THE AUSLANDER–REITEN CONJECTURE FOR NORMAL RINGS

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ABSTRACT. In this article, we consider the Auslander–Reiten conjecture, which is a celebrated long-standing conjecture in ring theory. One of the main results of this article asserts that the conjecture holds for an arbitrary normal ring.

Key Words: Auslander–Reiten conjecture, Ext module, Serre's condition.

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1. INTRODUCTION

We refer the reader to [7] (`arXiv:2304.03956`) for details on the contents of this article. Throughout this article, we assume that R is a commutative noetherian ring and that M is a finitely generated R-module.

Auslander and Reiten [3] proposed the *generalized Nakayama conjecture*, which is rooted in the *Nakayama conjecture* [9] and asserts that for any artin algebra Λ, any indecomposable injective Λ-module appears as a direct summand in the minimal injective resolution of Λ. In addition, they proposed another conjecture, characterizing the projectivity of a module in terms of vanishing of Ext modules, which is called the *Auslander–Reiten conjecture*, and proved that this conjecture is true if and only if the generalized Nakayama conjecture is true.

The Auslander–Reiten conjecture remains meaningful for arbitrary commutative noetherian rings for formalization by Auslander, Ding, and Solberg [2]. The conjecture is known as follows: if $\operatorname{Ext}^i_R(M, M \oplus R) = 0$ for all $i \geq 1$, then M is projective. This conjecture is known to hold true if R is a complete intersection [2], or if R is a locally excellent Cohen-Macaulay normal ring containing the field of rational numbers $\mathbb{Q}$ [6], or if R is a Gorenstein normal ring [1], or if R is a Cohen–Macaulay normal ring and M is a maximal Cohen–Macaulay module of rank one [5], or if R is a Cohen–Macaulay normal ring and M is a maximal Cohen–Macaulay module such that $\operatorname{Hom}_R(M, M)$ is projective [4]. Recently, Kimura, Otake, and Takahashi [8] proved the conjecture for every Cohen-Macaulay normal ring. Even if R is not Cohen–Macaulay, it is known that R satisfies the conjecture if it is a normal ring and either $\operatorname{Ext}^i_R(\operatorname{Hom}_R(M, M), R) = 0$ for all $2 \leq i \leq \operatorname{depth} R$ or $\operatorname{Hom}_R(M, M)$ has finite G-dimension [10], or if it is a quotient of a regular ring and is a normal ring containing $\mathbb{Q}$ [4].

We give the following answer to this conjecture. We say that R satisfies *Serre's condition* (S_2) if $\operatorname{depth} R_{\mathfrak{p}} \geq \inf\{2, \operatorname{ht} \mathfrak{p}\}$ for all prime ideals $\mathfrak{p}$ of R.

The detailed version [7] of this article will be submitted for publication elsewhere.

Theorem 1. *Suppose that R satisfies (S_2). Then the Auslander–Reiten conjecture holds for R if it holds for $R_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of R such that $\operatorname{ht}\mathfrak{p} \le 1$. In particular, the Auslander–Reiten conjecture holds true for every normal ring.*

The above result is discussed in Section 2. It is worth noting that we shall prove the result of Kimura, Otake and Takahashi [8] without assuming Cohen–Macaulayness of the ring. We extend the method over Cohen–Macaulay rings to the general case, using the dualizing complex instead of the canonical module.

2. Comments on Theorem 1

In this section, we provide sufficient conditions for finitely generated modules over a commutative noetherian ring to be projective in terms of vanishing of Ext modules and prove the theorem stated in the Introduction. We prepare several lemmas to state Theorem 1. See [7] for proofs.

Lemma 2. *Let N be an R-module, and let I be an injective R-module. Then there is an isomorphism $\operatorname{Tor}_i^R(M, \operatorname{Hom}(N, I)) \cong \operatorname{Hom}(\operatorname{Ext}_R^i(M, N), I)$ for every integer $i \ge 0$.*

Lemma 3. *Let F be an R-linear functor on the category of R-modules. Let $\mathfrak{p}$ be a prime ideal of R and $E_R(R/\mathfrak{p})$ the injective hull of $R/\mathfrak{p}$. If $F(E_R(R/\mathfrak{p}))_\mathfrak{p}$ is the zero module, then so is $F(E_R(R/\mathfrak{p}))$.*

We denote by $(-)^*$ the R-dual $\operatorname{Hom}_R(-, R)$. Let R be a local ring, and let $F = (\cdots \to F_2 \to F_1 \xrightarrow{\alpha} F_0 \to 0)$ be a minimal free resolution of M. The (Auslander) transpose $\operatorname{Tr} M$ of M is defined as $\operatorname{Coker}(\alpha^*)$.

Lemma 4. *Let $(R, \mathfrak{m}, k)$ be a local ring, and let N be an R-module such that $k \otimes_R N$ is nonzero. Suppose that $\operatorname{Tor}_1^R(\operatorname{Tr} M, M \otimes_R N) = 0$. Then M is a free R-module.*

This Lemma 4 also played an important role in the proof of the main result of [8]. However, compared to [8, Proposition 3.3(1)], the assumption that N is finitely generated is removed by assuming $k \otimes_R N \neq 0$.

One of the main results of this article is the theorem below.

Theorem 5. *Let $(R, \mathfrak{m}, k)$ be a local ring of depth t. Suppose that $\operatorname{Ext}_R^i(M, R) = 0$ for all $1 \le i \le t$ and $\operatorname{Ext}_R^{t+1}(\operatorname{Tr} M, M^*) = 0$, and that M is locally free on the punctured spectrum of R. Then M is free.*

Proof. Put $d = \dim R$. We may assume that R admits a dualizing complex $D = (\cdots \to 0 \to D^0 \to \cdots \to D^{d-1} \to D^d \to 0 \to \cdots)$. Set $K = \operatorname{Ker}(D^{d-t} \to D^{d-t+1})$. It follows from Lemma 2 that $\operatorname{Tor}_{t+1}^R(\operatorname{Tr} M, M \otimes_R D^d) = 0$. Lemma 3 implies that for any $i \neq 0$ and $j \neq d$, $\operatorname{Tor}_i^R(\operatorname{Tr} M, M \otimes_R D^j) = 0$. From the above, we have $\operatorname{Tor}_1^R(\operatorname{Tr} M, M \otimes_R K) = 0$ by Lemma 2. Noting that $k \otimes_R K \neq 0$, we see that Lemma 4 concludes that M is free. $\square$

Below is a direct corollary of Theorem 5.

Corollary 6. *Let R be a local ring of depth $t \ge 2$. Suppose that M is locally free on the punctured spectrum of R. Then M is free in each of the two cases below.*
(1) $\operatorname{Ext}_R^i(M, R) = 0 = \operatorname{Ext}_R^{t-1}(M^*, M^*)$ *for all* $1 \le i \le t$.
(2) $\operatorname{Ext}_R^i(M, R) = 0 = \operatorname{Ext}_R^{t-1}(M, M)$ *for all* $1 \le i \le 2t + 1$.

We obtain Theorem 1 as a corollary of Corollary 6. Indeed, applying the case (2) of Corollary 6, we can prove by induction on $\operatorname{ht}\mathfrak{p}$ that $M_\mathfrak{p}$ is free for any prime ideal $\mathfrak{p}$ of R.

3. Comparison with previous studies

The results obtained in this article refine (or recover) a lot of results in the literature.

Remark 7. (1) Corollary 6(2) is a non-Gorenstein version of [1, Corollary 10]. Indeed, let R be a Gorenstein ring of dimension $d \geq 2$. It is seen that $\operatorname{Ext}^i_R(M,R) = 0$ for all $i > d$ and that M is maximal Cohen–Macaulay if and only if for all $1 \leq j \leq d$, $\operatorname{Ext}^j_R(M,R) = 0$.

(2) The Auslander–Reiten conjecture is known to hold for every Cohen–Macaulay normal ring by virtue of [10, Corollary 1.3]. Theorem 1 shows that the conjecture also holds for an arbitrary normal ring, i.e. it refines [10, Corollary 1.3].

(3) The Auslander–Reiten conjecture holds true if R is a quotient of a regular local ring and is a normal ring containing $\mathbb{Q}$ [4, Theorem 3.14]. In particular, every complete normal local ring of equicharacteristic zero satisfies the conjecture. Note that since the normality is not necessarily stable under completion, it is not easy to remove from these results the assumption that R is a quotient of a regular local ring or R is complete. Theorem 1, however, does make it happen.

(4) As mentioned in the introduction, besides (2) and (3) above, there are many results that show that the Auslander–Reiten conjecture holds in normal rings when some conditions are imposed; see [1, 4, 5, 6, 10] for instance. Theorem 1 improves all of them.

(5) The Auslander–Reiten conjecture is known to hold true if R is a complete intersection [2, Proposition 1.9]. Using this result, we see that if R is a complete intersection, then R satisfies (S_2) and the Auslander–Reiten conjecture holds for $R_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of R such that $\operatorname{ht}\mathfrak{p} \leq 1$. In this sense, Theorem 1 refines [2, Proposition 1.9].

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