ON INTERVAL GLOBAL DIMENSION OF POSETS: A CHARACTERIZATION OF CASE 0

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Abstract. We study the relative homological algebra of posets with respect to the intervals. We introduce our recent research on the properties of the supports of interval approximations and interval resolution global dimension. We also provide necessary and sufficient conditions on a poset to ensure that any representation is interval-decomposable (i.e. a characterization of the case where interval resolution global dimension is equal to 0).

Key Words: Representation, Relative homological algebra, Persistence module, Interval module

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1. Introduction

We refer the reader to [3] (arXiv:2308.14979) for details on the contents of this article.

Topological data analysis is a rapidly growing field applying the ideas of algebraic topology for data analysis. One of its main tools is persistent homology [1], which can compactly summarize the birth and death parameters of topological features (e.g. connected components, rings, cavities, and so on) of data via the persistence diagram. This allows us to analyze hidden structures in data. Algebraically, one part of the persistent homology analysis can be formalized by using the so-called one-parameter persistence modules, which are just ("pointwise") finite dimensional modules over the incidence algebra of a totally ordered set. In this point of view, one-parameter persistence modules are guaranteed to decompose into the indecomposable modules called *interval modules*, which provide a multiset of intervals that are encoded by the persistence diagram.

As a generalization, multi-parameter persistence modules are proposed, understood as representations of n-dimensional grids, and are expected to provide richer information than the one-parameter setting. When dealing with multi-parameter settings, however, there are some difficulties with adapting the same techniques.

Recently, there has been an interest in the use of relative homological algebra in persistence theory. Especially, the notion of right minimal approximations and resolutions by interval-decomposable modules are developed, and the finiteness of the interval resolution global dimension has been confirmed [2].

The aim of this article is to introduce the properties of right minimal approximations and resolutions by interval-decomposable modules studied in [3].

The detailed version of this paper will be submitted for publication elsewhere.

2. Preliminaries

Let A be a finite dimensional algebra over a field k. We denote by $\mathsf{mod}\, A$ the category of finitely generated right A-modules. Throughout this article, we assume that all modules are finitely generated. For morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ of A-modules, we denote their composition by $gf\colon X \to Z$.

2.1. **Approximations and resolutions.** We recall the basic terminology of relative homological algebra. We consider the full subcategory $\mathcal{X} := \mathsf{add}\, \mathfrak{X}$ of $\mathsf{mod}\, A$ for a fixed finite collection $\mathfrak{X}$ of (isomorphism classes of) indecomposable A-modules including all the indecomposable projective modules.

Definition 1. For a morphism $f\colon X \to M$ of A-modules, we say that

(1) f is *right minimal* if any morphism $g : X \to X$ satisfying $fg = f$ is an isomorphism.
(2) f is a *right $\mathcal{X}$-approximation* of M if $X \in \mathcal{X}$ and $\mathrm{Hom}_A(Y, f)$ is surjective for any $Y \in \mathcal{X}$.
(3) f is a *right minimal $\mathcal{X}$-approximation* of M if it is a right $\mathcal{X}$-approximation which is right minimal.
(4) A *right minimal $\mathfrak{X}$-resolution* of M is an exact sequence

$$\cdots \longrightarrow J_m \xrightarrow{g_m} \cdots \xrightarrow{g_2} J_1 \xrightarrow{g_1} J_0 \xrightarrow{f} M \longrightarrow 0,$$

such that f is a right minimal $(\mathsf{add}\, \mathfrak{X})$-approximation of M, and for each $1 \leq i$, the morphism g_i is a right minimal $(\mathsf{add}\, \mathfrak{X})$-approximation of $\mathrm{Im}\, g_i = \mathrm{Ker}\, g_{i-1}$.
(5) If M has a right minimal $\mathfrak{X}$-resolution of the form

$$0 \longrightarrow J_m \xrightarrow{g_m} \cdots \xrightarrow{g_2} J_1 \xrightarrow{g_1} J_0 \xrightarrow{f} M \longrightarrow 0,$$

then we say that the *$\mathfrak{X}$-resolution dimension* of M is m and write $\mathfrak{X}\text{-res-dim}\, M = m$. Otherwise, we say that the $\mathfrak{X}$-resolution dimension of M is infinity. We set

$$\mathfrak{X}\text{-res-gldim}\, A := \sup\{\mathfrak{X}\text{-res-dim}\, M \mid M \in \mathsf{mod}\, A\}$$

and call *$\mathfrak{X}$-resolution global dimension* of A. Notice that it can be infinity.

2.2. **Partially ordered set and its representations.** Let P be a finite poset. We recall that the *Hasse diagram* of P is a directed graph whose vertices are in bijection with elements of P and there is an arrow $x \to y$ for $x, y \in P$ if $x < y$ and there is no $z \in P$ such that $x < z < y$. The *incidence algebra* $k[P]$ of a poset P is defined to be the quotient of the path algebra of the Hasse diagram of P modulo the two-sided ideal generated by all the commutative relations. The module category $\mathsf{mod}\, k[P]$ can be described in terms of a functor category as follows. Firstly, we regard P as a category whose objects are elements of P, and morphisms are defined by relations in P, i.e., there is a unique morphism $a \to b$ for $a, b \in P$ if and only if $a \leq b$. We denote by $\mathsf{rep}_k(P)$ the category of (covariant) functors from P to the category of finite dimensional vector spaces over k. For V in $\mathsf{rep}_k(P)$, the subset $\mathrm{supp}\, V := \{a \in P \mid V_a \neq 0\}$ is called the *support* of V.

It is well-known that there is an equivalence of abelian categories between $\mathsf{rep}_k(P)$ and the module category $\mathsf{mod}\, k[P]$ of the incidence algebra of P. In this sense, we identify objects V of $\mathsf{rep}_k(P)$ and $k[P]$-modules, and the *support* of a $k[P]$-module M is the subset

$\mathrm{supp}(M) = \{a \in P \mid Me_a \neq 0\}$, where e_a is a primitive idempotent of $k[P]$ corresponding to the element $a \in P$.

In our study, the following class of full subposets called *interval* is basic.

Definition 2. A *full subposet* of P is a subset $P' \subseteq P$ equipped with the induced partial order. Notice that it is completely determined by its elements. We say that

(1) P' is *convex* in P if, for any $x, y \in P'$ and any $z \in P$, $x < z < y$ implies $z \in P'$,
(2) P' is an *interval* of P if P' is connected as a poset and is convex in P.

We denote by $\mathbb{I}(P)$ the set of intervals of P.

The following special class of modules plays an important role in this article.

Definition 3. For an interval I of P, let k_I be a $k[P]$-module given as follows.

$$(k_I)_a = \begin{cases} k & \text{if } a \in I, \\ 0 & \text{otherwise,} \end{cases} \qquad k_I(a \leq b) = \begin{cases} 1_k & \text{if } a, b \in I, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

An *interval module* is a $k[P]$-module M such that $M \cong k_I$ for some interval $I \in \mathbb{I}(P)$. Clearly, every interval module is indecomposable.

We denote by $\mathfrak{I}_{k,P}$ the set of isomorphism classes of the interval $k[P]$-modules, which is in bijection with $\mathbb{I}(P)$ by $I \mapsto k_I$. Notice that $\mathbb{I}_P$ and $\mathfrak{I}_{P,k}$ are finite since so is P. Each module in $\mathsf{add}\,\mathfrak{I}_{P,k}$ is said to be *interval-decomposable*. In other words, a given $k[P]$-module M is interval-decomposable if and only if it can be written as

$$M \cong \bigoplus_{I \in \mathbb{I}(P)} k_I^{m(I)}$$

for some non-negative integers $m(I)$. We will write $\mathfrak{I}_P$ instead of $\mathfrak{I}_{k,P}$ when the base field k is clear.

Since $\mathfrak{I}_P$ contains all indecomposable projective $k[P]$-modules by definition, one can consider resolutions by interval modules. By *interval covers over* P (resp., *interval resolutions over* P), we mean right minimal $(\mathsf{add}\,\mathfrak{I}_P)$-approximations (resp., $\mathfrak{I}_P$-resolutions) of $k[P]$-modules. When the poset P is clear, we may omit it. In addition, we will write

$$\text{int-res-dim}\, M := \mathfrak{I}_P\text{-res-dim}\, M \quad \text{and} \quad \text{int-res-gldim}\, k[P] := \mathfrak{I}_P\text{-res-gldim}\, k[P],$$

and call them the *interval resolution dimension* of a module M and the *interval resolution global dimension* of $k[P]$ respectively. It has been shown in [2, Proposition 4.5] that the interval resolution global dimension is always finite. To show that, the next proposition is a key.

Proposition 4. [2, Lemma 4.4 and its dual] *The subcategory* $\mathsf{add}\,\mathfrak{I}_P$ *is closed under both submodules and quotients of indecomposable modules.*

Then, we can apply [10, Theorem in §5](cf. [8]) and obtain the following.

Theorem 5. [2, Proposition 4.5] *For any finite poset* P*,* $\text{int-res-gldim}(k[P]) < \infty$*.*

3. Results

In this section, we will give three results on interval covers and interval resolution dimensions (Theorems 6, 8, and 9). These results are motivated by topological data analysis, and they would be interesting from the perspective of representation theory of finite dimensional algebras.

3.1. **Result 1.** We show the following result.

Theorem 6. *Let P be a finite poset and $\mathcal{I}_P$ the set of isomorphism classes of interval modules. For a given $k[P]$-module M, we take its interval cover $f\colon X = \bigoplus_{i=1}^{m} X_i \to M$, where all the X_i's are interval modules. Then, the following holds.*

(1) *f is surjective.*
(2) *$f|_{X_i}\colon X_i \to M$ is injective for every $i \in \{1, \dots, m\}$.*
(3) $\operatorname{supp} X = \operatorname{supp} M$.

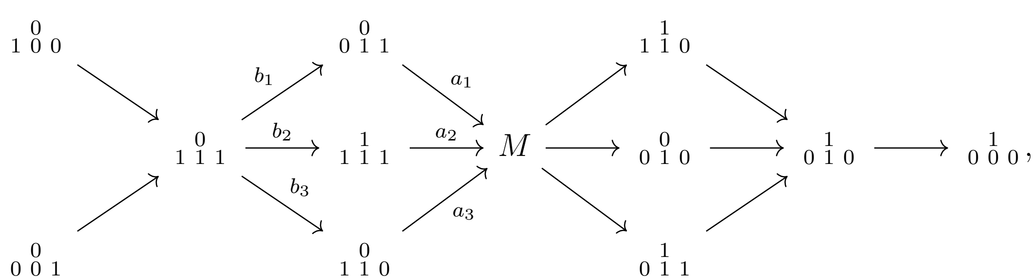
In particular, every X_i can be taken as an interval submodule of M.

An importance of Theorem 6 is that it provides one way to reduce the computational burden for computing interval resolutions. We note that [5, Proposition 4.8] show Theorem 6 in essentially the same way.

Example 7. We consider the D_4-type quiver $D_4(b)$ displayed below:

$$\begin{array}{ccccc} & & 1 & & \\ & & \downarrow & & \\ 2 & \longleftarrow & 3 & \longrightarrow & 4. \end{array}$$

Then, the incidence algebra is just a path algebra of type D_4. The Auslander-Reiten quiver is given by



where all indecomposable modules except for M are interval, but M is

$$\begin{array}{ccccc} & & k & & \\ & & \downarrow {\scriptstyle {}^t[1\,1]} & & \\ k & \xleftarrow{[1\,0]} & k^2 & \xrightarrow{[0\,1]} & k. \end{array}$$

Looking at the Auslander-Reiten quiver, we find that an interval resolution of M is

$$0 \longrightarrow \begin{smallmatrix} & 0 & \\ 1 & 1 & 1 \end{smallmatrix} \xrightarrow{{}^t[b_1,b_2,b_3]} \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \end{smallmatrix} \xrightarrow{[a_1,a_2,a_3]} M \longrightarrow 0,$$

and hence

$$\text{int-res-dim}\, M = 1.$$

Consequently, the interval resolution global dimension for $D_4(b)$ is 1. One can also show that any D_4-type quiver has the interval resolution global dimension 1.

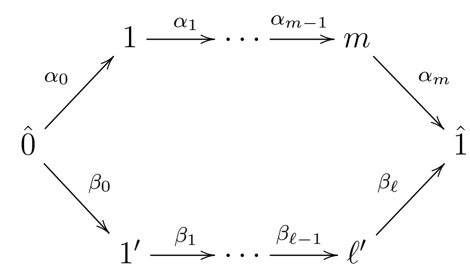
3.2. **Result 2.** We give a complete classification of posets whose modules are always interval-decomposable. This result generalizes the one-parameter settings of persistent homology.

Theorem 8. *Let P be a finite poset and $k[P]$ the incidence algebra of P. Then, the following conditions are equivalent.*

(a) $\text{int-res-gldim}\, k[P] = 0$.
(b) *Every $k[P]$-module is interval-decomposable.*
(c) *Each connected component of the Hasse diagram of P is one of $A_n(a)$ for some orientation a or $C_{m,\ell}$ displayed below, where the symbol $\leftrightarrow$ is either $\rightarrow$ or $\leftarrow$ assigned by its orientation a:*

$$A_n(a)\colon \qquad 1 \longleftrightarrow 2 \longleftrightarrow \cdots\cdots \longleftrightarrow n,$$

$C_{m,\ell}$:



In particular, these conditions do not depend on the characteristic of the base field k.

We note that equivalences among (a) and (b) in the statement are trivial.

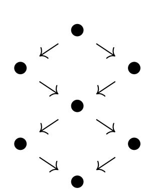
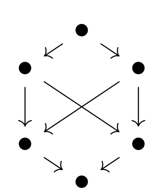
3.3. **Result 3.** Finally, we study a relationship between the interval resolution global dimensions of different posets. Our result is the following.

Theorem 9. *Let P be a finite poset and $k[P]$ the incidence algebra of P. For any full subposet P' of P, we have*

$$\text{int-res-gldim}\, k[P'] \leq \text{int-res-gldim}\, k[P]. \tag{3.1}$$

For the usual global dimension, we do not have the above monotonicity in general.

Example 10. Let P and P' be posets given by

P :  and P' : 

respectively. Then, P' is a full subposet of P, which is obtained by removing the point in the center. The global dimension of $k[P]$ is 2 but that of $k[P']$ is 3 (over an arbitrary field), see [7, Section 3].

On the other hands, we have $\operatorname{int-res-gldim} k[P'] = 2 \leq 3 = \operatorname{int-res-gldim} k[P]$ over a field with two elements.

In the rest, we give a sketch of a proof of Theorem 9. The main ingredient for its proof is a functor Θ_e defined as follows. Let A be a finite dimensional k-algebra. For a given idempotent $e \in A$, we consider the idempotent subalgebra $B := eAe$. It is well-known that the functors

$$\operatorname{Res}_e(-) := (-)e, \ \operatorname{Ind}_e(-) := - \otimes_B eA, \ \operatorname{Coind}_e(-) := \operatorname{Hom}_B(Ae, -),$$

respectively called the restriction, induction, and coinduction functors, provide a diagram

$$\begin{tikzcd} \mathsf{mod}\, A \arrow[r, "\operatorname{Res}_e"] & \mathsf{mod}\, B. \arrow[l, bend right, "\operatorname{Ind}_e"'] \arrow[l, bend left, "\operatorname{Coind}_e"] \end{tikzcd} \tag{3.2}$$

which gives an adjoint triple. Then, the identity 1_M is associated to the map θ_M by

$$\begin{array}{ccc} \operatorname{Hom}_A(\operatorname{Ind}_e(M), \operatorname{Coind}_e(M)) & \xleftarrow{\sim} & \operatorname{Hom}_B(M, M) \\ \Cup & & \Cup \\ \theta_M & \longleftarrow\!\!\shortmid & 1_M, \end{array}$$

and an A-module

$$\Theta_e(M) := \operatorname{Im} \theta_M \subseteq \operatorname{Coind}_e(M).$$

It gives rise to a functor Θ_e called *intermediate extension* in [9, Proposition 4.6], and *prolongedment intermédiare* in [4]. We have $\operatorname{Res}_e \circ \Theta_e \cong 1_{\mathsf{mod}\, B}$.

Let P be a finite poset and P' a full subposet of P. In this setting, the incidence algebra $k[P']$ can be obtained as an idempotent subalgebra. In fact, we have an isomorphism $k[P'] \cong ek[P]e$ of algebras, where $e := \sum_{x \in P'} e_x$. Due to the previous paragraph, we can define the functor $\Theta_e \colon \mathsf{mod}\, k[P'] \to \mathsf{mod}\, k[P]$.

The following is a key observation on interval modules.

Proposition 11. *The functor Θ_e sends interval modules to interval modules. More explicitly, for a given interval $I \in \mathbb{I}(P')$, we have $\Theta_e(k_I) \cong k_{\operatorname{conv}(I)}$, where $\operatorname{conv}(I)$ is the smallest interval of P containing I.*

Consequently, we find the exact functor Res_e and the functor Θ_e provides the diagram

$$\begin{array}{ccc} \mathsf{mod}\, k[P] & \underset{\operatorname{Res}_e}{\overset{\Theta_e}{\rightleftarrows}} & \mathsf{mod}\, k[P'] \\ \cup & & \cup \\ \mathsf{add}\, \mathfrak{I}_P & \underset{\operatorname{Res}_e|_{\mathsf{add}\, \mathfrak{I}_P}}{\overset{\Theta_e|_{\mathsf{add}\, \mathfrak{I}_{P'}}}{\rightleftarrows}} & \mathsf{add}\, \mathfrak{I}_{P'}, \end{array}$$

where $\mathcal{I}_P$ (resp., $\mathcal{I}_{P'}$) is the set of isomorphism classes of interval modules over P (resp., P'). Then, we can directly compare interval resolutions via these functors and obtain the following.

Proposition 12. *For any module $M \in k[P']$, we have the following inequality*

$$\mathcal{I}_{P'}\text{-res-dim}\, M \leq \mathcal{I}_P\text{-res-dim}\, \Theta_e(M). \tag{3.3}$$

Now, we are ready to prove Theorem 9.

Proof of Theorem 9. Since M is an arbitrary module in (3.3), we obtain the desired inequality (3.1) by Proposition 12. □

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