THE SPECTRUM OF THE CATEGORY OF MAXIMAL COHEN-MACAULAY MODULES

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ABSTRACT. We introduce a topology on the set of isomorphism classes of finitely generated maximal Cohen–Macaulay modules over a commutative complete Cohen–Macaulay ring, which is analogous to the Ziegler spectrum. We then calculate the Cantor–Bendixson rank of this topological space for rings of CM_+-finite representation type.

1. INTRODUCTION

The Ziegler spectrum of an associative algebra is a topological space whose points are the isomorphism classes of indecomposable pure-injective modules, whose topology is defined in terms of positive primitive formulas over the algebra. Many studies of Ziegler spectrums are given in the context of the representation theory of algebras [1, 2, 5] and so on. In this note, we consider an analog of the Ziegler spectrum for the (stable) category of maximal Cohen-Macaulay (abbr. MCM) modules over a complete Cohen-Macaulay local ring.

Let R be complete Cohen–Macaulay local ring with algebraic residue field k. We denote by $\mathcal{C}$ the category of MCM R-modules. We denote by $\mathrm{mod}(\mathcal{C})$ the category of finitely presented contravariant additive functors and also denote by $\underline{\mathrm{mod}}(\mathcal{C})$ the full subcategory of $\mathrm{mod}(\mathcal{C})$ consisting of functors with $F(R) = 0$. We denote $\mathsf{Sp}(\mathcal{C})$ the set of isomorphism classes of the indecomposable MCM R-modules except R and 0.

For a subset $\mathcal{X}$ of $\mathsf{Sp}(\mathcal{C})$, we denote by $\Sigma(\mathcal{X})$ the subcategory of $\underline{\mathrm{mod}}(\mathcal{C})$ formed by the functors F such that $F(X) = 0$ for all $X \in \mathcal{X}$. For a subcategory $\mathcal{F}$ of $\underline{\mathrm{mod}}(\mathcal{C})$, we denote by $\gamma(\mathcal{F})$ the subset of $\mathsf{Sp}(\mathcal{C})$ satisfying $F(X) = 0$ for all $F \in \mathcal{F}$.

Theorem 1. *Then the assignment $\mathcal{X} \mapsto \gamma \circ \Sigma(\mathcal{X})$ is a is a Kuratowski closure operator on $\mathsf{Sp}(\mathcal{C})$. In particular, it induces a topology on $\mathsf{Sp}(\mathcal{C})$.*

For some specific $\mathcal{C}$, we calculate a Cantor-Bendixson rank of $\mathsf{Sp}(\mathcal{C})$ with respect to the topology. The Cantor-Bendixson rank measures the complexity of the topology. It measures how far the topology is from the discrete topology.

We say that a Cohen–Macaulay local ring is $\mathcal{C}_+$-finite if there exist only finitely many isomorphism classes of indecomposable MCM modules that are not locally free on the punctured spectrum [7].

Theorem 2. *If R is $\mathcal{C}_+$-finite then $CB(\mathsf{Sp}(\mathcal{C})) \leq 1$.*

In this talk, we consider only finitely generated modules. Previous studies have also considered infinitely generated modules, which is different from our consideration.

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2. The spectrum of the category of maximal Cohen-Macaulay modules

In this note, R is a commutative complete Cohen–Macaulay local ring with algebraic residue field k and all modules are "finitely generated" R-modules. We denote by $\mathcal{C}$ the category of maximal Cohen-Macaulay (MCM) modules.

$$\mathcal{C} = \{M \mid \mathrm{Ext}^i_R(k, M) = 0 \text{ for } i < \dim R\}$$

We denote by $\underline{\mathcal{C}}$ the stable category of $\mathcal{C}$. The objects of $\underline{\mathcal{C}}$ are the same as those of $\mathcal{C}$, the morphisms of $\underline{\mathcal{C}}$ are elements of $\underline{\mathrm{Hom}}_R(M, N) := \mathrm{Hom}_A(M, N)/P(M, N)$ for $M, N \in \underline{\mathcal{C}}$, where $P(M, N)$ denote the set of morphisms from M to N factoring through free R-modules. Since R is complete, $\mathcal{C}$, thus $\underline{\mathcal{C}}$ are Krull-Schmidt categories. That is the endomorphism ring of the indecomposable module is local.

Let us recall the full subcategory of the functor category of $\mathcal{C}$ which is called the Auslander category. The Auslander category $\mathrm{mod}(\mathcal{C})$ is the category whose objects are finitely presented contravariant additive functors from $\mathcal{C}$ to a category of abelian groups and whose morphisms are natural transformations between functors. We denote by $\underline{\mathrm{mod}}(\mathcal{C})$ the full subcategory $\mathrm{mod}(\mathcal{C})$ consisting of functors F with $F(R) = 0$. The important fact is that $\mathrm{mod}(\mathcal{C})$ and $\underline{\mathrm{mod}}(\mathcal{C})$ are abelian categories.

Remark 3. It is nothing but $\underline{\mathrm{mod}}(\mathcal{C})$ is the Aunslander category of $\underline{\mathcal{C}}$ $\mathrm{mod}(\underline{\mathcal{C}})$. Actually, the category $\underline{\mathrm{mod}}(\mathcal{C})$ is equivalent to $\mathrm{mod}(\underline{\mathcal{C}})$;

$$\mathrm{mod}(\underline{\mathcal{C}}) \to \underline{\mathrm{mod}}(\mathcal{C}); \quad F \mapsto F \circ \iota,$$

where $\iota : \mathcal{C} \to \underline{\mathcal{C}}$. See [8, Remark 4.16]. So in the rest of this note, we denote $\mathrm{mod}(\underline{\mathcal{C}})$ instead of $\underline{\mathrm{mod}}(\mathcal{C})$.

Note that every object $F \in \mathrm{mod}(\underline{\mathcal{C}})$ is obtained from a short exact sequence in $\mathcal{C}$. Namely we have the short exact sequence $0 \to N \to M \to L \to 0$ such that

$$0 \to \mathrm{Hom}_R(\ , N) \to \mathrm{Hom}_R(\ , M) \to \mathrm{Hom}_R(\ , L) \to F \to 0$$

is exact in $\mathrm{mod}(\mathcal{C})$.

Definition 4. We denote by $\mathsf{Sp}(\mathcal{C})$ the set of isomorphism classes of the indecomposable MCM R-modules except R and 0. Namely,

$$\mathsf{Sp}(\mathcal{C}) := \{\text{the indecomposable MCM } R\text{-modules except } R \text{ and } 0\}/\cong.$$

The following assignments are introduced by Krause [2].

Definition 5. The assignments

$$\Sigma : \mathsf{Sp}(\mathcal{C}) \to \mathrm{mod}(\underline{\mathcal{C}}), \quad \gamma : \mathrm{mod}(\underline{\mathcal{C}}) \to \mathsf{Sp}(\mathcal{C})$$

are defined by

$$\Sigma(\mathcal{X}) := \{F \in \mathrm{mod}(\underline{\mathcal{C}}) \mid F(X) = 0 \text{ for } \forall X \in \mathcal{X}\},$$
$$\gamma(\mathcal{F}) := \{M \in \mathsf{Sp}(\mathcal{C}) \mid F(M) = 0 \text{ for } \forall F \in \mathcal{F}\}.$$

We state several basic properties of the assignments Σ and Γ.

Lemma 6. *Let $\mathcal{X}$, $\mathcal{Y}$ be subsets of $\mathsf{Sp}(\mathcal{C})$ and $\mathcal{F}$ and $\mathcal{G}$ be subcategories of $\mathrm{mod}(\underline{\mathcal{C}})$. For the assignments Σ and γ, the following statements hold.*

(1) $\mathcal{X} \subseteq \mathcal{Y} \Rightarrow \Sigma(\mathcal{X}) \supseteq \Sigma(\mathcal{Y})$.

(2) $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \gamma(\mathcal{F}) \supseteq \gamma(\mathcal{G})$.
(3) $\mathcal{X} \subseteq \gamma \circ \Sigma(\mathcal{X})$.
(4) $\mathcal{F} \subseteq \Sigma \circ \gamma(\mathcal{F})$. *Moreover* $\gamma(\mathcal{F}) = \gamma \circ \Sigma \circ \gamma(\mathcal{F})$.
(5) $\Sigma(\mathcal{X})$ *is a Serre subcategory in* $\mathrm{mod}(\underline{\mathcal{C}})$.

This is the main theorem of this note.

Theorem 7. *The assignment* $\mathcal{X} \mapsto \gamma \circ \Sigma(\mathcal{X})$ *is a Kuratowski closure operator. That is,*
(1) $\gamma \circ \Sigma(\emptyset) = \emptyset$,
(2) $\mathcal{X} \subseteq \gamma \circ \Sigma(\mathcal{X})$,
(3) $\gamma \circ \Sigma(\mathcal{X} \cup \mathcal{Y}) = \gamma \circ \Sigma(\mathcal{X}) \cup \gamma \circ \Sigma(\mathcal{Y})$,
(4) $\gamma \circ \Sigma(\gamma \circ \Sigma(\mathcal{X})) = \gamma \circ \Sigma(\mathcal{X})$
hold for all subsets $\mathcal{X}$, $\mathcal{Y}$ *in* $\mathsf{Sp}(\mathcal{C})$.

Proof. The assertions (i), (ii), and (iv) follow from the definition and the lemma above. To show (iii), we now notice that $\underline{\mathrm{Hom}}_R(-, M) \in \mathrm{mod}(\underline{\mathcal{C}})$ for $\forall M \in \mathcal{C}$. The inclusion $\gamma \circ \Sigma(\mathcal{X} \cup \mathcal{Y}) \supseteq \gamma \circ \Sigma(\mathcal{X}) \cup \gamma \circ \Sigma(\mathcal{Y}))$ follows from the fact that $\Sigma(\mathcal{X} \cup \mathcal{Y}) = \Sigma(\mathcal{X}) \cap \Sigma(\mathcal{Y})$, and the equality is clear. To show another inclusion, we take $M \in \gamma \circ \Sigma(\mathcal{X} \cup \mathcal{Y})$. Note that M is indecomposable. Assume that $M \notin \gamma \circ \Sigma(\mathcal{X}) \cup \gamma \circ \Sigma(\mathcal{Y})$. Then there exist $F \in \Sigma(\mathcal{X})$ and $G \in \Sigma(\mathcal{Y})$ such that $F(M) \neq 0$ and $G(M) \neq 0$. We construct the functor $H \in \Sigma(\mathcal{X} \cup \mathcal{Y})$ such that $H(M) \neq 0$ by using F and G. If such a functor exists we have a contradiction because M annihilates all functors in $\Sigma(\mathcal{X} \cup \mathcal{Y})$. By Yoneda's Lemma, we have nonzero morphisms $f : \underline{\mathrm{Hom}}_R(-, M) \to F$ and $g : \underline{\mathrm{Hom}}_R(-, M) \to G$. Take a pushout diagram in $\mathrm{mod}(\underline{\mathcal{C}})$:

$$\begin{array}{ccccc}
\underline{\mathrm{Hom}}_R(-, M) & \longrightarrow & \mathrm{Im}\ f & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\mathrm{Im}\ g & \longrightarrow & H & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
0 & & 0. & &
\end{array}$$

Since $\Sigma(\mathcal{X})$ and $\Sigma(\mathcal{Y})$ are Serre subcategories, $\mathrm{Im}\ f \in \Sigma(\mathcal{X})$, $\mathrm{Im}\ g \in \Sigma(\mathcal{Y})$. This implies that $H \in \Sigma(\mathcal{X} \cup \mathcal{Y})$. From the push out diagram we obtain the exact sequence $\underline{\mathrm{Hom}}_R(-, M) \to \mathrm{Im}\ f \oplus \mathrm{Im}\ g \to H \to 0$. Since $\underline{\mathrm{End}}_R(M)$ is local, $\underline{\mathrm{End}}_R(M)$ is an indecomposable $\underline{\mathrm{End}}_R(M)$-free module. Moreover $\mathrm{Im}\ f(M)$ and $\mathrm{Im}\ g(M)$ are cyclic modules. This concludes that $H(M)$ must be nonzero. Therefore we have $H \in \Sigma(\mathcal{X} \cup \mathcal{Y})$ such that $H(M) \neq 0$. This gives the contradiction that $M \in \gamma \circ \Sigma(\mathcal{X} \cup \mathcal{Y})$, so that M is in $\gamma \circ \Sigma(\mathcal{X}) \cup \gamma \circ \Sigma(\mathcal{Y})$. □

Corollary 8. *The assignment* $\mathcal{X} \mapsto \gamma \circ \Sigma(\mathcal{X})$ *defines a topology on* $\mathsf{Sp}(\mathcal{C})$. *That is a subset* $\mathcal{X}$ *of* $\mathsf{Sp}(\mathcal{C})$ *is closed if and only if* $\gamma \circ \Sigma(\mathcal{X}) = \mathcal{X}$.

For a locally coherent category $\mathcal{G}$, a bijective correspondence between closed subsets in $\mathsf{Sp}(\mathcal{G})$ and Serre subcategories in $\mathrm{mod}(\mathcal{G})$ is given in [1, 2]. In our setting, for a Serre subcategory $\mathcal{F} \subseteq \mathrm{mod}(\underline{\mathcal{C}})$, $\mathcal{F} = \Sigma \circ \gamma(\mathcal{F})$ does not hold in general.

Example 9. Let $R = k[[x, y]]/(x^2)$. The indecomposable MCM R-modules are R, $I = (x)R$ and $I_n = (x, y^n)R$ for $n > 0$. Since $\gamma(\underline{\mathrm{Hom}}_R(-, I_n)) = \emptyset$, $\Sigma \circ \gamma(\underline{\mathrm{Hom}}_R(-, I_n)) =$

$\Sigma(\emptyset) = \mathrm{mod}(\underline{\mathcal{C}})$. However $\mathcal{S}(\underline{\mathrm{Hom}}_R(-, I_n)) \neq \mathrm{mod}(\underline{\mathcal{C}})$. Here we denote by $\mathcal{S}(\underline{\mathrm{Hom}}_R(-, I_n))$ the smallest Serre subcategory which contains $\underline{\mathrm{Hom}}_R(-, I_n)$. Since KGdim $\underline{\mathrm{Hom}}_R(-, I_n) = 1$ [6, Proposition 3.8], KGdim $\mathcal{S}(\underline{\mathrm{Hom}}_R(-, I_n)) = 1$. Note that KGdim $\underline{\mathrm{Hom}}_R(-, I) = 2$. [6, Proposition 3.11]. Hence $\underline{\mathrm{Hom}}_R(-, I) \notin \mathcal{S}(\underline{\mathrm{Hom}}_R(-, I_n))$, so that $\mathcal{S}(\underline{\mathrm{Hom}}_R(-, I_n)) \neq \mathrm{mod}(\underline{\mathcal{C}})$.

Lemma 10. *Let $X, Y \in \mathsf{Sp}(\mathcal{C})$ with $X \ncong Y$. Suppose that $\underline{\mathrm{Hom}}_R(X, Y) \neq 0$. Then $Y \notin \gamma \circ \Sigma(X)$.*

By the lemma above, one can show the following.

Proposition 11. *We have $\gamma \circ \Sigma(X) = \{X\}$ for all $X \in \mathsf{Sp}(\mathcal{C})$. Hence $\mathsf{Sp}(\mathcal{C})$ is T_1-space.*

Proof. Let $Y \in \mathsf{Sp}(\mathcal{C})$ which is not isomorphic to X. Suppose that $\underline{\mathrm{Hom}}_R(X, Y) \neq 0$. Then $Y \notin \gamma \circ \Sigma(X)$ by the lemma. Suppose that $\underline{\mathrm{Hom}}_R(X, Y) = 0$. Then $\underline{\mathrm{Hom}}_R(-, Y)$ is contained in $\Sigma(X)$ Assume that $Y \in \gamma \circ \Sigma(X)$, and in the case $\underline{\mathrm{Hom}}_R(Y, Y) = 0$. So that Y is 0 or R. This never happens since $\mathsf{Sp}(\mathcal{C})$ does not contain 0 and R. ☐

Proposition 12. *Let $M \in \mathsf{Sp}(\mathcal{C})$. M is an isolated point, that is $\{M\}$ is open, if and only if there exists an Auslander-Reiten (AR) sequence ending in M.*

Proof. If there exists an AR sequence ending in M we can consider the functor S_M which is obtained from the AR sequence. Then $\gamma(S_M) = \mathsf{Sp}(\mathcal{C}) \backslash \{M\}$ is closed, so that $\{M\}$ is open.

Suppose that M is isolated, and then $\mathsf{Sp}(\mathcal{C}) \backslash \{M\}$ is closed. Notice that $\Sigma(\mathsf{Sp}(\mathcal{C}) \backslash \{M\})$ is not empty, and take $F \in \Sigma(\mathsf{Sp}(\mathcal{C}) \backslash \{M\})$. Then $F(M) \neq 0$ and $F(N) = 0$ if $N \ncong M$. By Yoneda's lemma, we have a nonzero morphism $\rho : \underline{\mathrm{Hom}}_R(-, M) \to F$. Since $\mathrm{Im} f$ is finitely presented and a subfunctor of F, by considering $\mathrm{Im}\rho$ instead of F, we may assume that F has a presentation: $\underline{\mathrm{Hom}}_R(-, M) \to F \to 0$. Take a generator $f_1, \cdots, f_m$ of $\underline{\mathrm{rad}}_R(M, M)$ as an R-module. Then the image of $\underline{\mathrm{Hom}}_R(M, (f_1, \cdots, f_m)) : \underline{\mathrm{Hom}}_R(M, M^{\oplus m}) \to \underline{\mathrm{Hom}}_R(M, M)$ is $\underline{\mathrm{rad}}_R(M, M)$. Consider the diagram:

$$
\begin{array}{ccccc}
0 & & 0 & & \\
\uparrow & & \uparrow & & \\
H_M & \longrightarrow & F/\mathrm{Im}\rho \circ f & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \\
\underline{\mathrm{Hom}}_R(-, M) & \xrightarrow{\rho} & F & \longrightarrow & 0 \\
\uparrow{\scriptstyle f := \underline{\mathrm{Hom}}_R(-, (f_1, \cdots, f_m))} & & \uparrow & & \\
\underline{\mathrm{Hom}}_R(-, M^{\oplus m}) & \longrightarrow & \mathrm{Im}\rho \circ f & \longrightarrow & 0 \\
 & & \uparrow & & \\
 & & 0. & &
\end{array}
$$

We should remark that $F/\mathrm{Im}\rho \circ f$ is finitely presented since $\mathrm{Im}\rho \circ f$ is so. By the construction, we have $H_M(M) = \underline{\mathrm{Hom}}_R(M, M)/\underline{\mathrm{rad}}_R(M, M) \cong k$. Moreover $\rho(f(M)) =$

$\rho(\underline{\mathrm{rad}}_R(M,M)) \subseteq \mathrm{rad}_R F(M)$, so that $F/\mathrm{Im}\rho \circ f(M) = F(M)/\mathfrak{m}F(M)$. This yields that $F/\mathrm{Im}\rho \circ f$ is a simple functor and we conclude that M admits an AR sequence. □

Corollary 13. *Let R be an isolated singularity. Then the topology of $\mathsf{Sp}(\mathcal{C})$ is discrete.*

The author thanks Tsutomu Nakamura for telling him the remark below.

Remark 14. Let $\mathrm{GProj}(R)$ be a category of Gorenstein-projective R-modules and $\mathrm{GProj}(R)^c$ the full subcategory consisting of compactly generated modules. It has been studied in [5] that the Ziegler spectrum is defined by using the functor category of the stable category of $\mathrm{GProj}(R)^c$. Suppose that R is Gorenstein. Then it is shown in [5, Theorem 2.33] that we have the triangulated equivalence $\underline{\mathcal{C}} \cong \underline{\mathrm{GProj}}(R)^c$. So if R is Gorenstein, the spectrum $\mathsf{Sp}(\mathcal{C})$ is nothing but the Ziegler spectrum which is considered in [5] restricted to finitely generated ones.

3. Cantor-Bendixson rank

In this section, we calculate a Cantor-Bendixson rank of $\mathsf{Sp}(\mathcal{C})$.

Definition 15 (Cantor-Bendixson rank)**.** Let $\mathcal{T}$ be a topological space. If $x \in \mathcal{T}$ is an isolated point, then $\mathrm{C}B(x) = 0$. Put $\mathcal{T}' \subset \mathcal{T}$ is a set of the **<u>non</u>**-isolated point. Define the induced topology on $\mathcal{T}'$. Set $\mathcal{T}^{(0)} = \mathcal{T}, \mathcal{T}^{(1)} = \mathcal{T}^{(0)\prime}, \cdots, \mathcal{T}^{(n+1)} = \mathcal{T}^{(n)\prime}$. We define $\mathrm{C}B(x) = n$ if $x \in \mathcal{T}^{(n)} \backslash \mathcal{T}^{(n+1)}$ If $\exists n$ such that $\mathcal{T}^{(n+1)} = \emptyset$ and $\mathcal{T}^{(n)} \neq \emptyset$, then $\mathrm{C}B(\mathcal{T}) = n$. Otherwise $\mathrm{C}B(\mathcal{T}) = \infty$.

Example 16. Let R be a DVR (e,g. $R = k[[x]]$). Then $\mathrm{C}B(\mathrm{Spec}R) = 1$ concerning the Zariski topology. Note that $\mathrm{Spec}R = \{(0), \mathfrak{m}\}$. (0) is an isolated point since $D(f) = \{(0)\}$ for some $f \in R\backslash\{0\}$. Thus $\mathrm{Spec}R' = \{\mathfrak{m}\} = \mathrm{Spec}R^{(1)}$, and $\mathfrak{m}$ is isolated in the induced topology. In the case $R = k[[x,y]]$, you can show that $\mathrm{C}B(\mathrm{Spec}R) = \infty$. Note that $\mathrm{Spec}R' = \mathrm{Spec}R$.

By the corollary, we know $\mathsf{Sp}(\mathcal{C})$ is a discrete topology if R is an isolated singularity.

Corollary 17. *Let R be an isolated singularity. Then* $\mathrm{CB}(\mathsf{Sp}(\mathcal{C})) = 0$.

The definition of CM_+-finite is introduced in [7].

Definition 18. We say that a Cohen–Macaulay local ring R is CM_+-finite if there exist only finitely many isomorphism classes of indecomposable MCM modules that are **<u>not</u>** locally free on the punctured spectrum.

Example 19. The following rings are CM_+-finite.

1. (1) A ring which is an isolated singularity. (Thus a ring which is of finite CM-representation type.)
2. (2) A hypersurface ring which is of countable CM-representation type.

Here we say that R is of finite (countable) CM-representation type if there exists only finitely (countably) many isomorphism classes of indecomposable MCM modules.

Theorem 20. *If R is CM_+-finite then* $\mathrm{C}B(\mathsf{Sp}(\mathcal{C})) \leq 1$.

Proof. We denote by $\mathcal{C}_0$ the subset of $\mathsf{Sp}(\mathcal{C})$ consisting of modules that are locally free on the punctured spectrum and put $\mathcal{C}_+$ as $\mathsf{Sp}(\mathcal{C})\backslash\mathcal{C}_0$. For all $M \in \mathcal{C}_0$, M is an isolated point since M admits an AR sequence. Thus $\mathrm{C}B(\mathcal{C}_0) = 0$.

On the other hand, for all $M \in \mathcal{C}_+$, M is not isolated. Since R is CM_+-finite, $\mathcal{C}_+$ is a finite set. Hence, for each $M \in \mathcal{C}_+$,

$$V_M := \bigcup_{X \neq M, X \in \mathcal{C}_+}^{finite} \gamma \circ \Sigma(X)$$

is closed in $\mathsf{Sp}(\mathcal{C})$. Thus $[\mathcal{C}_+] \bigcap [\mathsf{Sp}(\mathcal{C})\backslash V_M] = \{M\}$ is open in $\mathcal{C}_+ \cap \mathsf{Sp}(\mathcal{C})$. Therefore $\mathrm{C}B(\mathsf{Sp}(\mathcal{C})) \leq 1$. □

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