RESOLVING SUBCATEGORIES OF DERIVED CATEGORIES

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Abstract. Let R be a commutative noetherian ring. Denote by $\mathsf{D}^{\mathsf{b}}(R)$ the bounded derived category of finitely generated R-modules. In this article we classify the preaisles of $\mathsf{D}^{\mathsf{b}}(R)$ containing R and closed under direct summands, when R is a complete intersection. This classification includes as restrictions the classification of thick subcategories of the singularity category due to Stevenson, and the classification of resolving subcategories of the module category due to Dao and Takahashi.

1. Main result

Throughout this article, we assume that all subcategories are strictly full. First of all, we introduce a setup to explain our main result.

Setup 1. Let (R, V) be a pair, where R and V satisfy either of the following two conditions.

(1) R is a commutative noetherian ring which is locally a hypersurface, and V is the singular locus of R.
(2) R is a quotient ring of the form $S/(\boldsymbol{a})$ where S is a regular ring of finite Krull dimension and $\boldsymbol{a} = a_1, \dots, a_c$ is a regular sequence, and V is the singular locus of the zero subscheme of $a_1x_1 + \cdots + a_cx_c \in \Gamma(X, \mathcal{O}_X(1))$ where $X = \mathbb{P}_S^{c-1} = \operatorname{Proj}(S[x_1, \dots, x_c])$.

Here, a commutative noetherian ring R is said to be *locally* a *hypersurface* if the local ring $R_\mathfrak{p}$ is a hypersurface for every prime ideal $\mathfrak{p}$ of R. When R is a local ring with maximal ideal $\mathfrak{m}$, we say that R is a *hypersurface* if the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is a quotient of a regular local ring by a principal ideal. A *regular sequence* on R is a sequence $\boldsymbol{x} = x_1, \dots, x_n$ of elements of R such that the residue class of x_i in $R/(x_1, \dots, x_{i-1})$ is a non-zerodivisor for each $i = 1, \dots, n$ and that $(x_1, \dots, x_n)$ is not a unit ideal of R.

For a commutative noetherian ring R, we denote by $\operatorname{\mathsf{mod}} R$ the category of finitely generated R-modules, by $\mathsf{D}^{\mathsf{b}}(R)$ the bounded derived category of $\operatorname{\mathsf{mod}} R$, by $\mathsf{D}^{\mathsf{perf}}(R)$ the subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ consisting of perfect complexes, and by $\mathsf{D}_{\mathsf{sg}}(R)$ the singularity category of R, i.e.,

$$\mathsf{D}_{\mathsf{sg}}(R) = \mathsf{D}^{\mathsf{b}}(R) / \mathsf{D}^{\mathsf{perf}}(R).$$

Recall that a *thick subcategory* of a triangulated category is by definition a triangulated subcategory closed under direct summands. Under the above setup, Stevenson [2] proved the following classification theorem of thick subcategories.

The detailed version [3] of this article has been submitted for publication elsewhere.

Theorem 2 (Stevenson)**.** *Let (R, V) be as in Setup 1. Then there are one-to-one correspondences*

$$\left\{\begin{matrix}\textit{thick}\\ \textit{subcategories}\\ \textit{of } \mathsf{D}_{\mathsf{sg}}(R)\end{matrix}\right\} \cong \left\{\begin{matrix}\textit{thick}\\ \textit{subcategories}\\ \textit{of } \mathsf{D}^{\mathsf{b}}(R)\\ \textit{containing } R\end{matrix}\right\} \overset{\text{(a)}}{\cong} \left\{\begin{matrix}\textit{specialization-}\\ \textit{closed}\\ \textit{subsets of } V\end{matrix}\right\}.$$

Recall that a *resolving subcategory* of $\mathsf{mod}\, R$ is defined to be a subcategory of $\mathsf{mod}\, R$ containing R and closed under direct summands, extensions and kernels of epimorphisms. Dao and Takahashi [1] gave a complete classification of the resolving subcategories of $\mathsf{mod}\, R$ under the setup introduced above.

Theorem 3 (Dao–Takahashi)**.** *Let (R, V) be as in Setup 1. Then there is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{resolving}\\ \textit{subcategories}\\ \textit{of } \mathsf{mod}\, R\end{matrix}\right\} \overset{\text{(b)}}{\cong} \left\{\begin{matrix}\textit{grade-}\\ \textit{consistent}\\ \textit{functions}\\ \textit{on } \operatorname{Spec} R\end{matrix}\right\} \times \left\{\begin{matrix}\textit{specialization-}\\ \textit{closed}\\ \textit{subsets of } V\end{matrix}\right\}.$$

Here, a *grade-consistent function* on $\operatorname{Spec} R$ is defined as an order-preserving map $f : \operatorname{Spec} R \to \mathbb{N}$ which satisfies the inequality $f(\mathfrak{p}) \leqslant \operatorname{grade} \mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec} R$, where

$$\operatorname{grade} \mathfrak{p} = \inf\{i \in \mathbb{N} \mid \operatorname{Ext}^i_R(R/\mathfrak{p}, R) \neq 0\}.$$

Recall that a *preaisle* (resp. *precoaisle*) of a triangulated category is defined as a subcategory closed under extensions and positive (resp. negative) shifts. Mimicking the definition of a resolving subcategory of $\mathsf{mod}\, R$, we define a *resolving subcategory* of $\mathsf{D}^{\mathsf{b}}(R)$ as a subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ containing R and closed under direct summands, extensions and cocones.

The main result of this article is the following theorem. This theorem provides a classification of preaisles of $\mathsf{D}^{\mathsf{b}}(R)$ that satisfy some mild and natural conditions. Also, the theorem includes both the classification of thick subcategories by Stevenson and the classification of resolving subcategories by Dao and Takahashi.

Theorem 4. *Let (R, V) be a pair as in Setup 1. Then there are one-to-one correspondences*

$$\left\{\begin{matrix}\textit{preaisles}\\ \textit{of } \mathsf{D}^{\mathsf{b}}(R)\\ \textit{containing}\\ R \textit{ and closed}\\ \textit{under direct}\\ \textit{summands}\end{matrix}\right\} \cong \left\{\begin{matrix}\textit{resolving}\\ \textit{subcategories}\\ \textit{of } \mathsf{D}^{\mathsf{b}}(R)\end{matrix}\right\} \overset{(*)}{\cong} \left\{\begin{matrix}\textit{order-}\\ \textit{preserving}\\ \textit{maps}\\ \textit{from } \operatorname{Spec} R\\ \textit{to } \mathbb{N}\cup\{\infty\}\end{matrix}\right\} \times \left\{\begin{matrix}\textit{specialization-}\\ \textit{closed}\\ \textit{subsets of } V\end{matrix}\right\}.$$

The restriction of the bijection $(*)$ *to the thick subcategories of* $\mathsf{D}^{\mathsf{b}}(R)$ *containing* R *is identified with the bijection* (a) *in Theorem 2. The composition of the bijection* $(*)$ *with the map*

$$\mathcal{X} \mapsto \mathsf{res}_{\mathsf{D}^{\mathsf{b}}(R)}\, \mathcal{X}$$

coincides with the bijection (b) *in Theorem 3.*

Here, $\mathsf{res}_{\mathsf{D^b}(R)}\,\mathcal{X}$ stands for the *resolving closure* of $\mathcal{X}$ in $\mathsf{D^b}(R)$, that is, the smallest resolving subcategory of $\mathsf{D^b}(R)$ containing $\mathcal{X}$.

2. Outline

This section is devoted to (roughly) explaining how to deduce Theorem 4.

We say that R is *locally* a *Gorenstein ring* if the local ring $R_\mathfrak{p}$ is Gorenstein for every prime ideal $\mathfrak{p}$ of R. When R is a local ring, we say that R is *Gorenstein* if R has finite injective dimension as an R-module.

The following proposition is immediately obtained by using the fact that the functor $\mathbf{R}\mathrm{Hom}_R(-,R)$ gives a duality of $\mathsf{D^b}(R)$ if R is locally a Gorenstein ring, and comparing the definitions of a resolving subcategory and a precoaisle.

Proposition 5. *Let R be a commutative noetherian ring. Suppose that R is locally a Gorenstein ring. Assigning to each subcategory $\mathcal{X}$ of $\mathsf{D^b}(R)$ the subcategory*

$$\mathbf{R}\mathrm{Hom}_R(\mathcal{X},R)=\{\mathbf{R}\mathrm{Hom}_R(X,R)\mid X\in\mathcal{X}\}$$

of $\mathsf{D^b}(R)$, one obtains a one-to-one correspondence

$$\left\{\begin{array}{c}\textit{preaisles}\\ \textit{of } \mathsf{D^b}(R)\\ \textit{containing}\\ R \textit{ and closed}\\ \textit{under direct}\\ \textit{summands}\end{array}\right\}\cong\left\{\begin{array}{c}\textit{precoaisles}\\ \textit{of } \mathsf{D^b}(R)\\ \textit{containing}\\ R \textit{ and closed}\\ \textit{under direct}\\ \textit{summands}\end{array}\right\}=\left\{\begin{array}{c}\textit{resolving}\\ \textit{subcategories}\\ \textit{of } \mathsf{D^b}(R)\end{array}\right\}.$$

We say that R is *locally* a *complete intersection* if the local ring $R_\mathfrak{p}$ is a complete intersection for every prime ideal $\mathfrak{p}$ of R. When R is a local ring with maximal ideal $\mathfrak{m}$, we say that R is a *complete intersection* if the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is a quotient of a regular local ring by an ideal generated by a regular sequence. Denote by $\mathsf{D^{CM}}(R)$ the subcategory of $\mathsf{D^b}(R)$ consisting of *maximal Cohen–Macaulay complexes*, that is, complexes $C\in\mathsf{D^b}(R)$ such that

$$\operatorname{depth}_{R_\mathfrak{p}} C_\mathfrak{p}\geqslant\dim R_\mathfrak{p}$$

for all prime ideals $\mathfrak{p}$ of R, where dim denotes Krull dimension. When R is a local ring with residue field k, for each $X\in\mathsf{D^b}(R)$ we set

$$\operatorname{depth}_R X=\inf\{i\in\mathbb{Z}\mid \operatorname{Ext}^i_R(k,X)\neq 0\}.$$

The following two theorems are the most essential parts of our work. In the first theorem, $\mathsf{thick}_{\mathsf{D^b}(R)}\,\mathcal{X}$ stands for the *thick closure* of $\mathcal{X}$ in $\mathsf{D^b}(R)$, that is, the smallest thick subcategory of $\mathsf{D^b}(R)$ containing $\mathcal{X}$. The assumption of locally a complete intersection in the first theorem is necessary to deduce that each resolving subcategory of $\mathsf{D^b}(R)$ contained in $\mathsf{D^{CM}}(R)$ is closed under exact triangles of maximal Cohen–Macaulay complexes. The proof of the second theorem uses subtle arguments on Koszul complexes, and the notion of an *NE-locus*, which is a certain Zariski-closed subset of $\operatorname{Spec} R$.

Theorem 6. *Let R be a commutative noetherian ring. Suppose that R is locally a complete intersection.*

(1) *There are mutually inverse bijections*

$$\left\{\begin{array}{c}\textit{resolving}\\ \textit{subcategories}\\ \textit{of } \mathsf{D}^{\mathsf{b}}(R)\end{array}\right\} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \left\{\begin{array}{c}\textit{resolving}\\ \textit{subcategories}\\ \textit{of } \mathsf{D}^{\mathsf{b}}(R)\\ \textit{contained}\\ \textit{in } \mathsf{D}^{\mathsf{perf}}(R)\end{array}\right\} \times \left\{\begin{array}{c}\textit{resolving}\\ \textit{subcategories}\\ \textit{of } \mathsf{D}^{\mathsf{b}}(R)\\ \textit{contained}\\ \textit{in } \mathsf{D}^{\mathsf{CM}}(R)\end{array}\right\},$$

where the maps ϕ, ψ *are given by*

$$\phi(\mathcal{X}) = (\mathcal{X} \cap \mathsf{D}^{\mathsf{perf}}(R), \mathcal{X} \cap \mathsf{D}^{\mathsf{CM}}(R))$$

for each element $\mathcal{X}$ *of the left-hand side, and*

$$\psi(\mathcal{Y}, \mathcal{Z}) = \mathsf{res}_{\mathsf{D}^{\mathsf{b}}(R)}(\mathcal{Y} \cup \mathcal{Z})$$

for each element $(\mathcal{Y}, \mathcal{Z})$ *of the right-hand side.*

(2) *There are mutually inverse bijections*

$$\left\{\begin{array}{c}\textit{resolving}\\ \textit{subcategories}\\ \textit{of } \mathsf{D}^{\mathsf{b}}(R)\\ \textit{contained}\\ \textit{in } \mathsf{D}^{\mathsf{CM}}(R)\end{array}\right\} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \left\{\begin{array}{c}\textit{thick}\\ \textit{subcategories}\\ \textit{of } \mathsf{D}^{\mathsf{b}}(R)\\ \textit{containing } R\end{array}\right\} \cong \left\{\begin{array}{c}\textit{thick}\\ \textit{subcategories}\\ \textit{of } \mathsf{D}_{\mathsf{sg}}(R)\end{array}\right\}.$$

where the maps ϕ, ψ *are given by*

$$\phi(\mathcal{X}) = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\, \mathcal{X}$$

for each element $\mathcal{X}$ *of the left-hand side, and*

$$\psi(\mathcal{Y}) = \mathcal{Y} \cap \mathsf{D}^{\mathsf{CM}}(R)$$

for each element $\mathcal{Y}$ *of the right-hand side.*

Theorem 7. *Let* R *be any commutative noetherian ring. Then there are mutually inverse bijections*

$$\left\{\begin{array}{c}\textit{resolving}\\ \textit{subcategories}\\ \textit{of } \mathsf{D}^{\mathsf{b}}(R)\\ \textit{contained}\\ \textit{in } \mathsf{D}^{\mathsf{perf}}(R)\end{array}\right\} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \left\{\begin{array}{c}\textit{order-}\\ \textit{preserving}\\ \textit{maps from}\\ \operatorname{Spec} R \textit{ to}\\ \mathbb{N} \cup \{\infty\}\end{array}\right\}.$$

where the maps ϕ, ψ *are given by*

$$\phi(\mathcal{X})(\mathfrak{p}) = \sup_{X \in \mathcal{X}} \{\operatorname{pd}_{R_{\mathfrak{p}}} X_{\mathfrak{p}}\}$$

for each element $\mathcal{X}$ *of the left-hand side and each prime ideal* $\mathfrak{p}$ *of* R*, and*

$$\psi(f) = \{X \in \mathsf{D}^{\mathsf{b}}(R) \mid \operatorname{pd}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \leqslant f(\mathfrak{p}) \textit{ for all } \mathfrak{p} \in \operatorname{Spec} R\}$$

for each element f *of the right-hand side.*

Here, pd stands for projective dimension.

Taking the combination of Proposition 5 with Theorems 6 and 7, we obtain the following theorem.

Theorem 8. *Let R be a commutative noetherian ring which is locally a complete intersection. Then there are one-to-one correspondences*

$$\left\{\begin{array}{c}\textit{preaisles}\\ \textit{of } \mathsf{D}^{\mathsf{b}}(R)\\ \textit{containing}\\ R \textit{ and closed}\\ \textit{under direct}\\ \textit{summands}\end{array}\right\} \cong \left\{\begin{array}{c}\textit{resolving}\\ \textit{subcategories}\\ \textit{of } \mathsf{D}^{\mathsf{b}}(R)\end{array}\right\} \cong \left\{\begin{array}{c}\textit{order-}\\ \textit{preserving}\\ \textit{maps from}\\ \operatorname{Spec} R \textit{ to}\\ \mathbb{N}\cup\{\infty\}\end{array}\right\} \times \left\{\begin{array}{c}\textit{thick}\\ \textit{subcategories}\\ \textit{of } \mathsf{D}_{\mathsf{sg}}(R)\end{array}\right\}.$$

Finally, combining Theorem 8 with Theorems 2 and 3 completes the proof of Theorem 4.

References

[1] H. Dao; R. Takahashi, Classification of resolving subcategories and grade consistent functions, *Int. Math. Res. Not. IMRN* **2015**, no. 1, 119–149.

[2] G. Stevenson, Subcategories of singularity categories via tensor actions, *Compos. Math.* **150** (2014), no. 2, 229–272.

[3] R. Takahashi, Classifying preaisles of derived categories of complete intersections, preprint 2023, arXiv:2309.06074.

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