}_k(I) = \sum_{n=k}^{r_I} \binom{n}{k} \ell_A(I^n/QI^{n-1} + I^{n+1}) - \binom{s}{k}$ *for* $1 \leq k \leq d$,
(*iii*) $\mathrm{a}_{d-1}(G) := \sup\{n \in \mathbb{Z} \mid [\mathrm{H}_M^{d-1}(G)]_n \neq (0)\} = s - d$, *and* $\ell_A([\mathrm{H}_M^{d-1}(G)]_{s-d}) = 1$,
(*iv*) $\operatorname{depth} G = d - 1$.

We can get the following result as a corollary of Theorem 1.

Corollary 2. *Suppose that $\chi_1(a_1t, a_2t, \ldots, a_dt; G) \leq 1$, then* $\operatorname{depth} G \geq d - 1$.

2. The structure of Sally modules

In this report we need the notion of Sally modules which was introduced by W. V. Vasconcelos [12]. The purpose of this section is to summarize some results and techniques on the Sally modules which we need throughout this report. Remark that in this section $\mathfrak{m}$-primary ideals I are not necessarily stretched.

Let $T = \mathrm{R}(Q) = A[Qt] \subseteq A[t]$ denotes the Rees algebra of Q. Following Vasconcelos [12], we consider

$$S = \mathrm{S}_Q(I) = IR/IT \cong \bigoplus_{n \geq 1} I^{n+1}/Q^n I$$

the Sally module of I with respect to Q.

We give one remark about Sally modules. See [5, 12] for further information.

Remark 3 ([5, 12]). We notice that S is a finitely generated graded T-module and $\mathfrak{m}^n S = (0)$ for all $n \gg 0$. We have $\mathrm{Ass}_T S \subseteq \{\mathfrak{m}T\}$ so that $\dim_T S = d$ if $S \neq (0)$.

From now on, let us introduce some techniques, being inspired by [3, 4], which plays a crucial role throughout this report. See [7, Section 3] (also [6, Section 2] for the case where $I = \mathfrak{m}$) for the detailed proofs.

We denote by $E(m)$, for a graded module E and each $m \in \mathbb{Z}$, the graded module whose grading is given by $[E(m)]_n = E_{m+n}$ for all $n \in \mathbb{Z}$.

We have an exact sequence

$$0 \to K^{(-1)} \to F \overset{\varphi_{-1}}{\to} G \to R/IR + T \to 0 \qquad (\dagger_{-1})$$

of graded T-modules induced by tensoring the canonical exact sequence

$$0 \to T \overset{i}{\hookrightarrow} R \to R/T \to 0$$

of graded T-modules with A/I where $\varphi_{-1} = A/I \otimes i$, $K^{(-1)} = \mathrm{Ker}\, \varphi_{-1}$, and $F = T/IT \cong (A/I)[X_1, X_2, \cdots, X_d]$ is a polynomial ring with d indeterminates over the residue class ring A/I.

Lemma 4. ([7]) *There exists an exact sequence*

$$0 \to K^{(0)}(-1) \to ([R/IR + T]_1 \otimes F)(-1) \overset{\varphi_0}{\to} R/IR + T \to S/IS(-1) \to 0 \quad (\dagger_0)$$

of graded T-modules where $K^{(0)} = \mathrm{Ker}\, \varphi_0$.

Notice that $\mathrm{Ass}_T K^{(m)} \subseteq \{\mathfrak{m}T\}$ for all $m = -1, 0$, because $F \cong (A/I)[X_1, X_2, \cdots, X_d]$ is a polynomial ring over the residue ring A/I and $[R/IR+T]_1 \otimes F$ is a maximal Cohen-Macaulay module over F.

We then have the following proposition by the exact sequences $(\dagger_{-1})$ and $(\dagger_0)$.

Proposition 5. ([7, Lemma 3.3]) *We have*

$$\begin{aligned} \ell_A(I^n/I^{n+1}) &= \ell_A(A/[I^2+Q])\binom{n+d-1}{d-1} - \ell_A(I/[I^2+Q])\binom{n+d-2}{d-2} \\ &+ \ell_A([S/IS]_{n-1}) - \ell_A(K_n^{(-1)}) - \ell_A(K_{n-1}^{(0)}) \end{aligned}$$

for all $n \geq 0$.

We also need the notion of *filtration of the Sally module* which was introduced by M. Vaz Pinto [13] as follows.

Definition 6. ([13]) We set, for each $m \geq 1$,

$$S^{(m)} = I^m t^{m-1} R / I^m t^{m-1} T (\cong I^m R / I^m T(-m+1)).$$

We notice that $S^{(1)} = S$, and $S^{(m)}$ are finitely generated graded T-modules for all $m \geq 1$, since R is a module-finite extension of the graded ring T.

The following lemma follows by the definition of the graded module $S^{(m)}$.

Lemma 7. *Let $m \geq 1$ be an integer. Then the following assertions hold true.*

(1) $\mathfrak{m}^n S^{(m)} = (0)$ *for integers $n \gg 0$; hence* $\dim_T S^{(m)} \leq d$.

(2) *The homogeneous components $\{S_n^{(m)}\}_{n \in \mathbb{Z}}$ of the graded T-module $S^{(m)}$ are given by*

$$S_n^{(m)} \cong \begin{cases} (0) & \text{if } n \leq m-1, \\ I^{n+1}/Q^{n-m+1}I^m & \text{if } n \geq m. \end{cases}$$

Let $L^{(m)} = TS_m^{(m)}$ be a graded T-submodule of $S^{(m)}$ generated by $S_m^{(m)}$ and

$$\begin{aligned} D^{(m)} &= (I^{m+1}/QI^m) \otimes (A/\operatorname{Ann}_A(I^{m+1}/QI^m))[X_1, X_2, \cdots, X_d] \\ &\cong (I^{m+1}/QI^m)[X_1, X_2, \cdots, X_d] \end{aligned}$$

for $m \geq 1$ (c.f. [13, Section 2]).

We then have the following lemma.

Lemma 8. ([13, Section 2]) *The following assertions hold true for $m \geq 1$.*

(1) $S^{(m)}/L^{(m)} \cong S^{(m+1)}$ *so that the sequence*

$$0 \to L^{(m)} \to S^{(m)} \to S^{(m+1)} \to 0$$

is exact as graded T-modules.

(2) *There is a surjective homomorphism $\theta_m : D^{(m)}(-m) \to L^{(m)}$ graded T-modules.*

For each $m \geq 1$, tensoring the exact sequence

$$0 \to L^{(m)} \to S^{(m)} \to S^{(m+1)} \to 0$$

and the surjective homomorphism $\theta_m : D^{(m)}(-m) \to L^{(m)}$ of graded T-modules with A/I, we get the exact sequence

$$0 \to K^{(m)}(-m) \to D^{(m)}/ID^{(m)}(-m) \overset{\varphi_m}{\to} S^{(m)}/IS^{(m)} \to S^{(m+1)}/IS^{(m+1)} \to 0 \qquad (\dagger_m)$$

of graded F-modules where $K^{(m)} = \operatorname{Ker} \varphi_m$.

Notice here that, for all $m \geq 1$, we have $\operatorname{Ass}_T K^{(m)} \subseteq \{\mathfrak{m}T\}$ because $D^{(m)}/ID^{(m)} \cong (I^{m+1}/QI^m + I^{m+2})[X_1, X_2, \cdots, X_d]$ is a maximal Cohen-Macaulay module over F.

We then have the following result by Proposition 5 and exact sequences $(\dagger_m)$ for $m \geq 1$.

Proposition 9. *The following assertions hold true:*

(1) *We have*

$$\begin{aligned} \ell_A(I^n/I^{n+1}) &= \{\ell_A(A/I^2 + Q) + \sum_{m=1}^{r_I - 1} \ell_A(I^{m+1}/QI^m + I^{m+2})\}\binom{n+d-1}{d-1} \\ &+ \sum_{k=1}^{r_I}(-1)^k \left\{\sum_{m=k-1}^{r_I-1}\binom{m+1}{k}\ell_A(I^{m+1}/QI^m + I^{m+2})\right\}\binom{n+d-k-1}{d-k-1} \\ &- \sum_{m=-1}^{r_I-1} \ell_A(K_{n-m-1}^{(m)}) \end{aligned}$$

for all $n \geq \max\{0, r_I - d + 1\}$.

(2) $\mathrm{e}_0(I) = \ell_A(A/I^2+Q) + \sum_{m=1}^{r_I-1} \ell_A(I^{m+1}/QI^m + I^{m+2}) - \sum_{m=-1}^{r_I-1} \ell_{T_\mathcal{P}}(K_\mathcal{P}^{(m)})$ *where* $\mathcal{P} = \mathfrak{m}T$.

3. Proof of Main Theorem

In this section, let us introduce a proof of Theorem 1.

Let us begin with the following remark, where $\mathrm{e}(a_1t, a_2t, \cdots, a_dt; G)$ denotes the multiplicity of G with respect to $a_1t, a_2t, \cdots, a_dt$, and

$$\chi_1(a_1t, a_2t, \cdots, a_dt; G) = \ell_A(G/(a_1t, a_2t, \cdots, a_dt)G) - \mathrm{e}(a_1t, a_2t, \cdots, a_dt; G) \geq 0$$

is called the first Euler characteristic of G with respect to $a_1t, a_2t, \cdots, a_dt$.

Remark 10. We have, by Proposition 9,

$$\chi_1(a_1t, a_2t, \cdots, a_dt; G) = \sum_{m \geq -1} \ell_{T_\mathcal{P}}(K_\mathcal{P}^{(m)})$$

because $\mathrm{e}(a_1t, a_2t, \cdots, a_dt; G) = \mathrm{e}_0(I)$ and $[G/(a_1t, a_2t, \cdots, a_dt)G]_n \cong I^n/QI^{n-1} + I^{n+1}$ for all $n \geq 1$.

The following corollary seems well known by the basic properties of Cohen-Macaulay rings.

Corollary 11. *The following conditions are equivalent to each other;*

(1) $\chi_1(a_1t, a_2t, \cdots, a_dt; G) = 0$,
(2) $\mathrm{e}_0(I) = \ell_A(A/I) + \sum_{n \geq 1} \ell_A(I^n/QI^{n-1} + I^{n+1})$,
(3) *the Hilbert series* $HS_I(z)$ *of* I *is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \sum_{n=1}^{r_I} \ell_A(I^n/QI^{n-1} + I^{n+1})z^n}{(1-z)^d},$$

(4) G *is Cohen-Macaulay.*

Let $B = T/\mathfrak{m}T \cong (A/\mathfrak{m})[X_1, X_2, \cdots, X_d]$ which is a polynomial ring with d indeterminates over the field $A/\mathfrak{m}$.

The following proposition plays an important role for our proof of Theorem 1.

Proposition 12. *The following conditions are equivalent to each other, where* $s = \min\{n \geq 1 \mid QI^{n-1} \cap I^{n+1} \neq QI^n\}$.

(1) $\chi_1(a_1t, a_2t, \cdots, a_dt; G) = 1$,
(2) $K^{(m)} \cong B(-u)$ *as graded* T*-modules for some* $-1 \leq m \leq s-2$ *and* $1 \leq u \leq s$, *and* $K^{(n)} = (0)$ *for all* $n \neq m$.

When this is the case we have the following.

(*i*) $\mathrm{e}_k(I) = \sum_{n=k}^{r_I} \binom{n}{k} \ell_A(I^n/QI^{n-1} + I^{n+1}) - \binom{s}{k}$ *for* $1 \leq k \leq d$,
(*ii*) *the Hilbert series* $HS_I(z)$ *of* I *is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \sum_{n=1}^{r_I} \ell_A(I^n/QI^{n-1} + I^{n+1})z^n - z^s}{(1-z)^d},$$

(*iii*) $\mathrm{a}_{d-1}(G) = s - d$, *and* $\ell_A([\mathrm{H}_M^{d-1}(G)]_{s-d}) = 1$,
(*iv*) $\operatorname{depth} G = d - 1$.

4. Applications for stretched ideals

In this section let us introduce some applications of Theorem 1 for stretched ideals.

The notion of *stretched* Cohen-Macaulay local rings was introduced by J. Sally to extend the rings of minimal or almost minimal multiplicity.

We say that the ring A is *stretched* if $\ell_A(\mathfrak{m}^2 + Q/\mathfrak{m}^3 + Q) = 1$ holds true, i.e. the ideal $(\mathfrak{m}/Q)^2$ is principal, for some parameter ideal Q in A which forms a reduction of $\mathfrak{m}$ ([10]). We note here that this condition depends on the choice of a reduction Q (see [9, Example 2.3]).

In 2001, Rossi and Valla [9] gave the notion of stretched $\mathfrak{m}$-primary ideals. We say that the $\mathfrak{m}$-primary ideal I is stretched if the following two conditions

(1) $Q \cap I^2 = QI$ and
(2) $\ell_A(I^2 + Q/I^3 + Q) = 1$

hold true for some parameter ideal Q in A which forms a reduction of I. We notice that the first condition is naturally satisfied if $I = \mathfrak{m}$ so that this extends the classical definition of stretched local rings given in [10].

The following lemma which was essentially given by Rossi and Valla.

Lemma 13. ([9, Lemma 2.4]) *Suppose that I is stretched. Then we have the following.*

(1) *There exists $x, y \in I \backslash Q$ such that $I^{n+1} = QI^n + (x^n y)$ holds true for all $n \geq 1$.*
(2) *The map*

$$I^{n+1}/QI^n \stackrel{\widehat{x}}{\to} I^{n+2}/QI^{n+1}$$

is surjective for all $n \geq 1$. Therefore $\alpha_n \geq \alpha_{n+1}$ for all $n \geq 1$.

(3) $\mathfrak{m} x^n y \subseteq QI^n + I^{n+2}$ *and hence* $\ell_A(I^n/QI^{n-1} + I^{n+1}) \leq 1$ *for all* $n \geq 1$.

We set

$$\Lambda := \Lambda_I = \Lambda_Q(I) = \{n \geq 1 \mid QI^{n-1} \cap I^{n+1}/QI^n \neq (0)\}$$

and $|\Lambda|$ denotes the cardinality of the set Λ. Let

$$n_I = n_Q(I) = \min\{n \geq 0 \mid I^{n+1} \subseteq Q\}.$$

It is easy to see that the inequality $r_I \geq n_I$ holds true.

Then the following proposition is satisfied.

Proposition 14. *Suppose that I is stretched. Then $\chi_1(a_1 t, a_2 t, \cdots, a_d t; G) = |\Lambda| = r_I - n_I$.*

The following result was essentially given by Sally and Rossi-Valla.

Corollary 15. ([9, 10]) *Suppose that I is stretched, then the following conditions are equivalent to each other.*

(1) $r_I = n_I$,
(2) $\Lambda = \emptyset$,

(3) *the Hilbert series* $HS_I(z)$ *of* I *is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \{e_0(I) - \ell_A(A/I) - n_I + 1\}z + \sum_{2 \leq n \leq r_I} z^n}{(1-z)^d}$$

(4) G *is Cohen-Macaulay.*

We can get the following corollary for the case where the reduction number r_I attains almost minimal value $n_I + 1$.

Corollary 16. ([8, Theorem 1.1]) *Suppose that* I *is stretched, then the following conditions are equivalent to each other.*

(1) $r_I = n_I + 1$,
(2) $|\Lambda| = 1$,
(3) *the Hilbert series* $HS_I(z)$ *of* I *is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \{e_0(I) - \ell_A(A/I) - n_I + 1\}z + \sum_{2 \leq n \leq r_I, n \neq s} z^n}{(1-z)^d}$$

for some $s > 0$.

When this is the case, the following conditions also hold true.

(*i*) $\Lambda = \{s\}$,
(*ii*) $e_1(I) = e_0(I) - \ell_A(A/I) + \binom{n_I+1}{2} - s + 1$,
(*iii*) $e_k(I) = \binom{n_I+2}{k+1} - \binom{s}{k}$ *for all* $2 \leq k \leq d$,
(*iv*) $a_{d-1}(G) = s - d$ *and* $\ell_A([\mathrm{H}^{d-1}_M(G)]_{s-d}) = 1$, *and*
(*v*) $\operatorname{depth} G = d - 1$.

Corollary 17. ([8, Corollary 1.2]) *Suppose that* I *is stretched and assume that* $r_I \leq n_I + 1$. *Then* $\operatorname{depth} G \geq d - 1$.

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