A CLASSIFICATION OF T-STRUCTURES BY A LATTICE OF TORSION CLASSES

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Abstract. We introduce the notion of ICE sequences to investigate t-structures on the bounded derived category of the module categories $\mathsf{mod}\Lambda$ over a finite dimensional algebra Λ. We give a correspondence between bounded t-structures and ICE sequences. Moreover we give a description of ICE sequences in $\mathsf{mod}\Lambda$ in terms of the lattice consisting of torsion classes in $\mathsf{mod}\Lambda$.

1. Introduction

Let Λ be a finite dimensional algebra over a field k. We denote by $\mathsf{mod}\Lambda$ the category of finitely generated right Λ-modules and $D^b(\mathsf{mod}\Lambda)$ the bounded derived category of $\mathsf{mod}\Lambda$. It is one of the main subjects of representation theory of finite dimensional algebras to study subcategories of $\mathsf{mod}\Lambda$ and $D^b(\mathsf{mod}\Lambda)$. For example, torsion classes are studied actively, and correspond to intermediate t-structures on $D^b(\mathsf{mod}\Lambda)$ bijectively [6]. In this note, we always assume that all subcategories are full and closed under isomorphisms.

We focus on t-structures on $D^b(\mathsf{mod}\Lambda)$. For subcategories $\mathcal{U}$ and $\mathcal{V}$ of $D^b(\mathsf{mod}\Lambda)$, we denote by $\mathcal{U} * \mathcal{V}$ the subcategory of $D^b(\mathsf{mod}\Lambda)$ consisting of objects X such that there exists an exact triangle $U \to X \to V \to \Sigma U$ in $D^b(\mathsf{mod}\Lambda)$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

Definition 1. [2, Définition 1.3.1] A pair of subcategories $(\mathcal{U}, \mathcal{V})$ of $D^b(\mathsf{mod}\Lambda)$ is a *t-structure* on $D^b(\mathsf{mod}\Lambda)$ if it satisfies the following conditions:

(1) $\mathsf{Hom}(\mathcal{U}, \mathcal{V}) = 0$.
(2) $D^b(\mathsf{mod}\Lambda) = \mathcal{U} * \mathcal{V}$.
(3) $\Sigma\mathcal{U} \subseteq \mathcal{U}$.

We call $\mathcal{U}$ an *aisle*. A t-structure $(\mathcal{U}, \mathcal{V})$ is *bounded* if it satisfies

$$\bigcup_{n\in\mathbb{Z}} \Sigma^{-n}\mathcal{U} = D^b(\mathsf{mod}\Lambda) = \bigcup_{n\in\mathbb{Z}} \Sigma^{n}\mathcal{V}.$$

For a t-structure $(\mathcal{U}, \mathcal{V})$ on $D^b(\mathsf{mod}\Lambda)$, we have $\mathcal{U} = {}^{\perp}\mathcal{V}$, therefore a t-structure is determined by its aisle. Hence we focus on aisles, and we call a subcategory of $D^b(\mathsf{mod}\Lambda)$ an aisle if it is an aisle of a certain t-structure.

A subcategory $\mathcal{X}$ of $D^b(\mathsf{mod}\Lambda)$ is *closed under extensions* if it satisfies $\mathcal{X} * \mathcal{X} \subseteq \mathcal{X}$.

Definition 2. A subcategory $\mathcal{U}$ of $D^b(\mathsf{mod}\Lambda)$ is a *preaisle* if $\mathcal{U}$ is closed under extensions and positive shifts.

It is easy to check that an aisle of a t-structure is a preaisle. Actually, aisles are exactly contravariantly finite preaisles:

The detailed version of this paper will be submitted for publication elsewhere.

Proposition 3. [8, Proposition 1.3] *The following are equivalent for a subcategory $\mathcal{U}$ of $D^b(\mathsf{mod}\Lambda)$.*

(1) *$\mathcal{U}$ is an aisle.*
(2) *$\mathcal{U}$ is a coreflective preaisle, that is, $\mathcal{U}$ is a preaisle and the inclusion $\mathcal{U} \to D^b(\mathsf{mod}\Lambda)$ has a right adjoint functor.*
(3) *$\mathcal{U}$ is a contravariantly finite preaisle closed under direct summands.*

Proof. (1) $\Leftrightarrow$ (2): This is well-known.
(2) $\Leftrightarrow$ (2): This follows from [3, Corollary 4.5]. $\square$

At first, we deel with preaisles. In [10], homology-determined preaisles are classified by *narrow sequences*. We denote by H^k the k-th cohomology functor.

Definition 4. A preaisle $\mathcal{U}$ of $D^b(\mathsf{mod}\Lambda)$ is *homology-determined* if for any $X \in D^b(\mathsf{mod}\Lambda)$, we have $X \in \mathcal{U}$ if and only if $\Sigma^{-k}(H^k X) \in \mathcal{U}$ for any $k \in \mathbb{Z}$.

Note that if Λ is hereditary, then every aisle is homology-determined since every complex X in $D^b(\mathsf{mod}\Lambda)$ is isomorphic to a direct sum $\oplus\Sigma^{-k}(H^k X)$. For homology-determined preaisle $\mathcal{U}$ of $D^b(\mathsf{mod}\Lambda)$, we can consider a sequence $\{H^k\mathcal{U}\}_{k\in\mathbb{Z}}$ of subcategories of $\mathsf{mod}\Lambda$. In the next section, we give a characterization of the sequence.

2. Aisles and ICE sequences

In this section, we introduce ICE sequences to study preaisles. We recall basic definitions of subcategories of an abelian category.

Definition 5. Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a subcategory of $\mathcal{A}$.

(1) $\mathcal{C}$ is *closed under extensions* if for every short exact sequence
$$0 \to L \to M \to N \to 0$$
in $\mathcal{A}$ with $L, N \in \mathcal{C}$, we have $M \in \mathcal{C}$.
(2) $\mathcal{C}$ is *closed under quotients (resp. subobjects) in* $\mathcal{A}$ if, for every object $C \in \mathcal{C}$, every quotient (resp. subobject) of C in $\mathcal{A}$ belongs to $\mathcal{C}$.
(3) $\mathcal{C}$ is a *torsion class (resp. torsion-free class) in* $\mathcal{A}$ if $\mathcal{C}$ is closed under extensions and quotients in $\mathcal{A}$ (resp. extensions and subobjects).
(4) $\mathcal{C}$ is closed under *images (resp. kernels, cokernels)* if, for every map $\varphi\colon C_1 \to C_2$ with $C_1, C_2 \in \mathcal{C}$, we have $\mathsf{Im}\varphi \in \mathcal{C}$ (resp. $\mathsf{Ker}\varphi \in \mathcal{C}$, $\mathsf{Coker}\varphi \in \mathcal{C}$).
(5) $\mathcal{C}$ is a *wide subcategory of* $\mathcal{A}$ if $\mathcal{C}$ is closed under kernels, cokernels, and extensions.
(6) $\mathcal{C}$ is an *ICE-closed subcategory of* $\mathcal{A}$ if $\mathcal{C}$ is closed under images, cokernels and extensions.

It is easy to check that torsion classes and wide subcategories are ICE-closed subcategories. Moreover, every torsion class in a wide subcategory (viewed as an abelian category) is ICE-closed, see [5, Lemma 2.2]. In [7], Ingalls and Thomas introduced an operation α which associates to a torsion class a wide subcategory. In [4, Proposition 4.2], the operation was generalized to ICE-closed subcategories. The following is shown by the same argument of [7, Proposition 2.12].

Proposition 6. *Let $\mathcal{C}$ be an ICE-closed subcategory of $\mathcal{A}$. Define a subcategory of $\mathcal{C}$ by*

$$\alpha\mathcal{C} = \{A \in \mathcal{C} \mid {}^{\forall}(f\colon C \to A) \in \mathcal{C},\ \ker f \in \mathcal{C}\}.$$

Then $\alpha\mathcal{C}$ is a wide subcategory of $\mathcal{A}$.

Next we give a definition of ICE sequences. This is the key notion in this note.

Definition 7. A sequence $\{\mathcal{C}(k)\}_{k\in\mathbb{Z}}$ of subcategories of $\mathsf{mod}\Lambda$ is an *ICE sequence* if for any k, the subcategory $\mathcal{C}(k)$ is an ICE-closed subcategory of $\mathsf{mod}\Lambda$ and the subcategory $\mathcal{C}(k+1)$ is a torsion class in $\alpha(\mathcal{C}(k))$.

Clealy, we have $\mathcal{C}(k+1) \subseteq \mathcal{C}(k)$ for any $k \in \mathbb{Z}$. Actually, ICE sequnces are the same notion of narrow sequences introduced in [10, Definition 4.1], see [9, Proposition 4.2]. Combining this fact and the result [10, Theorem 4.11], we obtain the following result.

Theorem 8. [9, Theorem 4.5] *There exist mutually bijective correspondences between*

(1) *the set of homology-determined preaisles in $D^b(\mathsf{mod}\Lambda)$.*
(2) *the set of ICE sequences in $\mathsf{mod}\Lambda$,*

The map from (1) to (2) is given by $\mathcal{U} \mapsto \{H^k\mathcal{U}\}_{k\in\mathbb{Z}}$. The converse is given by $\{\mathcal{C}(k)\}_{k\in\mathbb{Z}} \mapsto \{X \in D^b(\mathsf{mod}\Lambda) \mid H^kX \in \mathcal{C}(k)$ for any $k\}$.

Finally, we restrict the above result to aisles of bounded t-structures.

Definition 9. Let $\{\mathcal{C}(k)\}_{k\in\mathbb{Z}}$ be an ICE sequence in $\mathsf{mod}\Lambda$.

(1) $\{\mathcal{C}(k)\}_{k\in\mathbb{Z}}$ is *contravariantly finite* if $\mathcal{C}(k)$ is contravariantly finite in $\mathsf{mod}\Lambda$ for any $k \in \mathbb{Z}$.
(2) $\{\mathcal{C}(k)\}_{k\in\mathbb{Z}}$ is *full* if there exist integers $m \leq n$ such that $\mathcal{C}(m) = 0$ and $\mathcal{C}(n) = \mathsf{mod}\Lambda$.
(3) For a positive integer n, we say that $\{\mathcal{C}(k)\}_{k\in\mathbb{Z}}$ is *of length* $n+1$ if we have $\mathcal{C}(1) = 0$ and $\mathcal{C}(-n) = \mathsf{mod}\Lambda$.

Note that an ICE-closed subcategory of $\mathsf{mod}\Lambda$ is contravariantly finite if and only if it is coreflective by [3, Corollary 7.2]. If Λ is τ-tilting finite, then every ICE-closed subcategory of $\mathsf{mod}\Lambda$ is contravariantly finite, see [5, Proposition 4.20].

The following is the main result in this section.

Theorem 10. [9, Theorem 5.5, Corollary 5.6] *There exist bijective correspondences between*

(1) *the set of contravariantly finite full ICE sequences in $\mathsf{mod}\Lambda$,*
(2) *the set of bounded t-structures on $D^b(\mathsf{mod}\Lambda)$ whose aisles are homology-determined.*

Let n be a positive integer. Then the above restrict to the following.

(1) *the set of contravariantly finite ICE sequences in $\mathsf{mod}\Lambda$ of length $n+1$,*
(2) *the set of $(n+1)$-intermediate t-structures on $D^b(\mathsf{mod}\Lambda)$ whose aisles are homology-determined.*

Thus we can construct t-structures on $D^b(\mathsf{mod}\Lambda)$ from ICE sequences in $\mathsf{mod}\Lambda$. In the next section, we give a description of ICE sequences by a lattice-theoretical notion.

3. A Lattice of torsion classes

In this section, we fix a positive integer n, and focus on $(n+1)$-intermediate t-structures. We give a description of ICE sequences of length $n+1$ in $\mathsf{mod}\Lambda$ from the viewpoint of a lattice consisting of torsion classes in $\mathsf{mod}\Lambda$. We denote by $\mathsf{tors}\Lambda$ the set of torsion classes in $\mathsf{mod}\Lambda$, which forms a partially ordered set by inclusion. Moreover $\mathsf{tors}\Lambda$ is a complete lattice since there are arbitrary intersections. We collect some definitions and results.

Definition 11. To $\mathcal{T},\mathcal{U} \in \mathsf{tors}\Lambda$, we associate the set

$$[\mathcal{U},\mathcal{T}] := \{\mathcal{C} \in \mathsf{tors}\Lambda \mid \mathcal{U} \subseteq \mathcal{C} \subseteq \mathcal{T}\}$$

called an *interval* in $\mathsf{tors}\Lambda$. To an interval $[\mathcal{U},\mathcal{T}]$, we associate a subcategory $\mathcal{H}_{[\mathcal{U},\mathcal{T}]} = \mathcal{T} \cap \mathcal{U}^{\perp}$ called the *heart* of $[\mathcal{U},\mathcal{T}]$. We call an interval $[\mathcal{U},\mathcal{T}]$ a *wide interval* if the heart is a wide subcategory of $\mathsf{mod}\Lambda$. We denote by $\mathsf{Hasse}(\mathsf{tors}\Lambda)$ the *Hasse quiver* of $\mathsf{tors}\Lambda$, the quiver whose vertex set is $\mathsf{tors}\Lambda$, and there is an arrow $\mathcal{T} \to \mathcal{U}$ in $\mathsf{tors}\Lambda$ if and only if $\mathcal{U} \subsetneq \mathcal{T}$ holds and there is no $\mathcal{C} \in \mathsf{tors}\Lambda$ satisfying $\mathcal{U} \subsetneq \mathcal{C} \subsetneq \mathcal{T}$.

Wide intervals are characterized as a lattice-theoretical property in $\mathsf{tors}\Lambda$ as follows:

Proposition 12. [1, Theorem 5.2] *Let $[\mathcal{U},\mathcal{T}]$ be an interval in $\mathsf{tors}\Lambda$. Then the following conditions are equivalent:*

1. *$[\mathcal{U},\mathcal{T}]$ is a wide interval.*
2. *$[\mathcal{U},\mathcal{T}]$ is a meet interval, that is, it holds*

$$\mathcal{U} = \mathcal{T} \bigcap \{\mathcal{C} \in [\mathcal{U},\mathcal{T}] \mid \text{there is an arrow } \mathcal{T} \to \mathcal{C} \text{ in } \mathsf{Hasse}(\mathsf{tors}\Lambda)\}.$$

The operation α is understood from the viewpoint of wide intervals:

Proposition 13. *Let $\mathcal{T}$ be a torsion class in $\mathsf{mod}\Lambda$. Then the following statements hold.*

1. [1, Proposition 6.3] *$\alpha\mathcal{T}$ equals to the heart of the interval $[\mathcal{T} \cap {}^{\perp}\alpha\mathcal{T}, \mathcal{T}]$.*
2. [5, Proposition 3.3] *We set*

$$\mathcal{T}^{-} = \mathcal{T} \bigcap \{\mathcal{C} \in \mathsf{tors}\Lambda \mid \text{there is an arrow } \mathcal{T} \to \mathcal{C} \text{ in } \mathsf{Hasse}(\mathsf{tors}\Lambda)\}.$$

Then we have $\mathcal{T}^{-} = \mathcal{T} \cap {}^{\perp}\alpha\mathcal{T}$ and $\mathcal{H}_{[\mathcal{T}^{-},\mathcal{T}]} = \alpha\mathcal{T}$.

Thus we can understand α in terms of $\mathsf{tors}\Lambda$. We introduce the following notion.

Definition 14.

1. We call an interval of the form $[\mathcal{T}^{-},\mathcal{T}]$ a *maximal meet interval* in $\mathsf{tors}\Lambda$. More generally, we call an interval $[\mathcal{U}',\mathcal{T}']$ contained in a wide interval $[\mathcal{U},\mathcal{T}]$ in $\mathsf{tors}\Lambda$ a *maximal meet interval* in $[\mathcal{U},\mathcal{T}]$ if we have

$$\mathcal{U}' = \mathcal{T}' \bigcap \{\mathcal{C} \in [\mathcal{U},\mathcal{T}] \mid \text{there is an arrow } \mathcal{T}' \to \mathcal{C} \text{ in } \mathsf{Hasse}(\mathsf{tors}\Lambda)\}.$$

2. We call a sequence $\{[\mathcal{U}_k,\mathcal{T}_k]\}_{k=1}^{n}$ of intervals in $\mathsf{tors}\Lambda$ a *decreasing sequence of maximal meet intervals* in $\mathsf{tors}\Lambda$ provided that $[\mathcal{U}_{k+1},\mathcal{T}_{k+1}]$ is a maximal meet interval in $[\mathcal{U}_k,\mathcal{T}_k]$ for any $k = 0,\dots,n-1$ where we set $\mathcal{U}_0 = 0$ and $\mathcal{T}_0 = \mathsf{mod}\Lambda$. We call n the *length* of the sequence.

Now we obtain a classification of $(n+1)$-intermediate t-structures whose aisles are homology-determined via ICE sequences and the lattice of torsion classes:

Theorem 15. *Let Λ be a τ-tilting finite algebra and* $\mathsf{tors}\Lambda$ *the lattice consisting of torsion classes in* $\mathsf{mod}\Lambda$*. Then there are one-to-one correspondences between*

(1) *the set of $(n+1)$-intermediate t-structures on $D^b(\mathsf{mod}\Lambda)$ whose aisles are homology-determined,*
(2) *the set of ICE sequences in* $\mathsf{mod}\Lambda$ *of length $n+1$,*
(3) *the set of decreasing sequences of maximal meet intervals in* $\mathsf{tors}\Lambda$ *of length n,*

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