THE CLASSIFICATION OF 3-DIMENSIONAL CUBIC AS-REGULAR ALGEBRAS OF TYPE P, S, T AND WL

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Abstract. Classification of AS-regular algebras is one of the most important projects in noncommutative algebraic geometry. In this paper, we extend the notion of geometric algebra to cubic algebras, and give a geometric condition for isomorphism and graded Morita equivalence. One of the main results is a complete list of defining relations of 3-dimensional cubic AS-regular algebras corresponding to $\mathbb{P}^1 \times \mathbb{P}^1$ or a union of irreducible divisors of bidegree $(1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, we classify them up to isomorphism and up to graded Morita equivalence in terms of their defining relations.

1. Artin-Schelter regular algebras

Throughout this report, let k be an algebraically closed field of characteristic 0, A a graded algebra finitely generated in degree 1 over k. That is, $A = k\langle x_1, \cdots, x_n\rangle/I$ where $\deg x_i = 1$ for any $i = 1, \cdots, n$, and I is a homogeneous two-sided ideal of $k\langle x_1, \cdots, x_n\rangle$ with $I_0 = I_1 = 0$. We call $A = \langle x_1, \cdots, x_n\rangle/I$ a *cubic algebra* if I is an two-sided ideal of $k\langle x_1, \cdots, x_n\rangle$ generated by homogeneous polynomials of degree three. We denote by $\operatorname{GrMod} A$ the category of graded right A-modules and graded right A-module homomorphisms preserving degrees. We say that two graded algebras A and A' are *graded Morita equivalent* if the categories $\operatorname{GrMod} A$ and $\operatorname{GrMod} B$ are equivalent.

Let A be a graded algebra. We recall that

$$\operatorname{GKdim} A := \inf\{\alpha \in \mathbb{R} \mid \dim(\textstyle\sum_{i=0}^{n} A_i) \le n^{\alpha} \text{ for all } n \gg 0\}$$

is called the *Gelfand-Kirillov dimension* of A. In noncommutative algebraic geometry, Artin-Schelter regular algebras are main objects to study.

Definition 1 ([1]). A graded algebra A is called a *d-dimensional Artin-Schelter regular* (*simply AS-regular*) *algebra* if A satisfies the following conditions:

(1) $\operatorname{gldim} A = d < \infty$,
(2) $\operatorname{GKdim} A < \infty$,
(3) $\operatorname{Ext}^i_A(k, A) = \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

It follows from [1, Theorem 1.5 (i)] that a 3-dimensional AS-regular algebra A finitely generated in degree 1 over k is one of the following forms:

$$A = k\langle x, y, z\rangle/(f_1, f_2, f_3)$$

where f_i are homogeneous polynomials of degree 2 (quadratic case), or

$$A = k\langle x, y\rangle/(g_1, g_2)$$

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where g_j are homogeneous polynomials of degree 3 (cubic case). In this report, we focus on studying 3-dimensional cubic AS-regular algebras.

2. 3-GEOMETRIC ALGEBRAS

Artin, Tate and Van den Bergh [2] found a nice one-to-one correspondence between 3-dimensional AS-regular algebras and pairs (E, σ) where E is a scheme and σ is an automorphism of E. Focusing on pairs (E, σ), Mori introduced the notion of geometric algebra which determines and is determined by a pair (E, σ) (see [3, Definition 4.3]). In this report, we extend the notion of geometric algebra to cubic algebras.

Let $A = k\langle x_1, \cdots, x_n\rangle/(R)$ be a cubic algebra where R is a subspace of $k\langle x_1, \cdots, x_n\rangle_3$. We denote by $\mathbb{P}^{n-1}$ the projective space of dimension $n-1$ over k. We define the zero set of R by

$$\mathcal{V}(R) := \{(p, q, r) \in (\mathbb{P}^{n-1})^{\times 3} \mid f(p, q, r) = 0 \ \ \forall f \in R\}.$$

Let $E \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ be a projective variety and $\pi_i : \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ i-th projections where $i = 1, 2$. We set the following notation:

$$\operatorname{Aut}_k^G E := \{\sigma \in \operatorname{Aut}_k E \mid \pi_1\sigma(p, q) = \pi_2(p, q) \ \ \forall (p, q) \in E\}.$$

We say that a pair (E, σ) is a 3-*geometric pair* if $\sigma \in \operatorname{Aut}_k^G E$.

Definition 2. Let $A = k\langle x_1, \cdots, x_n\rangle/(R)$ be a cubic algebra. We say that A is a 3-*geometric algebra* if there exists 3-geometric pair (E, σ) such that

(**G1**) $\mathcal{V}(R) = \{(p, q, \pi_2\sigma(p, q)) \mid (p, q) \in E\}$,

(**G2**) $R = \{f \in k\langle x_1, \cdots, x_n\rangle_3 \mid f(p, q, \pi_2\sigma(p, q)) = 0 \ \ \forall (p, q) \in E\}$.

In this case, we write $A = \mathcal{A}(E, \sigma)$.

The following theorem tells us that classifying geometric algebras is equivalent to classifying 3-geometric pairs.

Theorem 3 (cf. [4, Lemma 2.5]). *Let $A = \mathcal{A}(E, \sigma)$ and $A' = \mathcal{A}(E', \sigma')$ be 3-geometric algebras.*

(1) *$A \cong A'$ as graded algebras if and only if there exists an automorphism τ of $\mathbb{P}^{n-1}$ such that $(\tau \times \tau)(E) = E'$ and the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\tau\times\tau} & E' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma'} \\ E & \xrightarrow[\tau\times\tau]{} & E' \end{array}$$

commutes.

(2) *$\operatorname{GrMod} A \cong \operatorname{GrMod} A'$ if and only if there exists a sequence $\{\tau_i\}_{i\in\mathbb{Z}}$ of automorphisms of $\mathbb{P}^{n-1}$ such that $(\tau_i \times \tau_{i+1})(E) = E'$ and the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\tau_i\times\tau_{i+1}} & E' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma'} \\ E & \xrightarrow[\tau_{i+1}\times\tau_{i+2}]{} & E' \end{array}$$

commutes for all $i \in \mathbb{Z}$.

Definition 4. Let E and E' be projective varieties in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$.

(1) We say that E and E' are *equivalent*, denoted by $E \sim E'$, if $E' = (\tau_1 \times \tau_2)(E)$ for some $\tau_1, \tau_2 \in \mathrm{Aut}_k \mathbb{P}^{n-1}$.

(2) We say that E and E' are 2-*equivalent*, denoted by $E \sim_2 E'$, if $E' = (\tau \times \tau)(E)$ for some $\tau \in \mathrm{Aut}_k \mathbb{P}^{n-1}$.

It is clear that if E and E' are 2-equivalent, then they are equivalent. Let $A = \mathcal{A}(E, \sigma)$ and $A' = \mathcal{A}(E', \sigma')$ be 3-geometric algebras. If $A \cong A'$ (resp.GrMod $A \cong$ GrMod A'), then E and E' are 2-equivalent (resp.equivalent) by Theorem 3. As the first step of classification of geometric algebras up to graded algebra isomorphism (resp.graded Morita equivalence), we need to classify projective varieties up to 2-equivalence (resp.equivalence).

3. Main results

In [2], Artin-Tate-Van den Bergh found a nice geometric characterization of 3-dimensional AS-regular algebras finitely generated in degree 1 over k. In this report, we focus on the cubic case.

Theorem 5 ([3])**.** *Every 3-dimensional cubic AS-reguar algebra A is a 3-geometric algebra $A = \mathcal{A}(E, \sigma)$. Moreover, E is $\mathbb{P}^1 \times \mathbb{P}^1$ or a divisor of bidegree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$.*

In this report, we study two cases when $E = \mathbb{P}^1 \times \mathbb{P}^1$ and E is a union of two irreducible divisors of bidegree $(1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. For each case, we

(I) give a complete list of defining relations of 3-dimensional cubic AS-regular algebras,

(II) classify them up to isomorphism as graded algebras in terms of their defining relations, and

(III) classify them up to graded Morita equivalence in terms of their defining relations.

We first treat the case $E = \mathbb{P}^1 \times \mathbb{P}^1$. We denote by ν an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by $\nu(p, q) = (q, p)$ for $(p, q) \in \mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 6. $\mathrm{Aut}_k^G(\mathbb{P}^1 \times \mathbb{P}^1) = \{(\mathrm{id} \times \rho)\nu \mid \rho \in \mathrm{Aut}_k \mathbb{P}^1\}$.

Example 7. For $\rho \in \mathrm{Aut}_k \mathbb{P}^1 (\cong \mathrm{PGL}_2(k))$, we set

$$A_\rho := \mathcal{A}(\mathbb{P}^1 \times \mathbb{P}^1, (\mathrm{id} \times \rho)\nu).$$

By Theorem 3 (1), $A_\rho \cong A_{\rho'}$ if and only if there exists $\tau \in \mathrm{Aut}_k \mathbb{P}^1$ such that the diagram

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\tau \times \tau} & \mathbb{P}^1 \times \mathbb{P}^1 \\ {\scriptstyle (\mathrm{id} \times \rho)\nu} \downarrow & & \downarrow {\scriptstyle (\mathrm{id} \times \rho')\nu} \\ \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow[\tau \times \tau]{} & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

commutes if and only if there exists $\tau \in \mathrm{Aut}_k \mathbb{P}^1$ such that $\rho' \tau = \tau \rho$. Hence, A_ρ is isomorphic to A_{ρ_λ} or A_{ρ_J} where $\lambda \in k \setminus \{0\}$, $\rho_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ and $\rho_J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Moreover, $A_{\rho_\lambda} \cong A_{\rho_{\lambda'}}$ if and only if $\lambda' = \lambda^{\pm 1}$.

We next treat the case when E is a union of two irreducible divisors of bidegree $(1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 8. *Let $C = \mathcal{V}(f) \subset \mathbb{P}^1 \times \mathbb{P}^1$ where $f \in k[x_1, y_1] \circ k[x_2, y_2]$. Assume that C is irreducible. Then C is a divisor of bidegree $(1, 1)$ if and only if there exists $\tau \in \mathrm{Aut}_k \mathbb{P}^1$ such that $C = C_\tau := \{(p, \tau(p)) \mid p \in \mathbb{P}^1\}$.*

By Lemma 8, if E is a union of two irreducible divisors of bidegree $(1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, then $E = C_{\tau_1} \cup C_{\tau_2}$ for some $\tau_i \in \mathrm{Aut}_k \mathbb{P}^1$ $(i = 1, 2)$. The following result is one of our main results.

Theorem 9. *Let $E = C_{\tau_1} \cup C_{\tau_2}$ be a union of two irreducible divisors of bidegree $(1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Then one of the following statements holds:*

(1) $|C_{\tau_1} \cap C_{\tau_2}| = 2$ *(if and only if* $\tau_2^{-1}\tau_1 \sim \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ *for some* $\lambda \in k \setminus \{0, 1\}$*),*

(2) $|C_{\tau_1} \cap C_{\tau_2}| = 1$ *(if and only if* $\tau_2^{-1}\tau_1 \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$*),*

(3) $|C_{\tau_1} \cap C_{\tau_2}| = \infty$ *(if and only if* $\tau_2^{-1}\tau_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$*).*

In this report, we define the types of 3-geometric pairs (E, σ) as follows:

(1) Type P: $E = \mathbb{P}^1 \times \mathbb{P}^1$ and $\sigma = (\mathrm{id} \times \tau)\nu \in \mathrm{Aut}_k^G(\mathbb{P}^1 \times \mathbb{P}^1)$ (Type P is divided into Type P_i $(i = 1, 2)$ in terms of the Jordan canonical form of τ).

(2) Type S: $E = C_{\tau_1} \cup C_{\tau_2}$ is a union of two irreducible divisors of bidegree $(1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ such that $|C_{\tau_1} \cap C_{\tau_2}| = 2$. Type S is divided into Type S_i $(i = 1, 2)$; Type S_1: σ fixes each components and Type S_2: σ switches each components.

(3) Type T: $E = C_{\tau_1} \cup C_{\tau_2}$ is a union of two irreducible divisors of bidegree $(1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ such that $|C_{\tau_1} \cap C_{\tau_2}| = 1$. Type T is divided into Type T_i $(i = 1, 2)$; Type T_1: σ fixes each components and Type T_2: σ switches each components.

(4) Type WL: $E = C_{\tau_1} \cup C_{\tau_2}$ is a union of two irreducible divisors of bidegree $(1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ such that $|C_{\tau_1} \cap C_{\tau_2}| = \infty$. Type WL is divided into Type WL_i $(i = 1, 2)$ in terms of the Jordan canonical form of $\tau_1 (= \tau_2)$.

The following theorem lists all possible defining relations of algebras in each type up to isomorphism of graded algebra.

Theorem 10. *Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional cubic AS-regular algebra. For each type the following table describes*

(I) the defining relations of A, and

(II) the conditions to be isomorphic in terms of their defining relations.

Moreover, every algebra listed in the following table is AS-regular. In the following table, if $X \neq Y$ or $i \neq j$, then Type X_i algebra is not isomorphic to any Type Y_j algebra.

Type	*(I) defining relations* $(\alpha, \beta \in k)$	*(II) condition to be graded algebra isomorphic*
P_1	$\begin{cases} x^2y - \alpha yx^2, \\ xy^2 - \alpha y^2x \quad (\alpha \neq 0) \end{cases}$	$\alpha' = \alpha^{\pm 1}$
P_2	$\begin{cases} x^2y - yx^2 + yxy, \\ xy^2 - y^2x + y^3 \end{cases}$	————————

S_1	$\begin{cases} \alpha\beta x^2y + (\alpha+\beta)xyx + yx^2, \\ \alpha\beta xy^2 + (\alpha+\beta)yxy + y^2x \end{cases}$ *($\alpha\beta \neq 0,\ \alpha^2 \neq \beta^2$)*	$\{\alpha', \beta'\} = \{\alpha, \beta\}, \{\alpha^{-1}, \beta^{-1}\}$
S_2	$\begin{cases} xy^2 + y^2x + (\alpha+\beta)x^3, \\ x^2y + yx^2 + (\alpha^{-1}+\beta^{-1})y^3 \end{cases}$ *($\alpha\beta \neq 0,\ \alpha^2 \neq \beta^2$)*	$\dfrac{\alpha'}{\beta'} = \left(\dfrac{\alpha}{\beta}\right)^{\pm 1}$
T_1	$\begin{cases} x^2y - 2xyx + yx^2 - 2(2\beta-1)yxy \\ +2(2\beta-1)xy^2 + 2\beta(\beta-1)y^3, \\ xy^2 - 2yxy + y^2x \end{cases}$	$\beta' = \beta, 1-\beta$
T_2	$\begin{cases} x^2y + 2xyx + yx^2 + 2y^3, \\ xy^2 + 2yxy + y^2x \end{cases}$	————————
WL_1	$\begin{cases} \alpha^2xy^2 + y^2x - 2\alpha yxy, \\ yx^2 + \alpha^2x^2y - 2\alpha xyx \end{cases}$ *($\alpha \neq 0$)*	$\alpha' = \alpha^{\pm 1}$
WL_2	$\begin{cases} xy^2 + y^2x - 2yxy, \\ 4xy^2 + 2y^3 + yx^2 + x^2y \\ \qquad -4yxy - 2xyx \end{cases}$	————————

The following theorem lists all possible defining relations of algebras in each type up to graded Morita equivalence.

Theorem 11. *Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional cubic AS-regular algebra. For each type the following table describes*
(I) the defining relations of A, and
(III) the conditions to be graded Morita equivalent in terms of their defining relations. Moreover, every algebra listed in the following table is AS-regular. In the following table, if $X \neq Y$, then Type X algebra is not graded Morita equivalent to any Type Y algebra.

Type	*(I) defining relations* *($\alpha, \beta \in k$)*	*(III) condition to be graded Morita equivalent*
P	$\begin{cases} x^2y - yx^2, \\ xy^2 - y^2x \end{cases}$	————————
S	$\begin{cases} \alpha\beta x^2y + (\alpha+\beta)xyx + yx^2, \\ \alpha\beta xy^2 + (\alpha+\beta)yxy + y^2x \end{cases}$ *($\alpha\beta \neq 0,\ \alpha^2 \neq \beta^2$)*	$\dfrac{\alpha'}{\beta'} = \left(\dfrac{\alpha}{\beta}\right)^{\pm 1}$
T	$\begin{cases} x^2y + yx^2 + 2xy^2 \\ \quad -2xyx - 2yxy, \\ xy^2 + y^2x - 2yxy \end{cases}$	————————

WL	$\begin{cases} xy^2 + y^2x - 2yxy \\ yx^2 + x^2y - 2xyx \end{cases}$	————————

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