CLASSIFICATION OF TWISTED ALGEBRAS OF 3-DIMENSIONAL SKLYANIN ALGEBRAS

MASAKI MATSUNO

Abstract. A twisting system is one of the major tools to study graded algebras, however, it is often difficult to construct a (non-algebraic) twisting system if a graded algebra is given by generators and relations. In this paper, we show that a twisted algebra of a geometric algebra is determined by a certain automorphism of its point variety. As an application, we classify twisted algebras of 3-dimensional Sklyanin algebras up to graded algebra isomorphism.

1. Introduction

This paper is based on [5]. The notion of twisting system was introduced by Zhang in [8]. If there is a twisting system $\theta = \{\theta_n\}_{n \in \mathbb{Z}}$ for a graded algebra A, then we can define a new graded algebra A^θ, called a twisted algebra. Zhang gave a necessary and sufficient algebraic condition via a twisting system when two categories of graded right modules are equivalent ([8, Theorem 3.5]). Although a twisting system plays an important role to study a graded algebra, it is often difficult to construct a twisting system on a graded algebra if it is given by generators and relations.

Mori introduced the notion of geometric algebra $\mathcal{A}(E, \sigma)$ which is determined by a geometric data which consists of a projective variety E, called the point variety, and its automorphism σ. For these algebras, Mori gave a necessary and sufficient geometric condition when two categories of graded right modules are equivalent ([6, Theorem 4.7]). By using this geometric condition, we can easily construct a twisting system.

Cooney and Grabowski defined a groupoid whose objects are geometric noncommutative projective spaces and whose morphisms are isomorphisms between them. By studying a correspondence between the morphisms of this groupoid and a twisting system, they showed that the morphisms of this groupoid are parametrized by a set of certain automorphisms of the point variety ([1, Theorem 28]).

In this paper, we focus on studying a twisted algebra of a geometric algebra $\mathcal{A}(E, \sigma)$. For a twisting system θ on A, we set $\Phi(\theta) := \overline{(\theta_1|_{A_1})^*} \in \operatorname{Aut}_k \mathbb{P}(A_1^*)$ by dualization and projectivization. We find a subset $M(E, \sigma)$ of $\operatorname{Aut}_k \mathbb{P}(A_1^*)$ parametrizing twisted algebras of A up to isomorphism. We show that a twisted algebra of a geometric algebra is determined by a certain automorphism of its point variety. As an application, we classify twisted algebras of 3-dimensional Sklyanin algebras up to graded algebra isomorphism.

The detailed version of this paper is [5].

2. Twisting systems and twisted algebras

Throughout this paper, we fix an algebraically closed field k of characteristic zero and assume that a graded algebra is an $\mathbb{N}$-graded algebra $A = \bigoplus_{i\in\mathbb{N}} A_i$ over k. A graded algebra $A = \bigoplus_{i\in\mathbb{N}} A_i$ is called *connected* if $A_0 = k$. Let $\operatorname{GrAut}_k A$ denote the group of graded algebra automorphisms of A. We denote by $\operatorname{GrMod} A$ the category of graded right A-modules. We say that two graded algebras A and A' are *graded Morita equivalent* if two categories $\operatorname{GrMod} A$ and $\operatorname{GrMod} A'$ are equivalent.

Definition 1. Let A be a graded algebra. A set of graded k-linear automorphisms $\theta = \{\theta_n\}_{n\in\mathbb{Z}}$ of A is called a twisting system on A if

$$\theta_n(a\theta_m(b)) = \theta_n(a)\theta_{n+m}(b)$$

for any $l, m, n \in \mathbb{Z}$ and $a \in A_m$, $b \in A_l$. The twisted algebra of A by θ, denoted by A^θ, is a graded algebra A with a new multiplication $*$ defined by

$$a * b = a\theta_m(b)$$

for any $a \in A_m$, $b \in A_l$. A twisting system $\theta = \{\theta_n\}_{n\in\mathbb{Z}}$ is called *algebraic* if $\theta_{m+n} = \theta_m \circ \theta_n$ for every $m, n \in \mathbb{Z}$.

We denote by $\operatorname{TS}(A)$ the set of twisting systems on A. Zhang [8] found a necessary and sufficient algebraic condition for $\operatorname{GrMod} A \cong \operatorname{GrMod} A'$.

Theorem 2 ([8, Theorem 3.5])**.** *Let A and A' be graded algebras finitely generated in degree 1 over k. Then $\operatorname{GrMod} A \cong \operatorname{GrMod} A'$ if and only if A' is isomorphic to a twisted algebra A^θ by a twisting system $\theta \in \operatorname{TS}(A)$.*

Definition 3. For a graded algebra A, we define

$$\begin{aligned}
\operatorname{TS}_0(A) &:= \{\theta \in \operatorname{TS}(A) \mid \theta_0 = \operatorname{id}_A\} \\
\operatorname{TS}_{\text{alg}}(A) &:= \{\theta \in \operatorname{TS}_0(A) \mid \theta \text{ is algebraic }\} \\
\operatorname{Twist}(A) &:= \{A^\theta \mid \theta \in \operatorname{TS}(A)\}/_{\cong} \\
\operatorname{Twist}_{\text{alg}}(A) &:= \{A^\theta \mid \theta \in \operatorname{TS}_{\text{alg}}(A)\}/_{\cong}.
\end{aligned}$$

Lemma 4 ([8, Proposition 2.4])**.** *Let A be a graded algebra. For every $\theta \in \operatorname{TS}(A)$, there exists $\theta' \in \operatorname{TS}_0(A)$ such that $A^\theta \cong A^{\theta'}$.*

It follows from Lemma 4 that

$$\operatorname{Twist}(A) = \{A^\theta \mid \theta \in \operatorname{TS}_0(A)\}/_{\cong},$$

so we may assume that $\theta \in \operatorname{TS}_0(A)$ to study $\operatorname{Twist}(A)$. By the definition of twisting system, it follows that $\theta \in \operatorname{TS}_{\text{alg}}(A)$ if and only if $\theta_n = \theta_1^n$ for every $n \in \mathbb{Z}$ and $\theta_1 \in \operatorname{GrAut}_k A$, so

$$\operatorname{Twist}_{\text{alg}}(A) = \{A^\phi \mid \phi \in \operatorname{GrAut}_k A\}/_{\cong}$$

where A^ϕ means the twisted algebra of A by $\{\phi^n\}_{n\in\mathbb{Z}}$.

3. Twisted algebras of geometric algebras

Let V be a finite dimensional k-vector space and $A = T(V)/(R)$ be a quadratic algebra where $T(V)$ is a tensor algebra over k and R is a subspace of $V \otimes V$. Since an element of R defines a multilinear function on $V^* \times V^*$, we can define a zero set associated to R by

$$\mathcal{V}(R) = \{(p, q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid g(p, q) = 0 \text{ for any } g \in R\}.$$

Definition 5. Let $A = T(V)/(R)$ be a quadratic algebra. A geometric pair (E, σ) consists of a projective variety $E \subset \mathbb{P}(V^*)$ and $\sigma \in \mathrm{Aut}_k E$. We say that A is a *geometric algebra* if there exists a geometric pair (E, σ) such that

(G1) $\mathcal{V}(R) = \{(p, \sigma(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in E\}$,

(G2) $R = \{g \in V \otimes V \mid g(p, \sigma(p)) = 0 \text{ for all } p \in E\}$.

In this case, we call E the *point variety* of A, and write $A = \mathcal{A}(E, \sigma)$.

We use the following notations introduced in [1]:

Definition 6. Let $E \subset \mathbb{P}(V^*)$ be a projective variety and $\sigma \in \mathrm{Aut}_k E$. We define

$$\mathrm{Aut}_k(E \uparrow \mathbb{P}(V^*)) := \{\tau \in \mathrm{Aut}_k E \mid \tau = \overline{\tau}|_E \text{ for some } \overline{\tau} \in \mathrm{Aut}_k \mathbb{P}(V^*)\},$$
$$\mathrm{Aut}_k(\mathbb{P}(V^*) \downarrow E) := \{\tau \in \mathrm{Aut}_k \mathbb{P}(V^*) \mid \tau|_E \in \mathrm{Aut}_k E\},$$
$$Z(E, \sigma) := \{\tau \in \mathrm{Aut}_k(\mathbb{P}(V^*) \downarrow E) \mid \sigma\tau|_E\sigma^{-1} = \tau|_E\},$$
$$M(E, \sigma) := \{\tau \in \mathrm{Aut}_k(\mathbb{P}(V^*) \downarrow E) \mid (\tau|_E\sigma)^i\sigma^{-i} \in \mathrm{Aut}_k(E \uparrow \mathbb{P}(V^*)) \ \ \forall i \in \mathbb{Z}\},$$
$$N(E, \sigma) := \{\tau \in \mathrm{Aut}_k(\mathbb{P}(V^*) \downarrow E) \mid \sigma\tau|_E\sigma^{-1} \in \mathrm{Aut}_k(E \uparrow \mathbb{P}(V^*))\}.$$

Note that $Z(E, \sigma) \subset M(E, \sigma) \subset N(E, \sigma) \subset \mathrm{Aut}_k(\mathbb{P}(V^*) \downarrow E)$, and $Z(E, \sigma)$, $N(E, \sigma)$ are subgroups of $\mathrm{Aut}_k(\mathbb{P}(V^*) \downarrow E)$.

Let $A = \mathcal{A}(E, \sigma)$ be a geometric algebra. The map $\Phi : \mathrm{TS}_0(A) \to \mathrm{Aut}_k \mathbb{P}(A_1^*)$ is defined by $\Phi(\theta) := \overline{(\theta_1|_{A_1})^*}$. This map plays an important role to study twisted algebras of geometric algebras.

Lemma 7 ([5, Lemma 3.3 and Lemma 3.4])**.** *Let $A = \mathcal{A}(E, \sigma)$ be a geometric algebra.*

(1) $\Phi(\mathrm{TS}_0(A)) = M(E, \sigma)$.

(2) $\Phi(\mathrm{TS}_{\mathrm{alg}}(A)) = Z(E, \sigma)$.

The following is one of the main results.

Theorem 8 ([5, Theorem 3.5])**.** *Let $A = \mathcal{A}(E, \sigma)$ be a geometric algebra.*

(1) $\mathrm{Twist}(A) = \{\mathcal{A}(E, \tau|_E\sigma) \mid \tau \in M(E, \sigma)\}/_{\cong}$.

(2) $\mathrm{Twist}_{\mathrm{alg}}(A) = \{\mathcal{A}(E, \tau|_E\sigma) \mid \tau \in Z(E, \sigma)\}/_{\cong}$.

4. Twisted algebras of 3-dimensional Sklyanin algebras

In this section, we classify twisted algebras of 3-dimensional Sklyanin algebras. A 3-dimensional Sklyanin algebra is a typical example of 3-dimensional quadratic AS-regular algebras. It is known that every 3-dimensional Sklyanin algebra is a geometric algebra $\mathcal{A}(E, \sigma)$ where E is an elliptic curve in $\mathbb{P}^2$ and σ is a translation by some point $p \in E$.

First, we recall some properties of elliptic curves in $\mathbb{P}^2$. Let E be an elliptic curve in $\mathbb{P}^2$. We use a *Hesse form*

$$E = \mathcal{V}(x^3 + y^3 + z^3 - 3\lambda xyz)$$

where $\lambda \in k$ with $\lambda^3 \neq 1$. It is known that every elliptic curve in $\mathbb{P}^2$ can be written in this form (see [2, Corollary 2.18]). The j-invariant of a Hesse form E is given by

$$j(E) = \frac{27\lambda^3(\lambda^3+8)^3}{(\lambda^3-1)^3}$$

(see [2, Proposition 2.16]). The j-invariant $j(E)$ classifies elliptic curves in $\mathbb{P}^2$ up to projective equivalence (see [3, Theorem IV 4.1 (b)]). We fix the group structure on E with the zero element $o := (1, -1, 0) \in E$ (see [2, Theorem 2.11]). For a point $p \in E$, a *translation* by p, denoted by σ_p, is an automorphism of E defined by $\sigma_p(q) = p + q$ for every $q \in E$. We define $\operatorname{Aut}_k(E, o) := \{\sigma \in \operatorname{Aut}_k E \mid \sigma(o) = o\}$. It is known that $\operatorname{Aut}_k(E, o)$ is a finite cyclic subgroup of $\operatorname{Aut}_k E$ (see [3, Corollary IV 4.7]).

Lemma 9 ([4, Theorem 4.6]). *A generator of* $\operatorname{Aut}_k(E, o)$ *is given by*

(1) $\tau_E(a, b, c) := (b, a, c)$ *if* $j(E) \neq 0, 12^3$,

(2) $\tau_E(a, b, c) := (b, a, \varepsilon c)$ *if* $\lambda = 0$ *(so that* $j(E) = 0$*)*,

(3) $\tau_E(a, b, c) := (\varepsilon^2 a + \varepsilon b + c, \varepsilon a + \varepsilon^2 b + c, a + b + c)$ *if* $\lambda = 1 + \sqrt{3}$ *(so that* $j(E) = 12^3$*)*

where ε *is a primitive* 3*rd root of unity. In particular,* $\operatorname{Aut}_k(E, o)$ *is a subgroup of* $\operatorname{Aut}_k(E \uparrow \mathbb{P}^2) = \operatorname{Aut}_k(\mathbb{P}^2 \downarrow E)$.

Remark 10. When $j(E) = 0, 12^3$, we may fix $\lambda = 0, 1 + \sqrt{3}$ respectively as in Lemma 9, because if two elliptic curves E and E' in $\mathbb{P}^2$ are projectively equivalent, then for every $\mathcal{A}(E, \sigma)$, there exists an automorphism $\sigma' \in \operatorname{Aut}_k E'$ such that $\mathcal{A}(E, \sigma) \cong \mathcal{A}(E', \sigma')$ (see [7, Lemma 2.6]).

It follows from [4, Proposition 4.5] that every automorphism $\sigma \in \operatorname{Aut}_k E$ can be written as $\sigma = \sigma_p \tau_E^i$ where σ_p is a translation by a point $p \in E$, τ_E is a generator of $\operatorname{Aut}_k(E, o)$ and $i \in \mathbb{Z}_{|\tau_E|}$. For any $n \geq 1$, we call a point $p \in E$ *n-torsion* if $np = o$. We set $E[n] := \{p \in E \mid np = o\}$ and $T[n] := \{\sigma_p \in \operatorname{Aut}_k E \mid p \in E[n]\}$. It follows from [4, Theorem 4.12 (3)] that every automorphism $\sigma \in \operatorname{Aut}_k(\mathbb{P}^2 \downarrow E)$ can be written as $\sigma = \sigma_q \tau_E^i$ where $q \in E[3]$ and $i \in \mathbb{Z}_{|\tau_E|}$.

Let $E = \mathcal{V}(x^3 + y^3 + z^3 - 3\lambda xyz)$ be an elliptic curve in $\mathbb{P}^2$ and $p = (a, b, c) \in E \setminus E[3]$. Then $\mathcal{A}(E, \sigma_p)$ is called a 3-*dimensional Sklyanin algebra*, and

$$\mathcal{A}(E, \sigma_p) = k\langle x, y, z\rangle/(ayz + bzy + cx^2, azx + bxz + cy^2, axy + byx + cz^2).$$

Lemma 11 ([5, Lemma 4.10]). *Let* $A = \mathcal{A}(E, \sigma_p)$ *be a* 3*-dimensional Sklyanin algebra where* $p \in E \setminus E[3]$.

(1) *For* $\sigma_q \tau_E^i \in \operatorname{Aut}_k(\mathbb{P}^2 \downarrow E)$, $\sigma_q \tau_E^i \in Z(E, \sigma_p)$ *if and only if* $p - \tau_E^i(p) = o$.

(2) *For* $\sigma_q \tau_E^i \in \operatorname{Aut}_k(\mathbb{P}^2 \downarrow E)$, $\sigma_q \tau_E^i \in N(E, \sigma_p)$ *if and only if* $p - \tau_E^i(p) \in E[3]$.

(3) $M(E, \sigma_p) = N(E, \sigma_p)$.

By Theorem 8, to classify twisted algebras of 3-dimensional Sklyanin algebras $\mathcal{A}(E, \sigma_p)$ up to isomorphism of graded algebras, it is enough to classify subsets $Z(E, \sigma_p)$ and $M(E, \sigma_p)$ of $\operatorname{Aut}_k(\mathbb{P}^2 \downarrow E)$.

Theorem 12 ([5, Theorem 4.11]). *Let* $A = \mathcal{A}(E, \sigma_p)$ *be a* 3*-dimensional Sklyanin algebra. Then the following table gives* $Z(E, \sigma_p)$ *and* $M(E, \sigma_p)$*;*

Type	$j(E)$	$Z(E,\sigma_p)$	$M(E,\sigma_p)$
EC	$j(E)\neq 0,12^3$	$\begin{cases} T[3] & \textit{if } p\notin E[2] \\ \mathrm{Aut}_k(\mathbb{P}^2\downarrow E) & \textit{if } p\in E[2] \end{cases}$	$\begin{cases} T[3] & \textit{if } p\notin E[6] \\ \mathrm{Aut}_k(\mathbb{P}^2\downarrow E) & \textit{if } p\in E[6] \end{cases}$
	$j(E)=0$	$\begin{cases} T[3] & \textit{if } p\notin E[2] \\ T[3]\rtimes\langle\tau_E^3\rangle & \textit{if } p\in E[2] \end{cases}$	$\begin{cases} T[3] & \textit{if } p\notin \mathcal{E}\cup E[6] \\ T[3]\rtimes\langle\tau_E^2\rangle & \textit{if } p\in \mathcal{E} \\ T[3]\rtimes\langle\tau_E^3\rangle & \textit{if } p\in E[6] \end{cases}$
	$j(E)=12^3$	$\begin{cases} T[3] & \textit{if } p\notin E[2] \\ T[3]\rtimes\langle\tau_E^2\rangle & \textit{if } p\in E[2]\setminus\langle(1,1,\lambda)\rangle \\ \mathrm{Aut}_k(\mathbb{P}^2\downarrow E) & \textit{if } p=(1,1,\lambda) \end{cases}$	$\begin{cases} T[3] & \textit{if } p\notin E[6] \\ T[3]\rtimes\langle\tau_E^2\rangle & \textit{if } p\in E[6]\setminus\mathcal{F} \\ \mathrm{Aut}_k(\mathbb{P}^2\downarrow E) & \textit{if } p\in\mathcal{F} \end{cases}$

where $\mathcal{E}:=\{(a,b,c)\in E \mid a^9=b^9=c^9\}\subset E[9]\setminus E[6]$ *and* $\mathcal{F}:=\langle(1,1,\lambda)\rangle\oplus E[3]$.

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Katsushika division
Institute of Arts and Sciences
Tokyo University of Science
6-3-1 Niijuku, Katsushika-Ku, Tokyo, 125-8585, Japan
Email address: masaki.matsuno@rs.tus.ac.jp