FACES OF INTERVAL NEIGHBORHOODS OF SILTING CONES

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ABSTRACT. In the study of silting complexes for a finite dimensional algebra over a field, silting cones in the real Grothendieck group play an important role. The first named author defined the interval neighborhood of each silting cone so that it is compatible with τ-tilting reduction of Jasso. The closure of the interval neighborhood is a rational polyhedral cone in the real Grothendieck group. We have obtained many important properties of the faces of this rational polyhedral cone, and explain some of them in this proceeding.

1. Introduction

The representation theory of a finite dimensional algebra A over a field K studies the categories $\mathsf{mod}\, A$ and $\mathsf{proj}\, A$ of finitely generated (projective) A-modules, and its derived categories $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ and $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$.

Derived equivalences of algebras are characterized by the existence of *tilting complexes* in the category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ introduced by Rickard [20]. Keller-Vossieck [17] generalized tilting complexes to *silting complexes*, and silting complexes are equipped with the operation called *mutation* exchanging one indecomposable direct summand of a silting complex to obtain another one [3].

Among silting complexes, *2-term silting complexes* are strongly related to functorially finite torsion pairs [1, 5, 11], which is known as part of *τ-tilting theory*. It is natural to also consider direct summands of 2-term silting complexes, which are called *2-term presilting complexes*.

In the study of (pre)silting complexes, the *Grothendieck group* $K_0(\mathsf{proj}\, A)$ is important. Actually, $K_0(\mathsf{proj}\, A)$ is nothing but the free abelian group $\bigoplus_{i=1}^{n} \mathbb{Z}[P_i]$ whose canonical basis is given by the isoclasses of indecomposable projective modules $P_1, P_2, \ldots, P_n$.

Aihara-Iyama [3] proved that the indecomposable direct summands $S_1, S_2 \ldots, S_n$ of each basic silting complex $S = \bigoplus_{i=1}^{n} S_i$ give another free basis $[S_1], [S_2], \ldots, [S_n]$ of $K_0(\mathsf{proj}\, A)$. Then, for each basic 2-term presilting complex $U = \bigoplus_{i=1}^{m} U_i$ with U_i indecomposable, we have a *silting cone*

$$C^\circ(U) := \sum_{i=1}^{m} \mathbb{R}_{>0}[U_i],$$

in the *real Grothendieck group* $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$. The silting cone $C^\circ(U)$ is m-dimensional.

By [12], silting cones give a fan in $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ so that the intersection $C(U) \cap C(U')$ of the silting cones of basic 2-term presilting complexes U and U' coincides with the silting cone $C(U'')$ of the maximum common direct summand U'' of U and U'.

The detailed version of this paper will be submitted for publication elsewhere.

In general, this fan is not necessarily complete. In other words, there can be a region in $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ where no silting cones exist. To understand such a region more, it is helpful to consider *semistable subcategories* $\mathcal{W}_\theta$ of King [18] and *semistable torsion pairs* $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta), (\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ of Baumann-Kamnitzer-Tingley [9] in $\mathrm{mod}\, A$, given by certain linear conditions on subfactors of modules in $\mathrm{mod}\, A$ for elements θ in $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$.

By using semistable subcategories, Brüstle-Smith-Treffinger [10] introduced the *wall-chamber structure* in $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ whose walls are $\Theta_M := \{\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}} \mid M \in \mathcal{W}_\theta\}$ for all nonzero modules $M \in \mathsf{mod}\, A \setminus \{0\}$. Similarly, by semistable torsion pairs, the first named author [6] defined an equivalence relation called *TF equivalence* so that θ and η are TF equivalent if $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = (\overline{\mathcal{T}}_\eta, \mathcal{F}_\eta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\mathcal{T}_\eta, \overline{\mathcal{F}}_\eta)$.

Based on results of Brüstle-Smith-Treffinger [10] and Yurikusa [21], the first named author [6] proved that the silting cone $C^\circ(U)$ for each basic 2-term presilting complex U is a TF equivalence class. The semistable torsion pairs for $\theta \in C^\circ(U)$ are the functorially finite torsion pairs for U which have already been considered in [1, 8].

Sometimes, it is difficult to deal with all 2-term (pre)silting complexes at once. Then, one of the useful methods is *τ-tilting reduction* introduced by Jasso [16]. For a fixed basic 2-term presilting complex U, Jasso constructed a finite dimensional algebra $B = B_U$, and obtained that the basic 2-term (pre)silting complexes which have U as direct summands in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ are in bijections with the basic 2-term (pre)silting complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, B)$. Moreover, Jasso also proved that $\mathcal{W}_\theta$ for $\theta \in C^\circ(U)$ is equivalent to the module category $\mathsf{mod}\, B$.

The first named author introduced a subset N_U of $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ which connects the wall-chamber structure, TF equivalence and the τ-tilting reduction at U in [6]. The set N_U is an open neighborhood of the silting cone $C^\circ(U)$, so we decided to call N_U the *interval neighborhood* of $C^\circ(U)$.

By the constrution, the closure $\overline{N_U}$ is a rational polyhedral cone in $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$, so we are currently studying the faces of $\overline{N_U}$. We will state some of our results on the faces of $\overline{N_U}$ in this proceeding.

1.1. **Notation.** In this proceeding, K is a field, and A is a finite dimensional K-algebra. The symbol $\mathsf{proj}\, A$ denotes the category of finitely generated projective A-modules, and $\mathsf{mod}\, A$ denotes the category of finitely generated A-modules.

As usual, $K_0(\mathcal{C})$ is the *Grothendieck group* of an exact category $\mathcal{C}$. The *real Grothendieck group* means the $\mathbb{R}$-vector space $K_0(\mathcal{C})_{\mathbb{R}} := K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{R}$.

The Grothendieck group $K_0(\mathsf{proj}\, A)$ is nothing but $\bigoplus_{i=1}^n \mathbb{Z}[P_i]$, where $P_1, P_2, \ldots, P_n$ are the pairwise nonisomorphic indecomposable projective modules. Thus, $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ is the Euclidean space $\bigoplus_{i=1}^n \mathbb{R}[P_i]$. Similarly, $K_0(\mathsf{mod}\, A) = \bigoplus_{i=1}^n \mathbb{Z}[L_i]$ and $K_0(\mathsf{mod}\, A)_{\mathbb{R}} = \bigoplus_{i=1}^n \mathbb{R}[L_i]$ hold, where L_i is the simple top of P_i.

With respect to the *Euler form*, $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ can be seen as the dual $\mathbb{R}$-vector space of $K_0(\mathsf{mod}\, A)_{\mathbb{R}}$ up to scalar multiples. Namely, each $\theta = \sum_{i=1}^n a_i[P_i] \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ gives the $\mathbb{R}$-linear map $K_0(\mathsf{mod}\, A)_{\mathbb{R}} \to \mathbb{R}$ such that

$$\theta\left(\sum_{i=1}^n b_i[L_i]\right) = \sum_{i=1}^n a_i b_i \dim_K \mathsf{End}_A(L_i).$$

2. Silting cones and TF equivalence

We first recall some terminology on silting cones and TF equivalence.

Let U be a complex in the homotopy category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ of bounded complexes in $\mathsf{proj}\, A$. Since $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ is Krull-Schmidt, U is isomorphic to a direct sum of the form $\bigoplus_{i=1}^{m} U_i^{\oplus s_i}$ with $U_1, U_2, \dots, U_m$ indecomposable and pairwise nonisomorphic and all $s_i \geq 1$. In this case, we set $|U| := m$, and say that U is *basic* if all $s_i = 1$.

Then, we can define (pre)silting complexes as follows.

Definition 1. [17, 5.1][3, Theorem 2.27] Let $U \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$.

(1) We say that U is *presilting* if $\mathsf{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)}(U, U[>0]) = 0$.
(2) We say that U is *silting* if U is presilting and $|U| = |A|$.

Aihara [2, Proposition 2.16] proved that any presilting complex U is a direct summand of some silting complex S. By this and [3, Theorem 2.27], if $U = \bigoplus_{i=1}^{m} U_i$ with each U_i indecomposable is presilting, then $[U_1], [U_2], \dots, [U_m] \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ are linearly independent.

We say that $U \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ is *2-term* if the terms of U except the -1st and the 0th ones are zero. The result [2, Proposition 2.16] also says that any 2-term presilting complex U is a direct summand of some 2-term silting complex S.

We set $\mathsf{2\text{-}silt}\, A$ (resp. $\mathsf{2\text{-}psilt}\, A$) as the set of basic 2-term (pre)silting complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$. Thus, it is natural to consider the following notions.

Definition 2. Let $U = \bigoplus_{i=1}^{m} U_i \in \mathsf{2\text{-}psilt}\, A$ with U_i indecomposable. Then, we set the *silting cones* $C^\circ(U), C(U) \subset K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ as

$$C^\circ(U) = \sum_{i=1}^{m} \mathbb{R}_{>0}[U_i], \quad C(U) = \sum_{i=1}^{m} \mathbb{R}_{\geq 0}[U_i].$$

We will characterize the silting cone $C^\circ(U)$ by semistable torsion pairs, which are defined as follows.

Definition 3. Let $\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$.

(1) [9, Subsection 3.1] We set the *semistable torsion pairs* $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta), (\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ in $\mathsf{mod}\, A$ by

$$\begin{aligned}
\overline{\mathcal{T}}_\theta &:= \{M \in \mathsf{mod}\, A \mid \theta(N) \geq 0 \text{ for any factor module } N \text{ of } M\},\\
\mathcal{F}_\theta &:= \{M \in \mathsf{mod}\, A \mid \theta(L) < 0 \text{ for any submodule } L \neq 0 \text{ of } M\},\\
\mathcal{T}_\theta &:= \{M \in \mathsf{mod}\, A \mid \theta(N) > 0 \text{ for any factor module } N \neq 0 \text{ of } M\},\\
\overline{\mathcal{F}}_\theta &:= \{M \in \mathsf{mod}\, A \mid \theta(L) \leq 0 \text{ for any submodule } L \text{ of } M\}.
\end{aligned}$$

(2) [18, Definition 1.1] We set $\mathcal{W}_\theta := \overline{\mathcal{T}}_\theta \cap \overline{\mathcal{F}}_\theta$, and call it the *semistable subcategory*.

The semistable subcategory $\mathcal{W}_\theta$ is a *wide subcategory* of $\mathsf{mod}\, A$; that is, closed under taking kernels, cokernels, and extensions in $\mathsf{mod}\, A$. Therefore, the interval $[\mathcal{T}_\theta, \overline{\mathcal{T}}_\theta]$ in the poset $\mathsf{tors}\, A$ of torsion classes is a *wide interval* in [7]. Moreover, $\mathcal{W}_\theta$ is an abelian length category, and hence has the Jordan-Hölder property [14, Theorem 6.2].

Then, we can define TF equivalence.

Definition 4. [6, Definition 2.13] Let $\theta, \eta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$. We say that θ and η are *TF equivalent* if $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = (\overline{\mathcal{T}}_\eta, \mathcal{F}_\eta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\mathcal{T}_\eta, \overline{\mathcal{F}}_\eta)$.

The following result based on [10, Proposition 3.27] and [21, Proposition 3.3] is fundamental in our study.

Proposition 5. [6, Proposition 3.11] *Let $U \in \mathsf{2\text{-}psilt}\, A$. Then, $C^\circ(U)$ is a TF equivalence class. For any $\theta \in C^\circ(U)$, we have*

$$(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = ({}^\perp H^{-1}(\nu U), \mathsf{Sub}\, H^{-1}(\nu U)), \quad (\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\mathsf{Fac}\, H^0(U), H^0(U)^\perp).$$

The torsion pairs in the right-hand sides are classical functorially finite ones which were in [8, Theorem 5.10]. In the terminology of [1], the module $H^{-1}(\nu U)$ is *τ^{-1}-rigid*, and the module $H^0(U)$ is *τ-rigid*. See [1, 8] for details including the definitions of the symbols.

Definition 6. Let $U \in \mathsf{2\text{-}psilt}\, A$. Then, we set

$$(\overline{\mathcal{T}}_U, \mathcal{F}_U) := ({}^\perp H^{-1}(\nu U), \mathsf{Sub}\, H^{-1}(\nu U)), \quad (\mathcal{T}_U, \overline{\mathcal{F}}_U) := (\mathsf{Fac}\, H^0(U), H^0(U)^\perp),$$
$$\mathcal{W}_U := \overline{\mathcal{T}}_U \cap \overline{\mathcal{F}}_U.$$

Thus, $\mathcal{W}_U = \mathcal{W}_\theta$ for $\theta \in C^\circ(U)$ holds, so $\mathcal{W}_U$ is a wide subcategory of $\mathsf{mod}\, A$. This was shown by [16, Theorem 3.8] without using semistable torsion pairs. See also [13, Theorem 4.12].

3. Interval neighborhoods of silting cones

For each $U \in \mathsf{2\text{-}psilt}\, A$, we set

$$\mathsf{2\text{-}psilt}_U\, A := \{V \in \mathsf{2\text{-}psilt}\, A \mid U \in \mathsf{add}\, V\}.$$

This is the subset of $\mathsf{2\text{-}psilt}\, A$ consisting all $V \in \mathsf{2\text{-}psilt}\, A$ which have U as direct summands.

To study $\mathsf{2\text{-}psilt}_U\, A$, the first named author introduced the following set.

Definition 7. [6, Subsection 4.1] Let $U \in \mathsf{2\text{-}silt}\, A$. Then, we define the *interval neighborhood N_U* of $C^\circ(U)$ by

$$\begin{aligned} N_U &:= \{\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}} \mid H^0(U) \subset \mathcal{T}_\theta,\ H^{-1}(\nu U) \subset \mathcal{F}_\theta\} \\ &= \{\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}} \mid \mathcal{T}_U \subset \mathcal{T}_\theta,\ \mathcal{F}_U \subset \mathcal{F}_\theta\}. \end{aligned}$$

We first observe the following properties.

Lemma 8. *Let $U, V \in \mathsf{2\text{-}psilt}\, A$.*

(1) [6, Lemma 4.3] *The set N_U is an open neighborhood of $C^\circ(U)$.*
(2) *The set N_U is given by finitely many linear strict inequalities.*
(3) [6, Lemma 3.13] *The following conditions are equivalent:*
   (a) *$V \in \mathsf{2\text{-}psilt}_U\, A$;*
   (b) *$\mathcal{T}_V \supset \mathcal{T}_U$ and $\mathcal{F}_V \supset \mathcal{F}_U$;*
   (c) *$C^\circ(V) \subset N_U$;*
   (d) *$N_V \subset N_U$.*

Moreover, N_U satisfies the following minimality.

Lemma 9. *Let $U \in \textsf{2-psilt}\, A$. Then, N_U is the smallest set satisfying both the following conditions:*

(a) *N_U is a neighborhood of $C^\circ(U)$;*
(b) *N_U is a union of TF equivalence classes.*

We also focus on the closure $\overline{N_U} \subset K_0(\textsf{proj}\, A)_\mathbb{R}$.

Lemma 10. *Let $U, V \in \textsf{2-psilt}\, A$.*

(1) *We have $\overline{N_U} = \{\theta \in K_0(\textsf{proj}\, A)_\mathbb{R} \mid H^0(U) \subset \overline{\mathcal{T}}_\theta,\ H^{-1}(\nu U) \subset \overline{\mathcal{F}}_\theta\}$. In particular, $\overline{N_U}$ is a union of TF equivalence classes.*
(2) *We have $\overline{N_U} \supset C(U)$.*
(3) *The set $\overline{N_U}$ is a rational polyhedral cone in $K_0(\textsf{proj}\, A)_\mathbb{R}$.*
(4) *The following conditions are equivalent:*
   (a) *$U \oplus V$ is (not necessarily basic) presilting;*
   (b) *$N_U \cap N_V \neq \emptyset$;*
   (c) *$C(V) \subset \overline{N_U}$.*

   In this case, $N_{U \oplus V} = N_U \cap N_V$ holds.

4. Faces of interval neighborhoods

Let $U \in \textsf{2-psilt}\, A$. Since $\overline{N_U}$ is a rational polyhedral cone, we study the set $\textsf{Face}\, \overline{N_U}$ of its faces. If $U = \bigoplus_{i=1}^m U_i$ with U_i indecomposable, we set $U_I := \bigoplus_{i \in I} U_i$ for each subset $I \subset \{1, 2, \dots, m\}$. We have obtained the following properties in our study.

Definition-Proposition 11. *Let $U \in \textsf{2-psilt}\, A$ and $F \in \textsf{Face}\, \overline{N_U}$. Set $I_F := \{i \in \{1, 2, \dots, m\} \mid [U_i] \notin F\}$.*

(1) *We have $F \cap C(U) = C(U/U_{I_F})$.*
(2) *If $\textsf{dim}_\mathbb{R} F = n - 1$, then $\#I_F = 1$.*
(3) *For any $I \subset \{1, 2, \dots, m\}$, we define*

$$\textsf{Face}_I\, \overline{N_U} := \{F \in \textsf{Face}\, \overline{N_U} \mid I_F = I\}.$$

Then, we have a (not necessarily convex) subset

$$\partial_I := \bigcup_{F \in \textsf{Face}_I \overline{N_U}} F = \overline{N_U} \setminus \bigcup_{i \in I} N_{U_i} \subset \overline{N_U}.$$

To explain our main results, we need to recall some results in *τ-tilting reduction*.

Fix $U \in \textsf{2-psilt}\, A$. Then, we take the unique $S \in \textsf{2-silt}\, A$ such that $\overline{\mathcal{T}}_S = \overline{\mathcal{T}}_U$. This S is called the *Bongartz completion* of U. We define a finite dimensional algebra $B = B_U$ by $B := \textsf{End}_{\textsf{K}^\textsf{b}(\textsf{proj}\, A)}(S)/\langle e \rangle$, where e is the idempotent $S \to U \to S$.

Jasso [16] proved the following results. See also [13, Theorem 4.12] and [4, Theorem 4.9].

Proposition 12. *Let $U \in \textsf{2-psilt}\, A$.*

(1) [16, Theorem 3.8] *There exists a category equivalence*

$$\Phi := \textsf{Hom}_{\textsf{D}^\textsf{b}(\textsf{mod}\, A)}(S, ?) \colon \mathcal{W}_U \to \textsf{mod}\, B.$$

(2) [16, Theorems 3.16, 4.12] *There uniquely exist bijections*

$$p\colon \ \mathsf{2\text{-}silt}_U A \to \mathsf{2\text{-}silt}\, B, \quad p\colon \ \mathsf{2\text{-}psilt}_U A \to \mathsf{2\text{-}psilt}\, B$$

such that

$$(\Phi(\overline{\mathcal{T}}_U \cap \mathcal{W}_U), \Phi(\mathcal{F}_U \cap \mathcal{W}_U)) = (\overline{\mathcal{T}}_{p(U)}, \mathcal{F}_{p(U)}),$$
$$(\Phi(\mathcal{T}_U \cap \mathcal{W}_U), \Phi(\overline{\mathcal{F}}_U \cap \mathcal{W}_U)) = (\mathcal{T}_{p(U)}, \overline{\mathcal{F}}_{p(U)}).$$

In particular, $p(S) = B$.

The first named author found the corresponding results in $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$.

Definition-Proposition 13. [6, Lemma 4.4, Theorem 4.5] *Let* $U \in \mathsf{2\text{-}psilt}\, A$. *Then, there exists an* $\mathbb{R}$*-linear surjective map* $\pi\colon K_0(\mathsf{proj}\, A)_{\mathbb{R}} \to K_0(\mathsf{proj}\, B)_{\mathbb{R}}$ *satisfying the following conditions.*

(a) *The kernel* $\mathsf{Ker}\,\pi$ *is the* $\mathbb{R}$*-vector subspace* $\mathbb{R}C(U)$ *generated by* $C(U)$.
(b) *The resriction* $\pi|_{N_U}\colon N_U \to K_0(\mathsf{proj}\, B)_{\mathbb{R}}$ *is still surjective.*
(c) *For any* $\theta \in N_U$, *we have* $\Phi(\mathcal{T}_\theta \cap \mathcal{W}_U) = \mathcal{T}_{\pi(\theta)}$ *and* $\Phi(\mathcal{F}_\theta \cap \mathcal{W}_U) = \mathcal{F}_{\pi(\theta)}$. *In particular,* π *induces a bijection*

$$\{\text{TF equivalence classes in } N_U\} \to \{\text{TF equivalence classes in } K_0(\mathsf{proj}\, B)_{\mathbb{R}}\}.$$

(d) *For any* $V \in \mathsf{2\text{-}psilt}_U A$, *we have* $\pi(C^\circ(V)) = C^\circ(p(V))$.

Then, we can state our first main result.

Theorem 14. *Let* $U = \bigoplus_{i=1}^m U_i \in \mathsf{2\text{-}psilt}\, A$ *with* U_i *indecomposable, and* $I \subset \{1, 2, \dots, m\}$. *We set*

$$\Sigma_I := \{\pi(F) \mid F \in \mathsf{Face}_I\, \overline{N_U}\}.$$

(1) *We have a bijection* $\mathsf{Face}_I\, \overline{N_U} \to \Sigma_I$ *sending* F *to* $\pi(F)$. *The inverse is given by* $\sigma \mapsto \pi^{-1}(\sigma) \cap \partial_I$.
(2) *For any* $F \in \mathsf{Face}_I\, \overline{N_U}$, *we have* $\mathsf{dim}_{\mathbb{R}}\, \pi(F) = \mathsf{dim}_{\mathbb{R}}\, F - \#I$.
(3) Σ_I *is a finite complete rational polyhedral fan in* $K_0(\mathsf{proj}\, B)_{\mathbb{R}}$.

Before stating our second main result, we prepare some notions. Since $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta), (\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ are torsion pairs in $\mathsf{mod}\, A$, for any $M \in \mathsf{mod}\, A$ and $\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$, we have unique short exact sequences

$$0 \to \bar{\mathrm{t}}_\theta M \to M \to \mathrm{f}_\theta M \to 0 \quad (\bar{\mathrm{t}}_\theta M \in \overline{\mathcal{T}}_\theta,\ \mathrm{f}_\theta M \in \mathcal{F}_\theta),$$
$$0 \to \mathrm{t}_\theta M \to M \to \bar{\mathrm{f}}_\theta M \to 0 \quad (\mathrm{t}_\theta M \in \mathcal{T}_\theta,\ \bar{\mathrm{f}}_\theta M \in \overline{\mathcal{F}}_\theta)$$

with $\mathrm{t}_\theta M \subset \bar{\mathrm{t}}_\theta M \subset M$. Moreover, we set $\mathrm{w}_\theta M := \bar{\mathrm{t}}_\theta M / \mathrm{t}_\theta M \in \mathcal{W}_\theta$. Then, we introduce the following equivalence relation.

Definition 15. Let $M \in \mathsf{mod}\, A$, and $\theta, \eta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$. Then, we say that θ and η are *M-TF equivalent* if the following conditions hold:

(a) $\mathrm{t}_\theta M = \mathrm{t}_\eta M$ and $\mathrm{w}_\theta M = \mathrm{w}_\eta M$ and $\mathrm{f}_\theta M = \mathrm{f}_\eta M$;
(b) the composition factors of $\mathrm{w}_\theta M = \mathrm{w}_\eta M$ in $\mathcal{W}_\theta$ and $\mathcal{W}_\eta$ coincide.

Moreover, we set $\Sigma(M)$ as the set of the closures of all M-TF equivalence classes.

The condition (b) seems complicated, but it is necessary to make the following property hold.

Proposition 16. *Let $M \in \mathsf{mod}\, A$. Then, $\Sigma(M)$ is a finite complete rational polyhedral fan in $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$.*

We remark that $\Sigma(M)$ coincides with the complete rational polyhedral fan $\Sigma(\mathrm{N}(M))$ in [4, Theorem 5.22] constructed from the *Newton polytope* $\mathrm{N}(M)$ of M in $K_0(\mathrm{mod}\, A)_{\mathbb{R}}$.

Now, we can state our second main result.

Theorem 17. *Let $U = \bigoplus_{i=1}^{m} U_i \in 2\text{-}\mathsf{psilt}_U\, A$ with U_i indecomposable. Then, there exist $M_1, M_2, \ldots, M_m \in \mathsf{mod}\, B$ such that, for any subset $I \subset \{1, 2, \ldots, m\}$, the rational polyhedral fans $\Sigma(\bigoplus_{i \in I} M_i)$ and Σ_I in $K_0(\mathsf{proj}\, B)_{\mathbb{R}}$ coincide.*

We sketch the construction of $M_1, M_2, \ldots, M_m$ above. We take the unique $S, T \in 2\text{-}\mathsf{silt}\, A$ such that $\overline{\mathcal{T}}_S = \overline{\mathcal{T}}_U$ and $\overline{\mathcal{F}}_S = \overline{\mathcal{F}}_U$. Then, we can prove that T is the left simultaneous mutation of S at S/U. Thus, we can decompose S, T as $S = \bigoplus_{i=1}^{n} S_i$ and $T = \bigoplus_{i=1}^{n} T_i$ so that

(a) for any $i \in \{1, 2, \ldots, m\}$, we have $S_i = U_i = T_i$; and
(b) for each $j \in \{m+1, m+2, \ldots, n\}$, there exists a triangle $S_j \to U'_j \to T_j \to S_j$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ with $S_j \to U'_j$ a minimal left $(\mathsf{add}\, U)$-approximation.

Next, we take the 2-term simple-minded collections $X = \bigoplus_{i=1}^{n} X_i$ and $Y = \bigoplus_{i=1}^{n} Y_i$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ corresponding to S, T under the bijection in [19, Theorem 6.1] and [11, Corollary 4.3]. Then, we have proved that, for each $i \in \{1, 2, \ldots, m\}$, there exists a triangle $X_i[-1] \to W_i \to Y_i \to X_i$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ with $X_i[-1] \to W_i$ a minimal left $\mathcal{W}_U$-approximation by using [15, Proposition 4.8]. Now, $M_i := \Phi(W_i)$ is the desired B-module.

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