QUANTUM PROJECTIVE PLANES AND BEILINSON ALGEBRAS OF 3-DIMENSIONAL QUANTUM POLYNOMIAL ALGEBRAS FOR TYPE S'

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Abstract. Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quantum polynomial algebra where E is the projective plane $\mathbb{P}^2$ or a cubic divisor in $\mathbb{P}^2$, and $\sigma \in \operatorname{Aut}_k E$. In this report, we prove that, for a Type S' algebra $A = \mathcal{A}(E, \sigma)$, where $E \subset \mathbb{P}^2$ is a union of a line and a conic meeting at two points, and $\sigma \in \operatorname{Aut}_k E$, the following conditions are equivalent: (1) The noncommutative projective plane $\mathsf{Proj}_{\mathrm{nc}} A$ is finite over its center. (2) The Beilinson algebra ∇A of A is 2-representation tame. (3) The isomorphism classes of simple 2-regular modules over ∇A are parametrized by $\mathbb{P}^2$.

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1. Quantum polynomial algebras and quantum projective spaces

This report is based on [7]. Throughout this report, let k be an algebraically closed field of characteristic 0, and all algebras are defined over k. Unless otherwise described, let A be a connected graded k-algebra finitely generated in degree 1.

In noncommutative algebraic geometry, a quantum polynomial algebra defined by Artin and Schelter [2] is a basic and important research object, which is a noncommutative analogue of a commutative polynomial algebra.

Definition 1 ([2]). A right noetherian graded algebra A is called a *d-dimensional quantum polynomial algebra* if

(i) $\operatorname{gldim} A = d < \infty$,

(ii) $\operatorname{Ext}^i_A(k, A) \cong \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d, \end{cases}$ (*Gorenstein condition*)

(iii) $H_A(t) := \sum_{i=0}^{\infty} (\dim_k A_i) t^i = (1-t)^{-d}$ (*Hilbert series*).

A right noetherian graded algebra A is called a *d-dimensional AS-regular algebra* if the above conditions (i) and (ii) hold.

Artin and Schelter [2] gave the classifications of low dimensional quantum polynomial algebras as follows: For a 1-dimensional quantum polynomial algebra A, A is isomorphic to $k[x]$ as graded algebras up to isomorphism. For a 2-dimensional quantum polynomial algebra A, A is isomorphic to

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$$k\langle x,y\rangle/(-x^2+xy-yx), \text{ or } k_\lambda[x,y] := k\langle x,y\rangle/(xy-\lambda yx) \ \ (\lambda \in k \setminus \{0\})$$

as graded algebras up to isomorphism, where $k_\lambda[x,y] \cong k_{\lambda'}[x,y]$ if and only if $\lambda' = \lambda^{\pm 1}$. Moreover, Artin and Schelter [2] proved that every 3-dimensional quantum polynomial algebra is isomorphic to one of the following algebra as graded k-algebras:

$$A \cong k\langle x,y,z\rangle/(f_1,f_2,f_3), \text{ or } A \cong k\langle x,y\rangle/(g_1,g_2),$$

where, $f_i \in k\langle x,y,z\rangle_2$ and $g_i \in k\langle x,y\rangle_3$. Note that A is a 3-dimensional quantum polynomial algebra if and only if A is a 3-dimensional quadratic AS-regular algebra ([2]).

Artin, Tate and Van den Bergh [3] found a nice correspondence between 3-dimensional quantum polynomial algebras and geometric pair (E,σ), where E is the projective plane $\mathbb{P}^2$ or a cubic divisor in $\mathbb{P}^2$, and $\sigma \in \mathrm{Aut}_k E$. So, this result allows us to write a 3-dimensional quantum polynomial algebra A as the form $A = \mathcal{A}(E,\sigma)$. This result convinced us that algebraic geometry is very useful to study even noncommutative algebras.

Let A be a right noetherian graded algebra. The category of finitely generated graded right A-modules is denoted by $\mathsf{grmod}\, A$, and the full subcategory of $\mathsf{grmod}\, A$ consisting of finite dimensional modules over k id denoted by $\mathsf{tors}\, A$.

Definition 2 ([5])**.** (1) *The noncommutative projective scheme associated to* A is defined by $\mathsf{Proj}_{\mathrm{nc}}\, A = (\mathsf{tails}\, A, \pi A)$ where $\mathsf{tails}\, A := \mathsf{grmod}\, A/\mathsf{tors}\, A$ is the quotient category, $\pi : \mathsf{grmod}\, A \to \mathsf{tails}\, A$ is the quotient functor, and $A \in \mathsf{grmod}\, A$ is regular.

(2) If A is a d-dimensional quantum polynomial algebra. Then $\mathsf{Proj}_{\mathrm{nc}}\, A$ is called *a quantum* $\mathbb{P}^{d-1}$. In particular, if A is a 3-dimensional quantum polynomial algebra, then $\mathsf{Proj}_{\mathrm{nc}}\, A$ is called *a quantum projective plane.*

Note that, if A is commutative, then $\mathsf{Proj}_{\mathrm{nc}}\, A \cong (\mathsf{mod}\, X, \mathcal{O}_X)$, $X = \mathsf{Proj}\, A$. If A is a 2-dimensional quantum polynomial algebra, then $\mathsf{Proj}_{\mathrm{nc}}\, A \cong (\mathsf{coh}\, \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$.

2. Characterization of quantum projective planes finite their centers

For a 3-dimensional quantum polynomial algebra $A = \mathcal{A}(E,\sigma)$, Artin-Tate-Van den Bergh [4] gave the following geometric characterization when A is finite over its center $Z(A)$.

Theorem 3 ([4])**.** *For a 3-dimensional quantum polynomial algebra $A = \mathcal{A}(E,\sigma)$, then A is finite over its center $Z(A)$ if and only if the order $|\sigma|$ of σ is finite.*

To prove Theorem 3, "fat points of a quantum projective plane $\mathsf{Proj}_{\mathrm{nc}}\, A$"plays an essential role. By Artin [1], if A is finite over its center and $E \neq \mathbb{P}^2$, then $\mathsf{Proj}_{\mathrm{nc}}\, A$ has a fat point, however, the converse is not true.

Definition 4. Let A be a graded algebra.

(1) *A point of* $\mathsf{Proj}_{\mathrm{nc}}\, A$ is an isomorphism class of a simple object of the form $\pi M \in \mathsf{tails}\, A$ where $M \in \mathsf{grmod}\, A$ is a graded right A-module such that $\lim_{i\to\infty} \dim_k M_i < \infty$.

(2) A point πM is called *fat* if $\lim_{i\to\infty} \dim_k M_i > 1$ In this case, M is called *a fat point module over* A.

To check the existence of a fat point, the following was introduced by Mori [12].

Definition 5 ([12])**.** For a geometric pair (E, σ) where $E \subset \mathbb{P}^{n-1}$ and $\sigma \in \mathrm{Aut}_k E$,

$$\mathrm{Aut}_k(\mathbb{P}^{n-1}, E) := \{\phi|_E \in \mathrm{Aut}_k E \mid \phi \in \mathrm{Aut}_k \mathbb{P}^{n-1}\}$$

and $\|\sigma\| := \inf\{i \in \mathbb{N}^+ \mid \sigma^i \in \mathrm{Aut}_k(\mathbb{P}^{n-1}, E)\} = \tau|_E$ for some $\tau \in \mathrm{Aut}_k \mathbb{P}^{n-1}\}$, which is called *the norm of* σ.

For a geometric pair (E, σ), $\|\sigma\| \leq |\sigma|$ holds in general.

Lemma 6 ([12], [1])**.** *Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quantum polynomial algebra. Then the following hold*:

(1) $\|\sigma\| = 1$ *if and only if* $E = \mathbb{P}^2$.
(2) $1 < \|\sigma\| < \infty$ *if and only if* $\mathsf{Proj}_{\mathrm{nc}} A$ *has a fat point.*

For a d-dimensional quantum polynomial algebra, the following holds in general:

Lemma 7 ([13], [12])**.** *Let A and A' d-dimensional quantum polynomial algebras "satisfying the condition* (G1)*, where $\mathcal{P}(A) = (E, \sigma)$ and $\mathcal{P}(A') = (E', \sigma')$", respectively. Then the following hold*:

(1) *If $A \cong A'$, then $E \cong E'$ and $|\sigma| = |\sigma'|$.*
(2) *If $\mathsf{grmod}\, A \cong \mathsf{grmod}\, A'$, then $E \cong E'$, $\|\sigma\| = \|\sigma'\|$.*
In particular, when $d = 3$, if $\mathsf{Proj}_{\mathrm{nc}} A \cong \mathsf{Proj}_{\mathrm{nc}} A'$, then $E \cong E'$ and $\|\sigma\| = \|\sigma'\|$.

We remark that Lemma 7 (2) tells us that, for a 3-dimensional quantum polynomial algebra $A = \mathcal{A}(E, \sigma)$, the norm $\|\sigma\|$ of σ is a categorical invariant in $\mathsf{Proj}_{\mathrm{nc}} A$.

Definition 8 ([12], [10])**.** Let A be a d-dimensional quantum polynomial algebra. We say that $\mathsf{Proj}_{\mathrm{nc}} A$ is *finite over its center* if there exists a d-dimensional quantum polynomial algebra A' finite over its center such that $\mathsf{Proj}_{\mathrm{nc}} A \cong \mathsf{Proj}_{\mathrm{nc}} A'$.

For a 3-dimensional quantum polynomial algebra $A = \mathcal{A}(E, \sigma)$ the author and Mori [10] proved that the following results: This is a categorical analogue of Theorem 3.

Theorem 9 ([10])**.** *If $A = \mathcal{A}(E, \sigma)$ is a 3-dimensional Calabi-Yau quantum polynomial algebra, then $\|\sigma\| = |\sigma^3|$, so the following are equivalent*:

(1) $|\sigma| < \infty$.
(2) $||\sigma|| < \infty$.
(3) *A is finite over its center.*
(4) *$\mathsf{Proj}_{\mathrm{nc}} A$ is finite over its center.*

Theorem 10 ([10])**.** *If $A = \mathcal{A}(E, \sigma)$ is a 3-dimensional quantum polynomial algebra such that $E \neq \mathbb{P}^2$, and $\nu \in \mathrm{Aut}\, A$ the Nakayama automorphism of A. Then $\|\sigma\| = |\nu^* \sigma^3|$, so the following are equivalent:*

(1) $|\nu^* \sigma^3| < \infty$.
(2) $\|\sigma\| < \infty$.
(3) *$\mathsf{Proj}_{\mathrm{nc}} A$ is finite over its center.*
(4) *$\mathsf{Proj}_{\mathrm{nc}} A$ has a fat point.*

We apply the above results of the author and Mori [10] to representation theory of finite dimensional algebras.

Definition 11 ([6])**.** Let R be a finite dimensional algebra of $\mathrm{gldim} R = d < \infty$. An autoequivalence $\nu_d \in \operatorname{Aut} \mathsf{D}^{\mathrm{b}}(\mathsf{mod} R)$ is defined by $\nu_d(M) := M \otimes^{\mathsf{L}}_R DR[-d]$ where $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R)$ is the bounded derived category of $\mathsf{mod}\, R$ and $DR := \mathrm{Hom}_k(R, k)$. If $\nu_d^{-i}(R) \in \mathsf{mod}\, R$ for all $i \in \mathbb{N}$, then R is called *d-representation infinite*. In this case, we say that a module $M \in \mathsf{mod}\, R$ is *d-regular* if $\nu_d^i(M) \in \mathsf{mod}\, R$ for all $i \in \mathbb{Z}$.

In Minamoto-Mori [11], for a d-dimensional quantum polynomial algebra A, *the Beilinson algebra ∇A of A* is defined by

$$\nabla A := \begin{pmatrix} A_0 & A_1 & \cdots & A_{d-1} \\ 0 & A_0 & \cdots & A_{d-2} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}.$$

Theorem 12 ([11])**.** *If A is a d-dimensional quantum polynomial algebra A and the Beilinson algebra ∇A of A. Then ∇A is extremely Fano of global dimension of $d-1$, and there exists an equivalence of triangulated category $\mathsf{D}^{\mathrm{b}}(\mathsf{tails\,A}) \cong \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\nabla\mathsf{A})$.*

The Beilinson algebra is a typical example of $(d-1)$-representation infinite algebra in the sense of Herschend-Iyama-Oppermann [6] ([11]). To investigate representation theory of such an algebra, it is important to classify simple $(d-1)$-regular modules.

Remark 13. (1) If A is a 2-dimensional quantum polynomial algebra, then

$$\nabla A \cong \begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix} \cong k(\bullet \rightrightarrows \bullet),$$

that is, ∇A is isomorphic to a 2-Kronecker algebra, so ∇A is a finite dimensional hereditary algebra of tame representation type. It is known that the isomorphism classes of simple regular modules over ∇A are parameterized by $\mathbb{P}^1$ (cf. [12]).

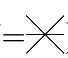
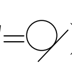
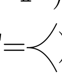
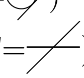
(2) For a 3-dimensional quantum polynomial algebra A, ∇A is a finite-dimensional algebra;

$$\nabla A \cong k\left(\bullet \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \bullet \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \bullet \right) \Big/ \text{(the same relations of } A).$$

Corollary 14 ([10])**.** *Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quantum polynomial algebra with the Nakayama automorphism $\nu \in \operatorname{Aut} A$. Then the following are equivalent:*

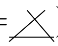
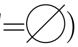
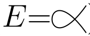
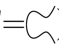
(1) $|\nu^*\sigma^3| (= \|\sigma\|) = 1$ *or* ∞.
(2) $\mathsf{Proj}_{\mathrm{nc}}\, A$ *has no fat point.*
(3) *The isomorphism classes of simple 2-regular modules over ∇A are parameterized by the set of closed points of $E \subset \mathbb{P}^2$.*

In particular, if $A = \mathcal{A}(E, \sigma)$ is one of the following types, then A satisfies all of the above conditions.

Type P ($E = \mathbb{P}^2$)	Type T ($E=$)	Type T' ($E=$)
Type CC ($E=$)	Type TL ($E=$)	Type WL ($E=$)

More precisely, if E is of Type P, then $\|\sigma\| = 1$ by Lemma 6, and if E is of Type T, Type T', Type CC, Type TL or Type WL, then $\|\sigma\|$ is infinite. The following types of

3-dimensional quantum polynomial algebras $A = \mathcal{A}(E, \sigma)$ have the case that $\|\sigma\|$ is finite.

Type S (E=) Type S' (E=) Type NC (E=) Type EC (E=)

In [10], for a 3-dimensional quantum polynomial algebra A, the author and Mori expect that the following are equivalent:

(1) $\mathsf{Proj}_{\mathrm{nc}}\, A$ is finite over its center.
(2) ∇A is 2-representation tame in the sense of Herschend-Iyama-Oppermann [6].
(3) The isomorphism classes of simple 2-regular modules over ∇A are parameterized by $\mathbb{P}^2$.

Note that these equivalences are shown for Type S in [12, Theorem 4.17, Theorem 4.21]. Do these equivalences in the above conjecture hold for Type S' in particular?

3. Main results

In this report, we prove the following results for Type S' algebra $A = \mathcal{A}(E, \sigma)$, where $E \subset \mathbb{P}^2$ is a union of a line and a conic meeting at two points, and $\sigma \in \mathrm{Aut}_k E$.

Let $A = \mathcal{A}(E, \sigma) = k\langle x, y, z\rangle/(f_1, f_2, f_3)$ be a 3-dimensional quantum polynomial algebra of Type S' where

$$\begin{cases} f_1 = yz - \alpha zy + x^2, \\ f_2 = zx - \beta xz, \\ f_3 = xy - \beta yx \quad (\alpha, \beta \in k,\ \alpha\beta^2 \neq 0, 1) \end{cases}$$

(see [8, Theorem 3.2], [9, Table 1 in Proposition 3.1]). For a 3-dimensional quantum polynomial algebra $A = \mathcal{A}(E, \sigma)$ of Type S', there exists the 3-dimensi onal Calabi-Yau quantum polynomial algebra A' of Type S' such that $\mathsf{grmod}\, A \cong \mathsf{grmod}\, A'$ so that $\mathsf{Proj}_{\mathrm{nc}}\, A \cong \mathsf{Proj}_{\mathrm{nc}}\, A'$ where $A' = \mathcal{A}(E, \sigma') = k\langle x, y, z\rangle/(g_1, g_2, g_3)$ is a 3-dimensional Calabi-Yau quantum polynomial algebra of Type S':

$$\begin{cases} g_1 = yz - \alpha zy + x^2, \\ g_2 = zx - \alpha xz, \\ g_3 = xy - \alpha yx \quad (\alpha^3 \neq 0, 1) \end{cases}$$

(see [9, Table 2 in Theorem 3.4]).

Proposition 15 ([7, Proposition 3.2]). *Let $A = \mathcal{A}(E, \sigma) = k\langle x, y, z\rangle/(g_1, g_2, g_3)$ be a 3-dimensional Calabi-Yau quantum polynomial algebra of Type S',*

where $\begin{cases} g_1 = yz - \alpha zy + x^2, \\ g_2 = zx - \alpha xz, \\ g_3 = xy - \alpha yx \quad (\alpha^3 \neq 0, 1). \end{cases}$ *Define $g := xyz + (1 - \alpha^3)^{-1}x^3 \in A_3$.*

(1) *If A is finite over its center $Z(A)$ (that is, $|\alpha|$ is finite), then $Z(A) = k[x^{|\alpha|}, y^{|\alpha|}, z^{|\alpha|}, g]$.*
(2) *If A is not finite over its center $Z(A)$ (that is, $|\alpha|$ is infinite), then $Z(A) = k[g]$.*

Theorem 16 ([12]). *Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quantum polynomial algebra. If the Beilinson algebra ∇A of A is not 2-representation tame, then the isomorphism classes of simple 2-regular modules over ∇A are parametrized by the set of points of $E \subsetneq \mathbb{P}^2$.*

Theorem 17 ([7, Theorem 4.3])**.** *Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quantum polynomial algebra of Type S'. If the Beilinson algebra ∇A of A is 2-representation tame, then the isomorphism classes of simple 2-regular modules over ∇A are parametrized by the set of points of $\mathbb{P}^2$.*

By using Proposition 15 and Theorems 16, 17, we have the following result:

Theorem 18 ([7, Theorem 4.4])**.** *For a 3-dimensional quantum polynomial algebra A of Type S', the following are equivalent*:

(1) *The noncommutative projective plane $\mathsf{Proj}_{\rm nc} A$ is finite over its center.*
(2) *The Beilinson algebra ∇A of A is 2-representation tame in the sense of Herschend, Iyama and Oppermann* [6]*.*
(3) *The isomorphism classes of simple 2-regular modules over ∇A are parameterized by $\mathbb{P}^2$.*

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