ON INDUCTIONS AND RESTRICTIONS OF SUPPORT τ-TILTING MODULES OVER GROUP ALGEBRAS

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ABSTRACT. Let G be a finite group, k an algebraically closed field of characteristic $p > 0$, and N a normal subgroup of G. Support τ-tilting modules over group algebras are under the one-to-one correspondences with many kinds of important objects for the representation theory. We will compare a certain subset of the support τ-tilting modules over kN and that of kG, and give a poset isomorphism between these two sets. Moreover, we introduce two applications of the results.

1. MOTIVATION

Since τ-tilting theory was introduced by Adachi-Iyama-Reiten in [2], classifications and features of the support τ-tilting modules have been given for many kinds of algebras. In particular, for group algebras and their block algebras, the considerations of the support τ-tilting modules are equivalent to those of two-term tilting complexes which control derived equivalences, hence they are expected to help the solution of the Broué's Abelian Defect Group Conjecture. For that perspective, it is important to consider the support τ-tilting modules for group algebras and their block algebras.

Let k be an algebraically closed field of characteristic $p > 0$, G a finite group, N a normal subgroup of G, and X a support τ-tilting kN-module. In [4], the authors showed that if N has a cyclic Sylow p-subgroup and if the index of N in G is a power of p, then the induction functor $\mathrm{Ind}_N^G := kG \otimes_{kN} -$ gives a poset isomorphism between the set of support τ-tilting modules over kN and that over kG. Also, in [3], the first author showed that if X is G-invariant, then $\mathrm{Ind}_N^G X$ is a support τ-tilting module over kG.

Naturally, we consider the following two questions.

- For the restriction functor Res_N^G, when is $\mathrm{Res}_N^G M$ a support τ-tilting kG-module for support τ-tilting kG-module M?
- Without the assumption that N has a cyclic Sylow p-subgroup and that the index of N in G is a power of p, can we determine the image of G-invariant support τ-tilting modules over kN under the induction functor Ind_N^G?

In this report, we give some results as positive answers of the above questions. Moreover we give applications of the results.

2. MAIN RESULTS

In this section, let k be an algebraically closed field of characteristic $p > 0$, G a finite group and N a normal subgroup of G. Moreover Ind_N^G means the induction functor and

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Res_N^G means the restriction functor. For $\Lambda \in \{kN, kG\}$ and Λ-module M, we denote by $\mathrm{add}\, M$ the set of all Λ-modules which are direct summand of $M^{\oplus r}$ for some integer $r \in \mathbb{Z}$.

First, we recall the definition of support τ-tilting modules introduced by Adachi-Iyama-Reiten [2]. For a finite dimensional algebra A and an A-module M, we denote by $|M|$ the number of pairwise non-isomorphic indecomposable direct summands of M and by $\mathrm{s}(M)$ the number of pairwise non-isomorphic composition factors of M.

Definition 1 ([2])**.** Let M be an A-module.

(1) We say that the A-module M is *τ-rigid* if $\mathrm{Hom}_A(M, \tau M) = 0$, here τ means the Auslander-Reiten translation.
(2) We say that the A-module M is a *support τ-tilting module* if M is τ-rigid and if $|M| = \mathrm{s}(M)$.

Here we remark that the above definition is different from the original one, but it is equivalent definition to the original one (see [1]).

Remark 2. Let M be a A-module. If A is a symmetric algebra, then τM is isomorphic to $\Omega^2 M$. In particular, if A is a group algebra or a block algebra of a group algebra, then the isomorphism holds.

2.1. **Restricted support τ-tilting modules.** As a answer to the first question in Section 1, we have one result. We recall that the relative projectivity of kG-modules.

Definition 3. Let G be a finite group, H a subgroup of G, and M a kG-module. We say that M is *relatively H-projective* if it holds that M is a direct summand of $\mathrm{Ind}_H^G \mathrm{Res}_H^G X$.

Now we state the first result.

Theorem 4. *Let k be an algebraically closed field of characteristic $p > 0$, G a finite group, N a normal subgroup of G, and M a support τ-tilting kG-module. If M is relatively N-projective and it holds that $\mathrm{Ind}_N^G \mathrm{Res}_N^G M \in \mathrm{add}\, M$, then the restricted module $\mathrm{Res}_N^G M$ is a support τ-tilting kN-module.*

Before stating the second result, we recall G-invariances of kN-modules.

Definition 5. Let G be a finite group, N a normal subgroup, and M a kN-module. For $g \in G$, we construct a kN-module gM by the following data.

- As a set $gM := \{gm \mid m \in M\}$.
- For $x \in N$ and $gm \in gM$, the action of x is given by $x(gm) := g(g^{-1}xgm)$.

We say that M is *G-invariant* if M is isomorphic to gM as kN-modules for any $g \in G$.

The next theorem explains how strong the assumption in Theorem 4 is in a sense.

Theorem 6. *Let k be an algebraically closed field of characteristic $p > 0$, G a finite group, N a normal subgroup of G, and M a support τ-tilting kG-module. The following conditions are equivalent:*

- *The support τ-tilting kG-module M is relatively N-projective and it holds that $\mathrm{Ind}_N^G \mathrm{Res}_N^G M \in \mathrm{add}\, M$.*
- $\mathrm{add}\, M = \mathrm{add}\, \mathrm{Ind}_N^G X$ *for some G-invariant support τ-tilting kN-module X.*
- *For each simple $k(G/N)$-module S, it holds that $S \otimes_k M \in \mathrm{add}\, M$.*

2.2. **The image of the induction functor Ind_N^G.** As a answer to the second question in Section 1, we introduce one result.

For $\Lambda \in \{kG, kN\}$ and Λ-modules X and Y, we say that X is add-*equivalent* to Y if $\mathrm{add}\, X = \mathrm{add}\, Y$, and we denote the set of add-equivalence classes of support τ-tilting Λ-modules by $\mathrm{s}\tau\text{-tilt}\Lambda$. Moreover we denote the set of add-equivalence classes of G-invariant support τ-tilting kN-modules by $(\mathrm{s}\tau\text{-tilt}kN)^G$.

We know that the induction functor Ind_N^G gives a well-defined map from $(\mathrm{s}\tau\text{-tilt}kN)^G$ to $\mathrm{s}\tau\text{-tilt}kG$ by the following result.

Theorem 7 ([3, Theorem 3.2])**.** *For $M \in (\mathrm{s}\tau\text{-tilt}kN)^G$, the induced module Ind_N^G is a support τ-tilting kG-module.*

As we stated in Section 1, we wonder if we describe the image of $(\mathrm{s}\tau\text{-tilt}kN)^G$ by the induction functor explicitly. The following is one answer to this question.

Theorem 8. *Let $(\mathrm{s}\tau\text{-tilt}kG)^\star$ be the set of* add*-equivalence classes of support τ-tilting kG-modules satisfying the equivalent conditions in Theorem 6. Then the induction functor $\mathrm{Ind}_N^G M$ gives a poset isomorphism between $(\mathrm{s}\tau\text{-tilt}kN)^G$ and $(\mathrm{s}\tau\text{-tilt}kG)^\star$:*

$$\mathrm{Ind}_N^G : (\mathrm{s}\tau\text{-tilt}kN)^G \xrightarrow{\sim} (\mathrm{s}\tau\text{-tilt}kG)^\star \ \ (M \mapsto \mathrm{Ind}_N^G M).$$

In particular, the image of $(\mathrm{s}\tau\text{-tilt}kN)^G$ by the induction functor is $(\mathrm{s}\tau\text{-tilt}kG)^\star$

2.3. **Applications.** We consider the case that the quotient group G/N is a p-group. Then the only simple $k(G/N)$-module is the trivial $k(G/N)$-module up to isomorphism, here trivial $k(G/N)$-module means one dimensional vector space on which any element $\overline{g} \in G/N$ acts trivially. Moreover we can easily check that for any kG-module M and the trivial $k(G/N)$-module $k_{G/N}$, the isomorphism $k_{G/N} \otimes_k M \cong M$ holds. By using these facts and Theorem 6, we have the following.

Theorem 9. *Let G/N be a p-group. Then we have the following isomorphism of the partially ordered sets by the induction functor Ind_N^G:*

$$\mathrm{Ind}_N^G : (\mathrm{s}\tau\text{-tilt}kN)^G \xrightarrow{\sim} \mathrm{s}\tau\text{-tilt}kG \ \ (M \mapsto \mathrm{Ind}_G^{\tilde{G}} M).$$

As a further application, we consider the vertex of an indecomposable τ-rigid kG-module. We recall the definition of the vertices of the indecomposable kG-modules.

Definition 10. Let M be an indecomposable kG-module. We say that a subgroup H of G is a *vertex* of M if H is a minimal subgroup of G with the property that M is relatively H-projective.

It is known that a vertex is unique up to conjugacy, and a p-subgroup of G. Also, a vertex of the trivial kG-module is a Sylow p-subgroup. We consider vertices of indecomposable τ-rigid kG-modules, and we have the following result by using Theorem 9.

Theorem 11. *Let G be a finite group and k an algebraically closed field of characteristic $p > 0$. Then any indecomposable τ-rigid kG-module has a vertex contained in a Sylow p-subgroup properly if and only if G has a proper normal subgroup of p-power index.*

References

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