PERIODIC DIMENSIONS OF MODULES AND ALGEBRAS

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Abstract. For an eventually periodic module, we obtain the degree n and the period p of its first periodic syzygy. In this note, in order to study the degree n, we introduce the notion of the periodic dimension of a module and report results on periodic dimensions obtained so far.

1. Introduction

Throughout this note, let k be a field, and we assume that all rings are left Noetherian semiperfect rings (that are associative and unital). By a module, we mean a finitely generated left module.

Homological algebra [7] has been playing an important role in the representation theory of rings, and one of the fundamental tools is a projective resolution of a module. So it is natural to study the behavior of projective resolutions. In this note, we are concerned with eventually periodic modules (i.e., modules whose minimal projective resolutions become periodic in sufficiently large degrees) and study when their minimal projective resolutions become periodic. For this, we will introduce the notion of the *periodic dimension* of a module. From the definition, a module M is of finite periodic dimension if and only if M is eventually periodic. In this case, the value of the periodic dimension equals the degree of the first periodic syzygy of M. We first provide some of the basic properties of periodic dimensions and then investigate the relationship between Gorenstein and periodic dimensions. Moreover, motivated by a recent result of Dotsenko-Gélinas-Tamaroff [9], we determine the bimodule periodic dimension of a finite dimensional eventually periodic Gorenstein algebra.

2. Eventually periodic modules

This section recalls the definition of eventually periodic modules and some related results. Let R be a ring. For an R-module M and an integer $i \geq 0$, we denote by $\Omega^i_R(M)$ the i-th syzygy of the R-module M. It is understood that $\Omega^0_R(M) = M$.

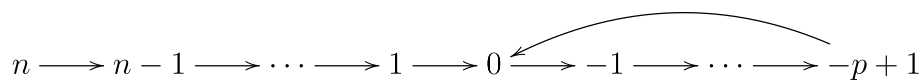
Definition 1. An R-module M is called *periodic* if there exists an integer $p > 0$ such that $\Omega^p_R(M) \cong M$ as R-modules. The smallest $p > 0$ with this property is called the *period* of M. We call M *eventually periodic* if there exists an integer $n \geq 0$ such that $\Omega^n_R(M)$ is periodic.

We say that an R-module M is (n, p)-*eventually periodic* if M is eventually periodic over R and satisfies that its n-th syzygy is the first periodic syzygy of period p. We call a $(0, p)$-eventually periodic module a p-*periodic* module.

The detailed version of this paper will be submitted for publication elsewhere.

Modules of finite projective dimension n are $(n+1, 1)$-eventually periodic. The following example exhibits (n, p)-eventually periodic modules (with infinite projective dimension).

Example 2. Fix two integers $n \geq 0$ and $p > 0$, and consider the finite dimensional radical square zero algebra $\Lambda = kQ/R_Q^2$, where Q is the following quiver:



and R_Q is the arrow ideal of the path algebra kQ. We denote by S_i the simple Λ-module associated with the vertex i. A direct calculation shows that S_i is (i, p)-eventually periodic if $1 \leq i \leq n$ and is p-periodic if $-p+1 \leq i \leq 0$. In particular, S_n is (n, p)-eventually periodic.

The integers n and p associated with an (n, p)-eventually periodic module are studied in the literature, for example [3, 6, 8, 10, 11]. We recall the following result of Avramov [3].

Theorem 3 ([3, Theorem 7.3.1])**.** *Let R be a commutative local ring, and let M be an R-module of finite complete intersection dimension. Then the following conditions are equivalent.*

(1) *M is (n, p)-eventually periodic with $n \leq \operatorname{depth} R - \operatorname{depth}_R M + 1$ and $p = 1$ or 2.*
(2) *M has bounded Betti numbers.*

Using [2, Lemma 1.2.6], one can check that any (n, p)-eventually periodic module M over a commutative local ring R satisfies that $\operatorname{depth} R - \operatorname{depth}_R M \leq n$. Thus, for any (n, p)-eventually periodic R-modules satisfying the assumption of Theorem 3, we obtain the following formula

$$\operatorname{depth} R - \operatorname{depth}_R M \leq n \leq \operatorname{depth} R - \operatorname{depth}_R M + 1. \tag{2.1}$$

3. Periodic dimensions

In this section, we will introduce the notion of the periodic dimension of a module and provide our main results. Throughout this section, let R denote a ring.

Observe that if M is a periodic module, then all its syzygies are periodic and have the same period as M. Thus it is natural to introduce the following notion.

Definition 4. Let M be an R-module. Then we define the *periodic dimension* of M by

$$\operatorname{per.dim}_R M := \inf \{\, n \geq 0 \mid \Omega_R^n(M) \text{ is periodic} \,\}.$$

By definition, M is eventually periodic if and only if $\operatorname{per.dim}_R M < \infty$. In this case, $\operatorname{per.dim}_R M$ equals the degree n of the first periodic syzygy $\Omega_R^n(M)$ of M. For instance, if M has finite projective dimension, then $\operatorname{per.dim}_R M = \operatorname{proj.dim}_R M + 1$. Also, if M is of finite periodic dimension n, then we have

$$\operatorname{per.dim}_R \Omega_R^i(M) = \begin{cases} n - i & \text{if } 0 \leq i \leq n, \\ 0 & \text{if } i > n\ . \end{cases}$$

Recall from [1, 4] that an R-module X, where R is an arbitrary ring, is called *totally reflexive* if $X \cong X^{**}$ and $\mathrm{Ext}^i_R(X,R) = 0 = \mathrm{Ext}^i_{R^{\mathrm{op}}}(X^*,R)$ for all $i > 0$, where we set $(-)^* := \mathrm{Hom}_R(-,R)$. The *Gorenstein dimension* $\mathrm{G\text{-}dim}_R M$ of an R-module M is defined to be the infimum of the length n of an exact sequence of R-modules

$$0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$

with each X_i totally reflexive. The following proposition states the property of periodic dimensions with respect to direct sums.

Proposition 5. *For any finite family $\{_R M_i\}_{i\in I}$ of R-modules, we have*

$$\mathrm{per.dim}_R \bigoplus_{i\in I} M_i \le \sup\{\, \mathrm{per.dim}_R M_i \mid i \in I \,\}$$

The equality holds if R is left artin, and $\mathrm{G\text{-}dim}_R M_i < \infty$ for all $i \in I$.

The following is our first main result.

Theorem 6. *Let M be an (n,p)-eventually periodic R-module of finite Gorenstein dimension r. Then we have $r \le n \le r+1$. If, furthermore, R is left artin, then the following assertions hold.*

1. *$n = r$ if and only if $\Omega^r_R(M)$ has no non-zero projective direct summand.*
2. *If $\Omega^{n-1}_R(M) = X \oplus Q$ for some R-module X without non-zero projective direct summand and some projective R-module Q, then $r = n-1$ if and only if $X \cong \Omega^{n+p-1}_R(M)$ as R-modules.*

Remark 7. (1) Let M be an (n,p)-eventually periodic R-module of finite complete intersection dimension, where R is a commutative local ring. Then, since we know from [3, Theorems 8.7 and 8.8] that $\mathrm{depth} R - \mathrm{depth}_R M = \mathrm{G\text{-}dim}_R M$, the obtained bounds $r \le n \le r+1$ in this case are noting but (2.1).

(2) If R is a CM-finite Gorenstein artin algebra, then any R-modules satisfy the assumption of the theorem. Here, CM-finite [5] means that there are only finitely many pairwise non-isomorphic indecomposable totally reflexive R-modules, and Gorenstein [4] means that the injective dimension of R is finite as a left and as a right R-module.

In what follows, let Λ be a finite dimensional algebra over the filed k. We say that Λ is *eventually periodic* if $\Omega^n_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}(\Lambda)$ is eventually periodic as a $\Lambda \otimes_k \Lambda^{\mathrm{op}}$-module for some $n \ge 0$. In case $n = 0$, we call Λ a *periodic* algebra. The following is a result of Dotsenko-Gélinas-Tamaroff [9].

Theorem 8 ([9, the proof of Theorem 6.3])**.** *Let Λ be a monomial Gorenstein algebra. Then $\mathrm{per.dim}_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}\Lambda$ is finite and at most $\mathrm{inj.dim}_\Lambda \Lambda + 1$, where $\mathrm{inj.dim}_\Lambda\Lambda$ stands for the injective dimension of the regular Λ-module Λ.*

Motivated by the theorem, we first obtain the following observation.

Proposition 9. *The following statements hold for a finite dimensional algebra Λ.*

1. *If Λ is eventually periodic, then $\mathrm{G\text{-}dim}_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}\Lambda < \infty$ if and only if Λ is Gorenstein.*
2. *If Λ is Gorenstein, then $\mathrm{G\text{-}dim}_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}\Lambda = \mathrm{inj.dim}_\Lambda\Lambda$.*

As a consequence of Theorem 6, we then have the following second main result of this note.

Theorem 10. *Let Λ be a finite dimensional eventually periodic Gorenstein algebra. Then we have*

$$\operatorname{inj.dim}_{\Lambda}\Lambda \leq \operatorname{per.dim}_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}\Lambda \leq \operatorname{inj.dim}_{\Lambda}\Lambda + 1.$$

Moreover, $\operatorname{per.dim}_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}\Lambda = \operatorname{inj.dim}_{\Lambda}\Lambda$ *if and only if* $\Omega^{\operatorname{inj.dim}_{\Lambda}\Lambda}_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}(\Lambda)$ *has no non-zero projective direct summand.*

We end this section by explaining that the bounds given in the theorem are the best possible.

Proposition 11 ([12, Proposition 4.3])**.** *Let Λ and Γ be finite dimensional algebras. Assume that Λ is periodic and Γ has finite global dimension d. Then the tensor product $A = \Lambda \otimes_k \Gamma$ is a Gorenstein algebra with* $\operatorname{per.dim}_{A\otimes_k A^{\mathrm{op}}}A = \operatorname{inj.dim}_{A}A$.

Example 12. Let Λ be the finite dimensional monomial algebra given by the following quiver with relations:

$$\beta \circlearrowright d \xrightarrow{\alpha_d} d-1 \xrightarrow{\alpha_{d-1}} \cdots \longrightarrow 1 \xrightarrow{\alpha_1} 0 \qquad \beta^2,\ \alpha_{i-1}\alpha_i \text{ for } 2 \leq i \leq d.$$

A direct calculation shows that Λ is a Gorenstein algebra with $\operatorname{per.dim}_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}\Lambda = \operatorname{inj.dim}_{\Lambda}\Lambda + 1$.

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