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Edited by
Kenta Ueyama
Shinshu University

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Organizing Committee of The Symposium on Ring Theory and Representation Theory

The Symposium on Ring Theory and Representation Theory has been held annually in Japan and the Proceedings have been published by the organizing committee. The first Symposium was organized in 1968 by H. Tominaga, H. Tachikawa, M. Harada and S. Endo. After their retirement a new committee has been formed in 1997 to manage the Symposium, and its committee members are listed in the web page

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The Symposium 2024 will be held at Tokyo Gakugei University in September. The program and local organizers are H. Nagase (Tokyo Gakugei Univ.) and T. Aihara (Tokyo Gakugei Univ.).

Shigeto Kawata
Nagoya, Japan
February, 2024

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Preface

The 55th Symposium on Ring Theory and Representation Theory was held at Osaka Metropolitan University on September 5th – 8th, 2023. The symposium was held with the support of

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Kenta Ueyama
Matsumoto, Japan
February, 2024

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ON TRIVIAL TILTING THEORY

TAKUMA AIHARA

ABSTRACT. We explore when an algebra has only trivial tilting module/complex.

INTRODUCTION

In mathematics, trivial cases are regularly trivial (insipid and uninteresting). For instance, we study that in group theory, a group with only trivial subgroup is a cyclic group of prime order, and in ring theory, a commutative ring with only trivial ideal is a field. These are first exercises for beginners. Neverthless, we cannot turn away from them.

In this note, we discuss trivial tilting thoery for a finite dimensional algebra Λ over an algebraically closed field. Tilting theory deals with (classical) tilting modules, one-sided tilting complexes, two-sided tilting complexes (derived Picard groups), support τ-tilting modules, Wakamatsu tilting modules and so on. For example, Λ_Λ is a trivial tilting module and the one-sided stalk complexes $\Lambda_\Lambda[m]$ are trivial tilting complexes. We will give answers to the question "when does Λ have only trivial tilting module/complex".

1. MODULE VERSION

The *right finitistic dimension* $\mathrm{r.fin.dim}\,\Lambda$ of Λ is defined to be the supremum of the projective dimensions of right modules with finite projective dimension. Dually, we define the *left finitistic dimension* $\mathrm{l.fin.dim}\,\Lambda$ of Λ. Note that $\mathrm{r.fin.dim}\,\Lambda$ and $\mathrm{l.fin.dim}\,\Lambda$ do not necessarily coincide. As is well-known, $\mathrm{r.fin.dim}\,\Lambda = 0$ if and only if there is a non-zero homomorphism from every simple module to Λ in $\mathsf{mod}\,\Lambda^{\mathrm{op}}$ [4]. Here, Λ^{op} stands for the opposite algebra of Λ.

A module T is said to be *tilting* if it has finite projective dimension satisfying $\mathrm{Ext}^n_\Lambda(T,T) = 0$ for any positive integer n and there exists an exact sequence $0 \to \Lambda \to T_0 \to T_1 \to \cdots \to T_\ell \to 0$ with $T_i \in \mathsf{add}\,T$; this is also called *Miyashita tilting*. When $\mathrm{proj.dim}\,T \leq 1$, we often call T *classical tilting*.

We state the first observation of this note; it seems to be well-known (or easy to show) for researchers who are familiar with tilting theory.

Theorem 1. *The following are equivalent for an algebra Λ:*

(1) $\mathrm{r.fin.dim}\,\Lambda = 0$;
(2) *The module Λ is the only (basic) Miyashita tilting module;*
(3) *It is the only (basic) classical tilting module.*

The detailed version of this paper will be submitted for publication elsewhere.

Remark 2. An affirmative answer to *the finitistic dimension conjecture*, which states that the dimension is always finite, would give us the fact that $\mathrm{r.fin.dim}\,\Lambda = \mathrm{proj.dim}\,T$ for some (possibly infinitely generated) tilting module T [3, Theorem 2.6].

2. Complex version

A *tilting* complex T is defined to be a perfect complex satisfying $\mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)}(T, T[n]) = 0$ for every nonzero integer n and $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda) = \mathsf{thick}\,T$. We denote by $\mathsf{tilt}\,\Lambda$ the set of isomorphism classes of (basic) tilting complexes of Λ. In this section, we explore when Λ has only trivial tilting complex. First, one gives well-known examples; (1) [6], (2) [1] and (3) by Ringel (unpublished paper).

Example 3. The following algebras have only trivial tilting complexes:

(1) local algebras;
(2) selfinjective algebras with cyclic Nakayama permutation;
(3) radical-square-zero algebras satisfying $\mathrm{Ext}^1(S, S') \neq 0$ for all simple modules S, S'.

All algebras above have left and right finitistic dimension zero. However, even if Λ satisfies the property, it does not necessarily hold that Λ admits no nontrivial tilting complex; many selfinjective algebras satisfy both the property and $\mathsf{tilt}\,\Lambda \neq \Lambda[\mathbb{Z}]$, so we should give an example of nonselfinjective algebras.

Example 4. Let Λ be the radical-square-zero algebra presented by the quiver:

$$\begin{tikzcd} 1 \arrow[loop left, "y"] \arrow[r, "x"] & 2 \arrow[loop right, "z"] \end{tikzcd}.$$

It is easy to check that $\mathrm{r.fin.dim}\,\Lambda = 0 = \mathrm{l.fin.dim}\,\Lambda$.

As is seen in [2, Example 5.10], we have only three indecomposable pretilting complexes up to shift: P_1, P_2 and $X := P_2 \xrightarrow{x} P_1$. This tells us that there precisely exist two types of nontrivial tilting complexes:

$$T := \bigoplus \left\{ \begin{array}{ccc} & & P_1 \\ P_2 & \xrightarrow[x]{} & P_1 \end{array} \right. , \qquad U := \bigoplus \left\{ \begin{array}{ccc} P_2 & & \\ P_2 & \xrightarrow[x]{} & P_1 \end{array} \right.$$

So, we obtain $\mathsf{tilt}\,\Lambda = \Lambda[\mathbb{Z}] \cup T[\mathbb{Z}] \cup U[\mathbb{Z}]$. Moreover, each component admits the endomorphism algebra isomorphic to Λ, Γ or Γ^{op} (mutually nonisomorphic). Here, Γ is given by the following quiver with relations $\alpha\beta\alpha = \alpha\gamma = \gamma\beta = \gamma^2 = 0$:

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \arrow[loop right, "\gamma"] \end{tikzcd}.$$

We say that a complex T is *two-term* provided it is of the form $T^{-1} \to T^0$. The subset of $\mathsf{tilt}\,\Lambda$ consisting of two-term tilting complexes is denoted by $\mathsf{2tilt}\,\Lambda$.

A full subcategory of $\mathsf{mod}\,\Lambda$ is said to be a *torsion class* if it is closed under extensions and factors. We say that a torsion class is *ν-stable* provided it is closed under taking the Nakayama functor $\nu := - \otimes_\Lambda D\Lambda$. Here is a useful observation.

Proposition 5. [5, Proposition 5.5] *Let T be a two-term perfect complex of Λ and put $X := H^0(T)$. Then T is tilting if and only if $\mathsf{Fac}\,X$ is a ν-stable functorially finite torsion class of* $\mathsf{mod}\,\Lambda$.

An *Iwanaga–Gorenstein* algebra is defined to have finite left and right selfinjective dimension. We can get a result similar to Theorem 1.

Theorem 6. *The following are equivalent for an Iwanaga–Gorenstein algebra* Λ*:*

(1) Λ *is a selfinjective algebra with cyclic Nakayama permutation;*
(2) $\mathsf{tilt}\,\Lambda = \{\Lambda[m] \mid m \in \mathbb{Z}\}$*;*
(3) $\mathsf{2tilt}\,\Lambda = \{\Lambda, \Lambda[1]\}$.

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Department of Mathematics
Tokyo Gakugei University
4-1-1 Nukuikita-machi, Koganei, Tokyo 184-8501, JAPAN
Email address: aihara@u-gakugei.ac.jp

ON INTERVAL GLOBAL DIMENSION OF POSETS: A CHARACTERIZATION OF CASE 0

TOSHITAKA AOKI, EMERSON G. ESCOLAR, AND SHUNSUKE TADA

Abstract. We study the relative homological algebra of posets with respect to the intervals. We introduce our recent research on the properties of the supports of interval approximations and interval resolution global dimension. We also provide necessary and sufficient conditions on a poset to ensure that any representation is interval-decomposable (i.e. a characterization of the case where interval resolution global dimension is equal to 0).

Key Words: Representation, Relative homological algebra, Persistence module, Interval module

2000 *Mathematics Subject Classification*: 16G20, 55N31, 18G25, 16E05

1. Introduction

We refer the reader to [3] (arXiv:2308.14979) for details on the contents of this article.

Topological data analysis is a rapidly growing field applying the ideas of algebraic topology for data analysis. One of its main tools is persistent homology [1], which can compactly summarize the birth and death parameters of topological features (e.g. connected components, rings, cavities, and so on) of data via the persistence diagram. This allows us to analyze hidden structures in data. Algebraically, one part of the persistent homology analysis can be formalized by using the so-called one-parameter persistence modules, which are just ("pointwise") finite dimensional modules over the incidence algebra of a totally ordered set. In this point of view, one-parameter persistence modules are guaranteed to decompose into the indecomposable modules called *interval modules*, which provide a multiset of intervals that are encoded by the persistence diagram.

As a generalization, multi-parameter persistence modules are proposed, understood as representations of n-dimensional grids, and are expected to provide richer information than the one-parameter setting. When dealing with multi-parameter settings, however, there are some difficulties with adapting the same techniques.

Recently, there has been an interest in the use of relative homological algebra in persistence theory. Especially, the notion of right minimal approximations and resolutions by interval-decomposable modules are developed, and the finiteness of the interval resolution global dimension has been confirmed [2].

The aim of this article is to introduce the properties of right minimal approximations and resolutions by interval-decomposable modules studied in [3].

The detailed version of this paper will be submitted for publication elsewhere.

2. Preliminaries

Let A be a finite dimensional algebra over a field k. We denote by $\mathsf{mod}\, A$ the category of finitely generated right A-modules. Throughout this article, we assume that all modules are finitely generated. For morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ of A-modules, we denote their composition by $gf\colon X \to Z$.

2.1. **Approximations and resolutions.** We recall the basic terminology of relative homological algebra. We consider the full subcategory $\mathcal{X} := \mathsf{add}\, \mathcal{X}$ of $\mathsf{mod}\, A$ for a fixed finite collection $\mathcal{X}$ of (isomorphism classes of) indecomposable A-modules including all the indecomposable projective modules.

Definition 1. For a morphism $f\colon X \to M$ of A-modules, we say that

(1) f is *right minimal* if any morphism $g : X \to X$ satisfying $fg = f$ is an isomorphism.
(2) f is a *right $\mathcal{X}$-approximation* of M if $X \in \mathcal{X}$ and $\mathrm{Hom}_A(Y, f)$ is surjective for any $Y \in \mathcal{X}$.
(3) f is a *right minimal $\mathcal{X}$-approximation* of M if it is a right $\mathcal{X}$-approximation which is right minimal.
(4) A *right minimal $\mathcal{X}$-resolution* of M is an exact sequence

$$\cdots \longrightarrow J_m \xrightarrow{g_m} \cdots \xrightarrow{g_2} J_1 \xrightarrow{g_1} J_0 \xrightarrow{f} M \longrightarrow 0,$$

such that f is a right minimal ($\mathsf{add}\, \mathcal{X}$)-approximation of M, and for each $1 \le i$, the morphism g_i is a right minimal ($\mathsf{add}\, \mathcal{X}$)-approximation of $\mathrm{Im}\, g_i = \mathrm{Ker}\, g_{i-1}$.
(5) If M has a right minimal $\mathcal{X}$-resolution of the form

$$0 \longrightarrow J_m \xrightarrow{g_m} \cdots \xrightarrow{g_2} J_1 \xrightarrow{g_1} J_0 \xrightarrow{f} M \longrightarrow 0,$$

then we say that the *$\mathcal{X}$-resolution dimension* of M is m and write $\mathcal{X}\text{-res-dim}\, M = m$. Otherwise, we say that the $\mathcal{X}$-resolution dimension of M is infinity. We set

$$\mathcal{X}\text{-res-gldim}\, A := \sup\{\mathcal{X}\text{-res-dim}\, M \mid M \in \mathsf{mod}\, A\}$$

and call *$\mathcal{X}$-resolution global dimension* of A. Notice that it can be infinity.

2.2. **Partially ordered set and its representations.** Let P be a finite poset. We recall that the *Hasse diagram* of P is a directed graph whose vertices are in bijection with elements of P and there is an arrow $x \to y$ for $x, y \in P$ if $x < y$ and there is no $z \in P$ such that $x < z < y$. The *incidence algebra* $k[P]$ of a poset P is defined to be the quotient of the path algebra of the Hasse diagram of P modulo the two-sided ideal generated by all the commutative relations. The module category $\mathsf{mod}\, k[P]$ can be described in terms of a functor category as follows. Firstly, we regard P as a category whose objects are elements of P, and morphisms are defined by relations in P, i.e., there is a unique morphism $a \to b$ for $a, b \in P$ if and only if $a \le b$. We denote by $\mathsf{rep}_k(P)$ the category of (covariant) functors from P to the category of finite dimensional vector spaces over k. For V in $\mathsf{rep}_k(P)$, the subset $\mathrm{supp}\, V := \{a \in P \mid V_a \neq 0\}$ is called the *support* of V.

It is well-known that there is an equivalence of abelian categories between $\mathsf{rep}_k(P)$ and the module category $\mathsf{mod}\, k[P]$ of the incidence algebra of P. In this sense, we identify objects V of $\mathsf{rep}_k(P)$ and $k[P]$-modules, and the *support* of a $k[P]$-module M is the subset

$\mathrm{supp}(M) = \{a \in P \mid Me_a \neq 0\}$, where e_a is a primitive idempotent of $k[P]$ corresponding to the element $a \in P$.

In our study, the following class of full subposets called *interval* is basic.

Definition 2. A *full subposet* of P is a subset $P' \subseteq P$ equipped with the induced partial order. Notice that it is completely determined by its elements. We say that

(1) P' is *convex* in P if, for any $x, y \in P'$ and any $z \in P$, $x < z < y$ implies $z \in P'$,

(2) P' is an *interval* of P if P' is connected as a poset and is convex in P.

We denote by $\mathbb{I}(P)$ the set of intervals of P.

The following special class of modules plays an important role in this article.

Definition 3. For an interval I of P, let k_I be a $k[P]$-module given as follows.

$$(k_I)_a = \begin{cases} k & \text{if } a \in I, \\ 0 & \text{otherwise,} \end{cases} \qquad k_I(a \leq b) = \begin{cases} 1_k & \text{if } a, b \in I, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

An *interval module* is a $k[P]$-module M such that $M \cong k_I$ for some interval $I \in \mathbb{I}(P)$. Clearly, every interval module is indecomposable.

We denote by $\mathcal{I}_{k,P}$ the set of isomorphism classes of the interval $k[P]$-modules, which is in bijection with $\mathbb{I}(P)$ by $I \mapsto k_I$. Notice that $\mathbb{I}_P$ and $\mathcal{I}_{P,k}$ are finite since so is P. Each module in $\mathsf{add}\,\mathcal{I}_{P,k}$ is said to be *interval-decomposable.* In other words, a given $k[P]$-module M is interval-decomposable if and only if it can be written as

$$M \cong \bigoplus_{I \in \mathbb{I}(P)} k_I^{m(I)}$$

for some non-negative integers $m(I)$. We will write $\mathcal{I}_P$ instead of $\mathcal{I}_{k,P}$ when the base field k is clear.

Since $\mathcal{I}_P$ contains all indecomposable projective $k[P]$-modules by definition, one can consider resolutions by interval modules. By *interval covers over* P (resp., *interval resolutions over* P), we mean right minimal ($\mathsf{add}\,\mathcal{I}_P$)-approximations (resp., $\mathcal{I}_P$-resolutions) of $k[P]$-modules. When the poset P is clear, we may omit it. In addition, we will write

$$\text{int-res-dim}\, M := \mathcal{I}_P\text{-res-dim}\, M \quad \text{and} \quad \text{int-res-gldim}\, k[P] := \mathcal{I}_P\text{-res-gldim}\, k[P],$$

and call them the *interval resolution dimension* of a module M and the *interval resolution global dimension* of $k[P]$ respectively. It has been shown in [2, Proposition 4.5] that the interval resolution global dimension is always finite. To show that, the next proposition is a key.

Proposition 4. [2, Lemma 4.4 and its dual] *The subcategory* $\mathsf{add}\,\mathcal{I}_P$ *is closed under both submodules and quotients of indecomposable modules.*

Then, we can apply [10, Theorem in § 5](cf. [8]) and obtain the following.

Theorem 5. [2, Proposition 4.5] *For any finite poset* P*,* $\text{int-res-gldim}(k[P]) < \infty$.

3. Results

In this section, we will give three results on interval covers and interval resolution dimensions (Theorems 6, 8, and 9). These results are motivated by topological data analysis, and they would be interesting from the perspective of representation theory of finite dimensional algebras.

3.1. **Result 1.** We show the following result.

Theorem 6. *Let P be a finite poset and $\mathfrak{I}_P$ the set of isomorphism classes of interval modules. For a given $k[P]$-module M, we take its interval cover $f\colon X = \bigoplus_{i=1}^{m} X_i \to M$, where all the X_i's are interval modules. Then, the following holds.*

(1) *f is surjective.*
(2) *$f|_{X_i}\colon X_i \to M$ is injective for every $i \in \{1, \dots, m\}$.*
(3) $\operatorname{supp} X = \operatorname{supp} M$.

In particular, every X_i can be taken as an interval submodule of M.

An importance of Theorem 6 is that it provides one way to reduce the computational burden for computing interval resolutions. We note that [5, Proposition 4.8] show Theorem 6 in essentially the same way.

Example 7. We consider the D_4-type quiver $D_4(b)$ displayed below:

$$\begin{tikzcd} & 1 \arrow[d] & \\ 2 & 3 \arrow[l] \arrow[r] & 4. \end{tikzcd}$$

Then, the incidence algebra is just a path algebra of type D_4. The Auslander-Reiten quiver is given by

$$\begin{tikzcd}
\begin{smallmatrix} & 0 & \\ 1 & 0 & 0 \end{smallmatrix} \arrow[dr] & & \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \end{smallmatrix} \arrow[dr, "a_1"] & & \begin{smallmatrix} & 1 & \\ 1 & 1 & 0 \end{smallmatrix} \arrow[dr] & & \\
& \begin{smallmatrix} & 0 & \\ 1 & 1 & 1 \end{smallmatrix} \arrow[ur, "b_1"] \arrow[r, "b_2"] \arrow[dr, "b_3"'] & \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \arrow[r, "a_2"] & M \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix} & 0 & \\ 0 & 1 & 0 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} & 1 & \\ 0 & 1 & 0 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} & 1 & \\ 0 & 0 & 0 \end{smallmatrix}, \\
\begin{smallmatrix} & 0 & \\ 0 & 0 & 1 \end{smallmatrix} \arrow[ur] & & \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \end{smallmatrix} \arrow[ur, "a_3"'] & & \begin{smallmatrix} & 1 & \\ 0 & 1 & 1 \end{smallmatrix} \arrow[ur] & &
\end{tikzcd}$$

where all indecomposable modules except for M are interval, but M is

$$\begin{tikzcd} & k \arrow[d, "{}^t[1\,1]"] & \\ k & k^2 \arrow[l, "{[1\,0]}"'] \arrow[r, "{[0\,1]}"] & k. \end{tikzcd}$$

Looking at the Auslander-Reiten quiver, we find that an interval resolution of M is

$$0 \longrightarrow \begin{smallmatrix} & 0 & \\ 1 & 1 & 1 \end{smallmatrix} \xrightarrow{{}^t[b_1, b_2, b_3]} \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \end{smallmatrix} \xrightarrow{[a_1, a_2, a_3]} M \longrightarrow 0,$$

and hence

$$\text{int-res-dim}\, M = 1.$$

Consequently, the interval resolution global dimension for $D_4(b)$ is 1. One can also show that any D_4-type quiver has the interval resolution global dimension 1.

3.2. **Result 2.** We give a complete classification of posets whose modules are always interval-decomposable. This result generalizes the one-parameter settings of persistent homology.

Theorem 8. *Let P be a finite poset and $k[P]$ the incidence algebra of P. Then, the following conditions are equivalent.*

(a) $\text{int-res-gldim}\, k[P] = 0$.
(b) *Every $k[P]$-module is interval-decomposable.*
(c) *Each connected component of the Hasse diagram of P is one of $A_n(a)$ for some orientation a or $C_{m,\ell}$ displayed below, where the symbol $\leftrightarrow$ is either $\rightarrow$ or $\leftarrow$ assigned by its orientation a:*

$$A_n(a)\colon \qquad 1 \longleftrightarrow 2 \longleftrightarrow \cdots\cdots \longleftrightarrow n,$$

$$C_{m,\ell}\colon \quad \begin{tikzcd} & 1 \arrow[r, "\alpha_1"] & \cdots \arrow[r, "\alpha_{m-1}"] & m \arrow[dr, "\alpha_m"] & \\ \hat{0} \arrow[ur, "\alpha_0"] \arrow[dr, "\beta_0"'] & & & & \hat{1} \\ & 1' \arrow[r, "\beta_1"] & \cdots \arrow[r, "\beta_{\ell-1}"] & \ell' \arrow[ur, "\beta_\ell"'] & \end{tikzcd}$$

In particular, these conditions do not depend on the characteristic of the base field k.

We note that equivalences among (a) and (b) in the statement are trivial.

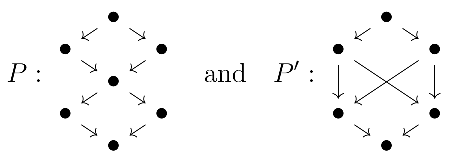
3.3. **Result 3.** Finally, we study a relationship between the interval resolution global dimensions of different posets. Our result is the following.

Theorem 9. *Let P be a finite poset and $k[P]$ the incidence algebra of P. For any full subposet P' of P, we have*

$$\text{int-res-gldim}\, k[P'] \leq \text{int-res-gldim}\, k[P]. \tag{3.1}$$

For the usual global dimension, we do not have the above monotonicity in general.

Example 10. Let P and P' be posets given by



respectively. Then, P' is a full subposet of P, which is obtained by removing the point in the center. The global dimension of $k[P]$ is 2 but that of $k[P']$ is 3 (over an arbitrary field), see [7, Section 3].

On the other hands, we have $\text{int-res-gldim}\, k[P'] = 2 \leq 3 = \text{int-res-gldim}\, k[P]$ over a field with two elements.

In the rest, we give a sketch of a proof of Theorem 9. The main ingredient for its proof is a functor Θ_e defined as follows. Let A be a finite dimensional k-algebra. For a given idempotent $e \in A$, we consider the idempotent subalgebra $B := eAe$. It is well-known that the functors

$$\mathrm{Res}_e(-) := (-)e,\ \mathrm{Ind}_e(-) := - \otimes_B eA,\ \mathrm{Coind}_e(-) := \mathrm{Hom}_B(Ae, -),$$

respectively called the restriction, induction, and coinduction functors, provide a diagram

$$\begin{tikzcd} \mathsf{mod}\, A \arrow[r, "\mathrm{Res}_e"] & \mathsf{mod}\, B. \arrow[l, bend right, "\mathrm{Ind}_e"'] \arrow[l, bend left, "\mathrm{Coind}_e"] \end{tikzcd} \tag{3.2}$$

which gives an adjoint triple. Then, the identity 1_M is associated to the map θ_M by

$$\begin{tikzcd} \mathrm{Hom}_A(\mathrm{Ind}_e(M), \mathrm{Coind}_e(M)) & \mathrm{Hom}_B(M, M) \arrow[l, "\sim"'] \\ \theta_M \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & 1_M, \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[l, mapsto] \end{tikzcd}$$

and an A-module

$$\Theta_e(M) := \mathrm{Im}\, \theta_M \subseteq \mathrm{Coind}_e(M).$$

It gives rise to a functor Θ_e called *intermediate extension* in [9, Proposition 4.6], and *prolongedment intermédiare* in [4]. We have $\mathrm{Res}_e \circ \Theta_e \cong 1_{\mathsf{mod}\, B}$.

Let P be a finite poset and P' a full subposet of P. In this setting, the incidence algebra $k[P']$ can be obtained as an idempotent subalgebra. In fact, we have an isomorphism $k[P'] \cong ek[P]e$ of algebras, where $e := \sum_{x \in P'} e_x$. Due to the previous paragraph, we can define the functor $\Theta_e \colon \mathsf{mod}\, k[P'] \to \mathsf{mod}\, k[P]$.

The following is a key observation on interval modules.

Proposition 11. *The functor Θ_e sends interval modules to interval modules. More explicitly, for a given interval $I \in \mathbb{I}(P')$, we have $\Theta_e(k_I) \cong k_{\mathrm{conv}(I)}$, where $\mathrm{conv}(I)$ is the smallest interval of P containing I.*

Consequently, we find the exact functor Res_e and the functor Θ_e provides the diagram

$$\begin{tikzcd} \mathsf{mod}\, k[P] \arrow[r, bend right, "\mathrm{Res}_e"'] & \mathsf{mod}\, k[P'] \arrow[l, bend right, "\Theta_e"'] \\ \mathsf{add}\, \mathcal{J}_P \arrow[u, phantom, "\cup"] \arrow[r, bend right, "\mathrm{Res}_e|_{\mathsf{add}\, \mathcal{J}_P}"'] & \mathsf{add}\, \mathcal{J}_{P'}, \arrow[u, phantom, "\cup"] \arrow[l, bend right, "\Theta_e|_{\mathsf{add}\, \mathcal{J}_{P'}}"'] \end{tikzcd}$$

where $\mathfrak{I}_P$ (resp., $\mathfrak{I}_{P'}$) is the set of isomorphism classes of interval modules over P (resp., P'). Then, we can directly compare interval resolutions via these functors and obtain the following.

Proposition 12. *For any module $M \in k[P']$, we have the following inequality*

$$\mathfrak{I}_{P'}\text{-res-dim}\, M \leq \mathfrak{I}_P\text{-res-dim}\, \Theta_e(M). \tag{3.3}$$

Now, we are ready to prove Theorem 9.

Proof of Theorem 9. Since M is an arbitrary module in (3.3), we obtain the desired inequality (3.1) by Proposition 12. □

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Graduate School of Human Development and Environment
Kobe University
3-11 Tsurukabuto, Nada-ku, Kobe 657-8501 Japan

Email address: toshitaka.aoki@people.kobe-u.ac.jp

Graduate School of Human Development and Environment
Kobe University
3-11 Tsurukabuto, Nada-ku, Kobe 657-8501 Japan

Email address: e.g.escolar@people.kobe-u.ac.jp

Graduate School of Human Development and Environment
Kobe University
3-11 Tsurukabuto, Nada-ku, Kobe 657-8501 Japan

Email address: 205d851d@stu.kobe-u.ac.jp

FACES OF INTERVAL NEIGHBORHOODS OF SILTING CONES

SOTA ASAI AND OSAMU IYAMA

Abstract. In the study of silting complexes for a finite dimensional algebra over a field, silting cones in the real Grothendieck group play an important role. The first named author defined the interval neighborhood of each silting cone so that it is compatible with τ-tilting reduction of Jasso. The closure of the interval neighborhood is a rational polyhedral cone in the real Grothendieck group. We have obtained many important properties of the faces of this rational polyhedral cone, and explain some of them in this proceeding.

1. Introduction

The representation theory of a finite dimensional algebra A over a field K studies the categories $\mathsf{mod}\, A$ and $\mathsf{proj}\, A$ of finitely generated (projective) A-modules, and its derived categories $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ and $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$.

Derived equivalences of algebras are characterized by the existence of *tilting complexes* in the category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ introduced by Rickard [20]. Keller-Vossieck [17] generalized tilting complexes to *silting complexes*, and silting complexes are equipped with the operation called *mutation* exchanging one indecomposable direct summand of a silting complex to obtain another one [3].

Among silting complexes, *2-term silting complexes* are strongly related to functorially finite torsion pairs [1, 5, 11], which is known as part of *τ-tilting theory*. It is natural to also consider direct summands of 2-term silting complexes, which are called *2-term presilting complexes*.

In the study of (pre)silting complexes, the *Grothendieck group* $K_0(\mathsf{proj}\, A)$ is important. Actually, $K_0(\mathsf{proj}\, A)$ is nothing but the free abelian group $\bigoplus_{i=1}^{n} \mathbb{Z}[P_i]$ whose canonical basis is given by the isoclasses of indecomposable projective modules $P_1, P_2, \ldots, P_n$.

Aihara-Iyama [3] proved that the indecomposable direct summands $S_1, S_2 \ldots, S_n$ of each basic silting complex $S = \bigoplus_{i=1}^{n} S_i$ give another free basis $[S_1], [S_2], \ldots, [S_n]$ of $K_0(\mathsf{proj}\, A)$. Then, for each basic 2-term presilting complex $U = \bigoplus_{i=1}^{m} U_i$ with U_i indecomposable, we have a *silting cone*

$$C^\circ(U) := \sum_{i=1}^{m} \mathbb{R}_{>0}[U_i],$$

in the *real Grothendieck group* $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$. The silting cone $C^\circ(U)$ is m-dimensional.

By [12], silting cones give a fan in $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ so that the intersection $C(U) \cap C(U')$ of the silting cones of basic 2-term presilting complexes U and U' coincides with the silting cone $C(U'')$ of the maximum common direct summand U'' of U and U'.

The detailed version of this paper will be submitted for publication elsewhere.

In general, this fan is not necessarily complete. In other words, there can be a region in $K_0(\mathsf{proj}\,A)_{\mathbb{R}}$ where no silting cones exist. To understand such a region more, it is helpful to consider *semistable subcategories* $\mathcal{W}_\theta$ of King [18] and *semistable torsion pairs* $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta), (\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ of Baumann-Kamnitzer-Tingley [9] in $\mathsf{mod}\,A$, given by certain linear conditions on subfactors of modules in $\mathsf{mod}\,A$ for elements θ in $K_0(\mathsf{proj}\,A)_{\mathbb{R}}$.

By using semistable subcategories, Brüstle-Smith-Treffinger [10] introduced the *wall-chamber structure* in $K_0(\mathsf{proj}\,A)_{\mathbb{R}}$ whose walls are $\Theta_M := \{\theta \in K_0(\mathsf{proj}\,A)_{\mathbb{R}} \mid M \in \mathcal{W}_\theta\}$ for all nonzero modules $M \in \mathsf{mod}\,A \setminus \{0\}$. Similarly, by semistable torsion pairs, the first named author [6] defined an equivalence relation called *TF equivalence* so that θ and η are TF equivalent if $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = (\overline{\mathcal{T}}_\eta, \mathcal{F}_\eta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\mathcal{T}_\eta, \overline{\mathcal{F}}_\eta)$.

Based on results of Brüstle-Smith-Treffinger [10] and Yurikusa [21], the first named author [6] proved that the silting cone $C^\circ(U)$ for each basic 2-term presilting complex U is a TF equivalence class. The semistable torsion pairs for $\theta \in C^\circ(U)$ are the functorially finite torsion pairs for U which have already been considered in [1, 8].

Sometimes, it is difficult to deal with all 2-term (pre)silting complexes at once. Then, one of the useful methods is *τ-tilting reduction* introduced by Jasso [16]. For a fixed basic 2-term presilting complex U, Jasso constructed a finite dimensional algebra $B = B_U$, and obtained that the basic 2-term (pre)silting complexes which have U as direct summands in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ are in bijections with the basic 2-term (pre)silting complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,B)$. Moreover, Jasso also proved that $\mathcal{W}_\theta$ for $\theta \in C^\circ(U)$ is equivalent to the module category $\mathsf{mod}\,B$.

The first named author introduced a subset N_U of $K_0(\mathsf{proj}\,A)_{\mathbb{R}}$ which connects the wall-chamber structure, TF equivalence and the τ-tilting reduction at U in [6]. The set N_U is an open neighborhood of the silting cone $C^\circ(U)$, so we decided to call N_U the *interval neighborhood* of $C^\circ(U)$.

By the constrution, the closure $\overline{N_U}$ is a rational polyhedral cone in $K_0(\mathsf{proj}\,A)_{\mathbb{R}}$, so we are currently studying the faces of $\overline{N_U}$. We will state some of our results on the faces of $\overline{N_U}$ in this proceeding.

1.1. **Notation.** In this proceeding, K is a field, and A is a finite dimensional K-algebra. The symbol $\mathsf{proj}\,A$ denotes the category of finitely generated projective A-modules, and $\mathsf{mod}\,A$ denotes the category of finitely generated A-modules.

As usual, $K_0(\mathcal{C})$ is the *Grothendieck group* of an exact category $\mathcal{C}$. The *real Grothendieck group* means the $\mathbb{R}$-vector space $K_0(\mathcal{C})_{\mathbb{R}} := K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{R}$.

The Grothendieck group $K_0(\mathsf{proj}\,A)$ is nothing but $\bigoplus_{i=1}^n \mathbb{Z}[P_i]$, where $P_1, P_2, \ldots, P_n$ are the pairwise nonisomorphic indecomposable projective modules. Thus, $K_0(\mathsf{proj}\,A)_{\mathbb{R}}$ is the Euclidean space $\bigoplus_{i=1}^n \mathbb{R}[P_i]$. Similarly, $K_0(\mathsf{mod}\,A) = \bigoplus_{i=1}^n \mathbb{Z}[L_i]$ and $K_0(\mathsf{mod}\,A)_{\mathbb{R}} = \bigoplus_{i=1}^n \mathbb{R}[L_i]$ hold, where L_i is the simple top of P_i.

With respect to the *Euler form*, $K_0(\mathsf{proj}\,A)_{\mathbb{R}}$ can be seen as the dual $\mathbb{R}$-vector space of $K_0(\mathsf{mod}\,A)_{\mathbb{R}}$ up to scalar multiples. Namely, each $\theta = \sum_{i=1}^n a_i[P_i] \in K_0(\mathsf{proj}\,A)_{\mathbb{R}}$ gives the $\mathbb{R}$-linear map $K_0(\mathsf{mod}\,A)_{\mathbb{R}} \to \mathbb{R}$ such that

$$\theta\left(\sum_{i=1}^n b_i[L_i]\right) = \sum_{i=1}^n a_i b_i \dim_K \mathsf{End}_A(L_i).$$

2. Silting cones and TF equivalence

We first recall some terminology on silting cones and TF equivalence.

Let U be a complex in the homotopy category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ of bounded complexes in $\mathsf{proj}\, A$. Since $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ is Krull-Schmidt, U is isomorphic to a direct sum of the form $\bigoplus_{i=1}^{m} U_i^{\oplus s_i}$ with $U_1, U_2, \ldots, U_m$ indecomposable and pairwise nonisomorphic and all $s_i \geq 1$. In this case, we set $|U| := m$, and say that U is *basic* if all $s_i = 1$.

Then, we can define (pre)silting complexes as follows.

Definition 1. [17, 5.1][3, Theorem 2.27] Let $U \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$.

(1) We say that U is *presilting* if $\mathsf{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)}(U, U[>0]) = 0$.
(2) We say that U is *silting* if U is presilting and $|U| = |A|$.

Aihara [2, Proposition 2.16] proved that any presilting complex U is a direct summand of some silting complex S. By this and [3, Theorem 2.27], if $U = \bigoplus_{i=1}^{m} U_i$ with each U_i indecomposable is presilting, then $[U_1], [U_2], \ldots, [U_m] \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ are linearly independent.

We say that $U \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ is *2-term* if the terms of U except the -1st and the 0th ones are zero. The result [2, Proposition 2.16] also says that any 2-term presilting complex U is a direct summand of some 2-term silting complex S.

We set $\mathsf{2\text{-}silt}\, A$ (resp. $\mathsf{2\text{-}psilt}\, A$) as the set of basic 2-term (pre)silting complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$. Thus, it is natural to consider the following notions.

Definition 2. Let $U = \bigoplus_{i=1}^{m} U_i \in \mathsf{2\text{-}psilt}\, A$ with U_i indecomposable. Then, we set the *silting cones* $C^\circ(U), C(U) \subset K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ as

$$C^\circ(U) = \sum_{i=1}^{m} \mathbb{R}_{>0}[U_i], \quad C(U) = \sum_{i=1}^{m} \mathbb{R}_{\geq 0}[U_i].$$

We will characterize the silting cone $C^\circ(U)$ by semistable torsion pairs, which are defined as follows.

Definition 3. Let $\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$.

(1) [9, Subsection 3.1] We set the *semistable torsion pairs* $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta), (\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ in $\mathsf{mod}\, A$ by

$$\begin{aligned}
\overline{\mathcal{T}}_\theta &:= \{M \in \mathsf{mod}\, A \mid \theta(N) \geq 0 \text{ for any factor module } N \text{ of } M\},\\
\mathcal{F}_\theta &:= \{M \in \mathsf{mod}\, A \mid \theta(L) < 0 \text{ for any submodule } L \neq 0 \text{ of } M\},\\
\mathcal{T}_\theta &:= \{M \in \mathsf{mod}\, A \mid \theta(N) > 0 \text{ for any factor module } N \neq 0 \text{ of } M\},\\
\overline{\mathcal{F}}_\theta &:= \{M \in \mathsf{mod}\, A \mid \theta(L) \leq 0 \text{ for any submodule } L \text{ of } M\}.
\end{aligned}$$

(2) [18, Definition 1.1] We set $\mathcal{W}_\theta := \overline{\mathcal{T}}_\theta \cap \overline{\mathcal{F}}_\theta$, and call it the *semistable subcategory*.

The semistable subcategory $\mathcal{W}_\theta$ is a *wide subcategory* of $\mathsf{mod}\, A$; that is, closed under taking kernels, cokernels, and extensions in $\mathsf{mod}\, A$. Therefore, the interval $[\mathcal{T}_\theta, \overline{\mathcal{T}}_\theta]$ in the poset $\mathsf{tors}\, A$ of torsion classes is a *wide interval* in [7]. Moreover, $\mathcal{W}_\theta$ is an abelian length category, and hence has the Jordan-Hölder property [14, Theorem 6.2].

Then, we can define TF equivalence.

Definition 4. [6, Definition 2.13] Let $\theta, \eta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$. We say that θ and η are *TF equivalent* if $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = (\overline{\mathcal{T}}_\eta, \mathcal{F}_\eta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\mathcal{T}_\eta, \overline{\mathcal{F}}_\eta)$.

The following result based on [10, Proposition 3.27] and [21, Proposition 3.3] is fundamental in our study.

Proposition 5. [6, Proposition 3.11] *Let $U \in \mathsf{2\text{-}psilt}\, A$. Then, $C^\circ(U)$ is a TF equivalence class. For any $\theta \in C^\circ(U)$, we have*

$$(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = ({}^\perp H^{-1}(\nu U), \mathsf{Sub}\, H^{-1}(\nu U)), \quad (\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\mathsf{Fac}\, H^0(U), H^0(U)^\perp).$$

The torsion pairs in the right-hand sides are classical functorially finite ones which were in [8, Theorem 5.10]. In the terminology of [1], the module $H^{-1}(\nu U)$ is τ^{-1}*-rigid*, and the module $H^0(U)$ is τ*-rigid*. See [1, 8] for details including the definitions of the symbols.

Definition 6. Let $U \in \mathsf{2\text{-}psilt}\, A$. Then, we set

$$(\overline{\mathcal{T}}_U, \mathcal{F}_U) := ({}^\perp H^{-1}(\nu U), \mathsf{Sub}\, H^{-1}(\nu U)), \quad (\mathcal{T}_U, \overline{\mathcal{F}}_U) := (\mathsf{Fac}\, H^0(U), H^0(U)^\perp),$$
$$\mathcal{W}_U := \overline{\mathcal{T}}_U \cap \overline{\mathcal{F}}_U.$$

Thus, $\mathcal{W}_U = \mathcal{W}_\theta$ for $\theta \in C^\circ(U)$ holds, so $\mathcal{W}_U$ is a wide subcategory of $\mathsf{mod}\, A$. This was shown by [16, Theorem 3.8] without using semistable torsion pairs. See also [13, Theorem 4.12].

3. Interval neighborhoods of silting cones

For each $U \in \mathsf{2\text{-}psilt}\, A$, we set

$$\mathsf{2\text{-}psilt}_U\, A := \{V \in \mathsf{2\text{-}psilt}\, A \mid U \in \mathsf{add}\, V\}.$$

This is the subset of $\mathsf{2\text{-}psilt}\, A$ consisting all $V \in \mathsf{2\text{-}psilt}\, A$ which have U as direct summands.

To study $\mathsf{2\text{-}psilt}_U\, A$, the first named author introduced the following set.

Definition 7. [6, Subsection 4.1] Let $U \in \mathsf{2\text{-}silt}\, A$. Then, we define the *interval neighborhood* N_U of $C^\circ(U)$ by

$$\begin{aligned} N_U &:= \{\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}} \mid H^0(U) \subset \mathcal{T}_\theta,\ H^{-1}(\nu U) \subset \mathcal{F}_\theta\} \\ &= \{\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}} \mid \mathcal{T}_U \subset \mathcal{T}_\theta,\ \mathcal{F}_U \subset \mathcal{F}_\theta\}. \end{aligned}$$

We first observe the following properties.

Lemma 8. *Let $U, V \in \mathsf{2\text{-}psilt}\, A$.*

(1) [6, Lemma 4.3] *The set N_U is an open neighborhood of $C^\circ(U)$.*
(2) *The set N_U is given by finitely many linear strict inequalities.*
(3) [6, Lemma 3.13] *The following conditions are equivalent:*
  (a) *$V \in \mathsf{2\text{-}psilt}_U\, A$;*
  (b) *$\mathcal{T}_V \supset \mathcal{T}_U$ and $\mathcal{F}_V \supset \mathcal{F}_U$;*
  (c) *$C^\circ(V) \subset N_U$;*
  (d) *$N_V \subset N_U$.*

Moreover, N_U satisfies the following minimality.

Lemma 9. *Let $U \in \mathsf{2\text{-}psilt}\, A$. Then, N_U is the smallest set satisfying both the following conditions:*

(a) *N_U is a neighborhood of $C^\circ(U)$;*
(b) *N_U is a union of TF equivalence classes.*

We also focus on the closure $\overline{N_U} \subset K_0(\mathsf{proj}\, A)_{\mathbb{R}}$.

Lemma 10. *Let $U, V \in \mathsf{2\text{-}psilt}\, A$.*

(1) *We have $\overline{N_U} = \{\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}} \mid H^0(U) \subset \overline{\mathcal{T}}_\theta,\ H^{-1}(\nu U) \subset \overline{\mathcal{F}}_\theta\}$. In particular, $\overline{N_U}$ is a union of TF equivalence classes.*
(2) *We have $\overline{N_U} \supset C(U)$.*
(3) *The set $\overline{N_U}$ is a rational polyhedral cone in $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$.*
(4) *The following conditions are equivalent:*
   (a) *$U \oplus V$ is (not necessarily basic) presilting;*
   (b) *$N_U \cap N_V \neq \emptyset$;*
   (c) *$C(V) \subset \overline{N_U}$.*

   In this case, $N_{U \oplus V} = N_U \cap N_V$ holds.

4. Faces of interval neighborhoods

Let $U \in \mathsf{2\text{-}psilt}\, A$. Since $\overline{N_U}$ is a rational polyhedral cone, we study the set $\mathsf{Face}\, \overline{N_U}$ of its faces. If $U = \bigoplus_{i=1}^m U_i$ with U_i indecomposable, we set $U_I := \bigoplus_{i \in I} U_i$ for each subset $I \subset \{1, 2, \ldots, m\}$. We have obtained the following properties in our study.

Definition-Proposition 11. *Let $U \in \mathsf{2\text{-}psilt}\, A$ and $F \in \mathsf{Face}\, \overline{N_U}$. Set $I_F := \{i \in \{1, 2, \ldots, m\} \mid [U_i] \notin F\}$.*

(1) *We have $F \cap C(U) = C(U/U_{I_F})$.*
(2) *If $\dim_{\mathbb{R}} F = n - 1$, then $\# I_F = 1$.*
(3) *For any $I \subset \{1, 2, \ldots, m\}$, we define*

$$\mathsf{Face}_I\, \overline{N_U} := \{F \in \mathsf{Face}\, \overline{N_U} \mid I_F = I\}.$$

Then, we have a (not necessarily convex) subset

$$\partial_I := \bigcup_{F \in \mathsf{Face}_I \overline{N_U}} F = \overline{N_U} \setminus \bigcup_{i \in I} N_{U_i} \subset \overline{N_U}.$$

To explain our main results, we need to recall some results in *τ-tilting reduction.*

Fix $U \in \mathsf{2\text{-}psilt}\, A$. Then, we take the unique $S \in \mathsf{2\text{-}silt}\, A$ such that $\overline{\mathcal{T}}_S = \overline{\mathcal{T}}_U$. This S is called the *Bongartz completion* of U. We define a finite dimensional algebra $B = B_U$ by $B := \mathsf{End}_{\mathsf{K^b}(\mathsf{proj}\, A)}(S)/\langle e \rangle$, where e is the idempotent $S \to U \to S$.

Jasso [16] proved the following results. See also [13, Theorem 4.12] and [4, Theorem 4.9].

Proposition 12. *Let $U \in \mathsf{2\text{-}psilt}\, A$.*

(1) [16, Theorem 3.8] *There exists a category equivalence*

$$\Phi := \mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(S, ?) \colon \mathcal{W}_U \to \mathsf{mod}\, B.$$

(2) [16, Theorems 3.16, 4.12] *There uniquely exist bijections*

$$p\colon \textsf{2-silt}_U A \to \textsf{2-silt}\, B, \quad p\colon \textsf{2-psilt}_U A \to \textsf{2-psilt}\, B$$

such that

$$(\Phi(\overline{\mathcal{T}}_U \cap \mathcal{W}_U), \Phi(\mathcal{F}_U \cap \mathcal{W}_U)) = (\overline{\mathcal{T}}_{p(U)}, \mathcal{F}_{p(U)}),$$
$$(\Phi(\mathcal{T}_U \cap \mathcal{W}_U), \Phi(\overline{\mathcal{F}}_U \cap \mathcal{W}_U)) = (\mathcal{T}_{p(U)}, \overline{\mathcal{F}}_{p(U)}).$$

In particular, $p(S) = B$.

The first named author found the corresponding results in $K_0(\textsf{proj}\, A)_{\mathbb{R}}$.

Definition-Proposition 13. [6, Lemma 4.4, Theorem 4.5] *Let $U \in \textsf{2-psilt}\, A$. Then, there exists an $\mathbb{R}$-linear surjective map $\pi\colon K_0(\textsf{proj}\, A)_{\mathbb{R}} \to K_0(\textsf{proj}\, B)_{\mathbb{R}}$ satisfying the following conditions.*

(a) *The kernel $\textsf{Ker}\,\pi$ is the $\mathbb{R}$-vector subspace $\mathbb{R}C(U)$ generated by $C(U)$.*
(b) *The resriction $\pi|_{N_U}\colon N_U \to K_0(\textsf{proj}\, B)_{\mathbb{R}}$ is still surjective.*
(c) *For any $\theta \in N_U$, we have $\Phi(\mathcal{T}_\theta \cap \mathcal{W}_U) = \mathcal{T}_{\pi(\theta)}$ and $\Phi(\mathcal{F}_\theta \cap \mathcal{W}_U) = \mathcal{F}_{\pi(\theta)}$. In particular, π induces a bijection*

$$\{\text{TF equivalence classes in } N_U\} \to \{\text{TF equivalence classes in } K_0(\textsf{proj}\, B)_{\mathbb{R}}\}.$$

(d) *For any $V \in \textsf{2-psilt}_U A$, we have $\pi(C^\circ(V)) = C^\circ(p(V))$.*

Then, we can state our first main result.

Theorem 14. *Let $U = \bigoplus_{i=1}^m U_i \in \textsf{2-psilt}\, A$ with U_i indecomposable, and $I \subset \{1, 2, \dots, m\}$. We set*

$$\Sigma_I := \{\pi(F) \mid F \in \textsf{Face}_I\, \overline{N_U}\}.$$

(1) *We have a bijection $\textsf{Face}_I\, \overline{N_U} \to \Sigma_I$ sending F to $\pi(F)$. The inverse is given by $\sigma \mapsto \pi^{-1}(\sigma) \cap \partial_I$.*
(2) *For any $F \in \textsf{Face}_I\, \overline{N_U}$, we have $\textsf{dim}_{\mathbb{R}}\, \pi(F) = \textsf{dim}_{\mathbb{R}}\, F - \#I$.*
(3) *Σ_I is a finite complete rational polyhedral fan in $K_0(\textsf{proj}\, B)_{\mathbb{R}}$.*

Before stating our second main result, we prepare some notions. Since $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta), (\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ are torsion pairs in $\textsf{mod}\, A$, for any $M \in \textsf{mod}\, A$ and $\theta \in K_0(\textsf{proj}\, A)_{\mathbb{R}}$, we have unique short exact sequences

$$0 \to \bar{\mathrm{t}}_\theta M \to M \to \mathrm{f}_\theta M \to 0 \quad (\bar{\mathrm{t}}_\theta M \in \overline{\mathcal{T}}_\theta,\ \mathrm{f}_\theta M \in \mathcal{F}_\theta),$$
$$0 \to \mathrm{t}_\theta M \to M \to \bar{\mathrm{f}}_\theta M \to 0 \quad (\mathrm{t}_\theta M \in \mathcal{T}_\theta,\ \bar{\mathrm{f}}_\theta M \in \overline{\mathcal{F}}_\theta)$$

with $\mathrm{t}_\theta M \subset \bar{\mathrm{t}}_\theta M \subset M$. Moreover, we set $\mathrm{w}_\theta M := \bar{\mathrm{t}}_\theta M / \mathrm{t}_\theta M \in \mathcal{W}_\theta$. Then, we introduce the following equivalence relation.

Definition 15. Let $M \in \textsf{mod}\, A$, and $\theta, \eta \in K_0(\textsf{proj}\, A)_{\mathbb{R}}$. Then, we say that θ and η are *M-TF equivalent* if the following conditions hold:

(a) $\mathrm{t}_\theta M = \mathrm{t}_\eta M$ and $\mathrm{w}_\theta M = \mathrm{w}_\eta M$ and $\mathrm{f}_\theta M = \mathrm{f}_\eta M$;
(b) the composition factors of $\mathrm{w}_\theta M = \mathrm{w}_\eta M$ in $\mathcal{W}_\theta$ and $\mathcal{W}_\eta$ coincide.

Moreover, we set $\Sigma(M)$ as the set of the closures of all M-TF equivalence classes.

The condition (b) seems complicated, but it is necessary to make the following property hold.

Proposition 16. *Let $M \in \mathsf{mod}\, A$. Then, $\Sigma(M)$ is a finite complete rational polyhedral fan in $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$.*

We remark that $\Sigma(M)$ coincides with the complete rational polyhedral fan $\Sigma(\mathrm{N}(M))$ in [4, Theorem 5.22] constructed from the *Newton polytope* $\mathrm{N}(M)$ of M in $K_0(\mathrm{mod}\, A)_{\mathbb{R}}$.

Now, we can state our second main result.

Theorem 17. *Let $U = \bigoplus_{i=1}^{m} U_i \in \mathsf{2\text{-}psilt}_U\, A$ with U_i indecomposable. Then, there exist $M_1, M_2, \dots, M_m \in \mathsf{mod}\, B$ such that, for any subset $I \subset \{1, 2, \dots, m\}$, the rational polyhedral fans $\Sigma(\bigoplus_{i \in I} M_i)$ and Σ_I in $K_0(\mathsf{proj}\, B)_{\mathbb{R}}$ coincide.*

We sketch the construction of $M_1, M_2, \dots, M_m$ above. We take the unique $S, T \in \mathsf{2\text{-}silt}\, A$ such that $\overline{\mathcal{T}}_S = \overline{\mathcal{T}}_U$ and $\overline{\mathcal{F}}_S = \overline{\mathcal{F}}_U$. Then, we can prove that T is the left simultaneous mutation of S at S/U. Thus, we can decompose S, T as $S = \bigoplus_{i=1}^{n} S_i$ and $T = \bigoplus_{i=1}^{n} T_i$ so that

(a) for any $i \in \{1, 2, \dots, m\}$, we have $S_i = U_i = T_i$; and
(b) for each $j \in \{m+1, m+2, \dots, n\}$, there exists a triangle $S_j \to U'_j \to T_j \to S_j$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ with $S_j \to U'_j$ a minimal left $(\mathsf{add}\, U)$-approximation.

Next, we take the 2-term simple-minded collections $X = \bigoplus_{i=1}^{n} X_i$ and $Y = \bigoplus_{i=1}^{n} Y_i$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ corresponding to S, T under the bijection in [19, Theorem 6.1] and [11, Corollary 4.3]. Then, we have proved that, for each $i \in \{1, 2, \dots, m\}$, there exists a triangle $X_i[-1] \to W_i \to Y_i \to X_i$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ with $X_i[-1] \to W_i$ a minimal left $\mathcal{W}_U$-approximation by using [15, Proposition 4.8]. Now, $M_i := \Phi(W_i)$ is the desired B-module.

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Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan
Email address: sotaasai@g.ecc.u-tokyo.ac.jp, iyama@ms.u-tokyo.ac.jp

THE SPECTRUM OF THE CATEGORY OF MAXIMAL COHEN-MACAULAY MODULES

NAOYA HIRAMATSU

Abstract. We introduce a topology on the set of isomorphism classes of finitely generated maximal Cohen–Macaulay modules over a commutative complete Cohen–Macaulay ring, which is analogous to the Ziegler spectrum. We then calculate the Cantor–Bendixson rank of this topological space for rings of CM_+-finite representation type.

1. Introduction

The Ziegler spectrum of an associative algebra is a topological space whose points are the isomorphism classes of indecomposable pure-injective modules, whose topology is defined in terms of positive primitive formulas over the algebra. Many studies of Ziegler spectrums are given in the context of the representation theory of algebras [1, 2, 5] and so on. In this note, we consider an analog of the Ziegler spectrum for the (stable) category of maximal Cohen-Macaulay (abbr. MCM) modules over a complete Cohen-Macaulay local ring.

Let R be complete Cohen–Macaulay local ring with algebraic residue field k. We denote by $\mathcal{C}$ the category of MCM R-modules. We denote by $\mathrm{mod}(\mathcal{C})$ the category of finitely presented contravariant additive functors and also denote by $\underline{\mathrm{mod}}(\mathcal{C})$ the full subcategory of $\mathrm{mod}(\mathcal{C})$ consisting of functors with $F(R) = 0$. We denote $\mathsf{Sp}(\mathcal{C})$ the set of isomorphism classes of the indecomposable MCM R-modules except R and 0.

For a subset $\mathcal{X}$ of $\mathsf{Sp}(\mathcal{C})$, we denote by $\Sigma(\mathcal{X})$ the subcategory of $\underline{\mathrm{mod}}(\mathcal{C})$ formed by the functors F such that $F(X) = 0$ for all $X \in \mathcal{X}$. For a subcategory $\mathcal{F}$ of $\underline{\mathrm{mod}}(\mathcal{C})$, we denote by $\gamma(\mathcal{F})$ the subset of $\mathsf{Sp}(\mathcal{C})$ satisfying $F(X) = 0$ for all $F \in \mathcal{F}$.

Theorem 1. *Then the assignment $\mathcal{X} \mapsto \gamma \circ \Sigma(\mathcal{X})$ is a is a Kuratowski closure operator on $\mathsf{Sp}(\mathcal{C})$. In particular, it induces a topology on $\mathsf{Sp}(\mathcal{C})$.*

For some specific $\mathcal{C}$, we calculate a Cantor-Bendixson rank of $\mathsf{Sp}(\mathcal{C})$ with respect to the topology. The Cantor-Bendixson rank measures the complexity of the topology. It measures how far the topology is from the discrete topology.

We say that a Cohen–Macaulay local ring is $\mathcal{C}_+$-finite if there exist only finitely many isomorphism classes of indecomposable MCM modules that are not locally free on the punctured spectrum [7].

Theorem 2. *If R is $\mathcal{C}_+$-finite then $CB(\mathsf{Sp}(\mathcal{C})) \leq 1$.*

In this talk, we consider only finitely generated modules. Previous studies have also considered infinitely generated modules, which is different from our consideration.

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2. The spectrum of the category of maximal Cohen-Macaulay modules

In this note, R is a commutative complete Cohen–Macaulay local ring with algebraic residue field k and all modules are "finitely generated" R-modules. We denote by $\mathcal{C}$ the category of maximal Cohen-Macaulay (MCM) modules.

$$\mathcal{C} = \{M \mid \mathrm{Ext}^i_R(k, M) = 0 \text{ for } i < \dim R\}$$

We denote by $\underline{\mathcal{C}}$ the stable category of $\mathcal{C}$. The objects of $\underline{\mathcal{C}}$ are the same as those of $\mathcal{C}$, the morphisms of $\underline{\mathcal{C}}$ are elements of $\underline{\mathrm{Hom}}_R(M, N) := \mathrm{Hom}_A(M, N)/P(M, N)$ for $M, N \in \underline{\mathcal{C}}$, where $P(M, N)$ denote the set of morphisms from M to N factoring through free R-modules. Since R is complete, $\mathcal{C}$, thus $\underline{\mathcal{C}}$ are Krull-Schmidt categories. That is the endomorphism ring of the indecomposable module is local.

Let us recall the full subcategory of the functor category of $\mathcal{C}$ which is called the Auslander category. The Auslander category $\mathrm{mod}(\mathcal{C})$ is the category whose objects are finitely presented contravariant additive functors from $\mathcal{C}$ to a category of abelian groups and whose morphisms are natural transformations between functors. We denote by $\underline{\mathrm{mod}}(\mathcal{C})$ the full subcategory $\mathrm{mod}(\mathcal{C})$ consisting of functors F with $F(R) = 0$. The important fact is that $\mathrm{mod}(\mathcal{C})$ and $\underline{\mathrm{mod}}(\mathcal{C})$ are abelian categories.

Remark 3. It is nothing but $\underline{\mathrm{mod}}(\mathcal{C})$ is the Aunslander category of $\underline{\mathcal{C}}$ $\mathrm{mod}(\underline{\mathcal{C}})$. Actually, the category $\underline{\mathrm{mod}}(\mathcal{C})$ is equivalent to $\mathrm{mod}(\underline{\mathcal{C}})$;

$$\mathrm{mod}(\underline{\mathcal{C}}) \to \underline{\mathrm{mod}}(\mathcal{C}); \quad F \mapsto F \circ \iota,$$

where $\iota : \mathcal{C} \to \underline{\mathcal{C}}$. See [8, Remark 4.16]. So in the rest of this note, we denote $\mathrm{mod}(\underline{\mathcal{C}})$ instead of $\underline{\mathrm{mod}}(\mathcal{C})$.

Note that every object $F \in \mathrm{mod}(\underline{\mathcal{C}})$ is obtained from a short exact sequence in $\mathcal{C}$. Namely we have the short exact sequence $0 \to N \to M \to L \to 0$ such that

$$0 \to \mathrm{Hom}_R(\ , N) \to \mathrm{Hom}_R(\ , M) \to \mathrm{Hom}_R(\ , L) \to F \to 0$$

is exact in $\mathrm{mod}(\mathcal{C})$.

Definition 4. We denote by $\mathsf{Sp}(\mathcal{C})$ the set of isomorphism classes of the indecomposable MCM R-modules except R and 0. Namely,

$$\mathsf{Sp}(\mathcal{C}) := \{\text{the indecomposable MCM } R\text{-modules except } R \text{ and } 0\}/\cong .$$

The following assignments are introduced by Krause [2].

Definition 5. The assignments

$$\Sigma : \mathsf{Sp}(\mathcal{C}) \to \mathrm{mod}(\underline{\mathcal{C}}), \quad \gamma : \mathrm{mod}(\underline{\mathcal{C}}) \to \mathsf{Sp}(\mathcal{C})$$

are defined by

$$\Sigma(\mathcal{X}) := \{F \in \mathrm{mod}(\underline{\mathcal{C}}) \mid F(X) = 0 \text{ for } \forall X \in \mathcal{X}\},$$
$$\gamma(\mathcal{F}) := \{M \in \mathsf{Sp}(\mathcal{C}) \mid F(M) = 0 \text{ for } \forall F \in \mathcal{F}\}.$$

We state several basic properties of the assignments Σ and Γ.

Lemma 6. *Let $\mathcal{X}$, $\mathcal{Y}$ be subsets of $\mathsf{Sp}(\mathcal{C})$ and $\mathcal{F}$ and $\mathcal{G}$ be subcategories of $\mathrm{mod}(\underline{\mathcal{C}})$. For the assignments Σ and γ, the following statements hold.*

(1) $\mathcal{X} \subseteq \mathcal{Y} \Rightarrow \Sigma(\mathcal{X}) \supseteq \Sigma(\mathcal{Y})$.

(2) $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \gamma(\mathcal{F}) \supseteq \gamma(\mathcal{G})$.
(3) $\mathcal{X} \subseteq \gamma \circ \Sigma(\mathcal{X})$.
(4) $\mathcal{F} \subseteq \Sigma \circ \gamma(\mathcal{F})$. *Moreover* $\gamma(\mathcal{F}) = \gamma \circ \Sigma \circ \gamma(\mathcal{F})$.
(5) $\Sigma(\mathcal{X})$ *is a Serre subcategory in* $\mathrm{mod}(\underline{\mathcal{C}})$.

This is the main theorem of this note.

Theorem 7. *The assignment* $\mathcal{X} \mapsto \gamma \circ \Sigma(\mathcal{X})$ *is a Kuratowski closure operator. That is,*
(1) $\gamma \circ \Sigma(\emptyset) = \emptyset$,
(2) $\mathcal{X} \subseteq \gamma \circ \Sigma(\mathcal{X})$,
(3) $\gamma \circ \Sigma(\mathcal{X} \cup \mathcal{Y}) = \gamma \circ \Sigma(\mathcal{X}) \cup \gamma \circ \Sigma(\mathcal{Y})$,
(4) $\gamma \circ \Sigma(\gamma \circ \Sigma(\mathcal{X})) = \gamma \circ \Sigma(\mathcal{X})$
hold for all subsets $\mathcal{X}$, $\mathcal{Y}$ *in* $\mathsf{Sp}(\mathcal{C})$.

Proof. The assertions (i), (ii), and (iv) follow from the definition and the lemma above. To show (iii), we now notice that $\underline{\mathrm{Hom}}_R(-, M) \in \mathrm{mod}(\underline{\mathcal{C}})$ for $\forall M \in \mathcal{C}$. The inclusion $\gamma \circ \Sigma(\mathcal{X} \cup \mathcal{Y}) \supseteq \gamma \circ \Sigma(\mathcal{X}) \cup \gamma \circ \Sigma(\mathcal{Y}))$ follows from the fact that $\Sigma(\mathcal{X} \cup \mathcal{Y}) = \Sigma(\mathcal{X}) \cap \Sigma(\mathcal{Y})$, and the equality is clear. To show another inclusion, we take $M \in \gamma \circ \Sigma(\mathcal{X} \cup \mathcal{Y})$. Note that M is indecomposable. Assume that $M \notin \gamma \circ \Sigma(\mathcal{X}) \cup \gamma \circ \Sigma(\mathcal{Y})$. Then there exist $F \in \Sigma(\mathcal{X})$ and $G \in \Sigma(\mathcal{Y})$ such that $F(M) \neq 0$ and $G(M) \neq 0$. We construct the functor $H \in \Sigma(\mathcal{X} \cup \mathcal{Y})$ such that $H(M) \neq 0$ by using F and G. If such a functor exists we have a contradiction because M annihilates all functors in $\Sigma(\mathcal{X} \cup \mathcal{Y})$. By Yoneda's Lemma, we have nonzero morphisms $f : \underline{\mathrm{Hom}}_R(-, M) \to F$ and $g : \underline{\mathrm{Hom}}_R(-, M) \to G$. Take a pushout diagram in $\mathrm{mod}(\underline{\mathcal{C}})$:

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_R(-, M) \arrow[r] \arrow[d] & \mathrm{Im}\ f \arrow[r] \arrow[d] & 0 \\
\mathrm{Im}\ g \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 0 \\
0 & 0. &
\end{tikzcd}$$

Since $\Sigma(\mathcal{X})$ and $\Sigma(\mathcal{Y})$ are Serre subcategories, $\mathrm{Im}\ f \in \Sigma(\mathcal{X})$, $\mathrm{Im}\ g \in \Sigma(\mathcal{Y})$. This implies that $H \in \Sigma(\mathcal{X} \cup \mathcal{Y})$. From the push out diagram we obtain the exact sequence $\underline{\mathrm{Hom}}_R(-, M) \to \mathrm{Im}\ f \oplus \mathrm{Im}\ g \to H \to 0$. Since $\underline{\mathrm{End}}_R(M)$ is local, $\underline{\mathrm{End}}_R(M)$ is an indecomposable $\underline{\mathrm{End}}_R(M)$-free module. Moreover $\mathrm{Im}\ f(M)$ and $\mathrm{Im}\ g(M)$ are cyclic modules. This concludes that $H(M)$ must be nonzero. Therefore we have $H \in \Sigma(\mathcal{X} \cup \mathcal{Y})$ such that $H(M) \neq 0$. This gives the contradiction that $M \in \gamma \circ \Sigma(\mathcal{X} \cup \mathcal{Y})$, so that M is in $\gamma \circ \Sigma(\mathcal{X}) \cup \gamma \circ \Sigma(\mathcal{Y})$. ∎

Corollary 8. *The assignment* $\mathcal{X} \mapsto \gamma \circ \Sigma(\mathcal{X})$ *defines a topology on* $\mathsf{Sp}(\mathcal{C})$. *That is a subset* $\mathcal{X}$ *of* $\mathsf{Sp}(\mathcal{C})$ *is closed if and only if* $\gamma \circ \Sigma(\mathcal{X}) = \mathcal{X}$.

For a locally coherent category $\mathcal{G}$, a bijective correspondence between closed subsets in $\mathsf{Sp}(\mathcal{G})$ and Serre subcategories in $\mathrm{mod}(\mathcal{G})$ is given in [1, 2]. In our setting, for a Serre subcategory $\mathcal{F} \subseteq \mathrm{mod}(\underline{\mathcal{C}})$, $\mathcal{F} = \Sigma \circ \gamma(\mathcal{F})$ does not hold in general.

Example 9. Let $R = k[[x, y]]/(x^2)$. The indecomposable MCM R-modules are R, $I = (x)R$ and $I_n = (x, y^n)R$ for $n > 0$. Since $\gamma(\underline{\mathrm{Hom}}_R(-, I_n)) = \emptyset$, $\Sigma \circ \gamma(\underline{\mathrm{Hom}}_R(-, I_n)) =$

$\Sigma(\emptyset) = \mathrm{mod}(\underline{\mathcal{C}})$. However $\mathcal{S}(\underline{\mathrm{Hom}}_R(-, I_n)) \neq \mathrm{mod}(\underline{\mathcal{C}})$. Here we denote by $\mathcal{S}(\underline{\mathrm{Hom}}_R(-, I_n))$ the smallest Serre subcategory which contains $\underline{\mathrm{Hom}}_R(-, I_n)$. Since KGdim $\underline{\mathrm{Hom}}_R(-, I_n) = 1$ [6, Proposition 3.8], KGdim $\mathcal{S}(\underline{\mathrm{Hom}}_R(-, I_n)) = 1$. Note that KGdim $\underline{\mathrm{Hom}}_R(-, I) = 2$. [6, Proposition 3.11]. Hence $\underline{\mathrm{Hom}}_R(-, I) \notin \mathcal{S}(\underline{\mathrm{Hom}}_R(-, I_n))$, so that $\mathcal{S}(\underline{\mathrm{Hom}}_R(-, I_n)) \neq \mathrm{mod}(\underline{\mathcal{C}})$.

Lemma 10. *Let $X, Y \in \mathsf{Sp}(\mathcal{C})$ with $X \ncong Y$. Suppose that $\underline{\mathrm{Hom}}_R(X, Y) \neq 0$. Then $Y \notin \gamma \circ \Sigma(X)$.*

By the lemma above, one can show the following.

Proposition 11. *We have $\gamma \circ \Sigma(X) = \{X\}$ for all $X \in \mathsf{Sp}(\mathcal{C})$. Hence $\mathsf{Sp}(\mathcal{C})$ is T_1-space.*

Proof. Let $Y \in \mathsf{Sp}(\mathcal{C})$ which is not isomorphic to X. Suppose that $\underline{\mathrm{Hom}}_R(X, Y) \neq 0$. Then $Y \notin \gamma \circ \Sigma(X)$ by the lemma. Suppose that $\underline{\mathrm{Hom}}_R(X, Y) = 0$. Then $\underline{\mathrm{Hom}}_R(-, Y)$ is contained in $\Sigma(X)$ Assume that $Y \in \gamma \circ \Sigma(X)$, and in the case $\underline{\mathrm{Hom}}_R(Y, Y) = 0$. So that Y is 0 or R. This never happens since $\mathsf{Sp}(\mathcal{C})$ does not contain 0 and R. $\square$

Proposition 12. *Let $M \in \mathsf{Sp}(\mathcal{C})$. M is an isolated point, that is $\{M\}$ is open, if and only if there exists an Auslander-Reiten (AR) sequence ending in M.*

Proof. If there exists an AR sequence ending in M we can consider the functor S_M which is obtained from the AR sequence. Then $\gamma(S_M) = \mathsf{Sp}(\mathcal{C})\backslash\{M\}$ is closed, so that $\{M\}$ is open.

Suppose that M is isolated, and then $\mathsf{Sp}(\mathcal{C})\backslash\{M\}$ is closed. Notice that $\Sigma(\mathsf{Sp}(\mathcal{C})\backslash\{M\})$ is not empty, and take $F \in \Sigma(\mathsf{Sp}(\mathcal{C})\backslash\{M\})$. Then $F(M) \neq 0$ and $F(N) = 0$ if $N \ncong M$. By Yoneda's lemma, we have a nonzero morphism $\rho : \underline{\mathrm{Hom}}_R(-, M) \to F$. Since $\mathrm{Im} f$ is finitely presented and a subfunctor of F, by considering $\mathrm{Im}\rho$ instead of F, we may assume that F has a presentation: $\underline{\mathrm{Hom}}_R(-, M) \to F \to 0$. Take a generator $f_1, \cdots, f_m$ of $\underline{\mathrm{rad}}_R(M, M)$ as an R-module. Then the image of $\underline{\mathrm{Hom}}_R(M, (f_1, \cdots, f_m)) : \underline{\mathrm{Hom}}_R(M, M^{\oplus m}) \to \underline{\mathrm{Hom}}_R(M, M)$ is $\underline{\mathrm{rad}}_R(M, M)$. Consider the diagram:

$$\begin{tikzcd}
0 & 0 & \\
H_M \arrow[u] \arrow[r] & F/\mathrm{Im}\rho \circ f \arrow[u] \arrow[r] & 0 \\
\underline{\mathrm{Hom}}_R(-, M) \arrow[u] \arrow[r, "\rho"] & F \arrow[u] \arrow[r] & 0 \\
\underline{\mathrm{Hom}}_R(-, M^{\oplus m}) \arrow[u, "f:=\underline{\mathrm{Hom}}_R(-,(f_1,\cdots,f_m))"'] \arrow[r] & \mathrm{Im}\rho \circ f \arrow[u] \arrow[r] & 0 \\
 & 0. \arrow[u] &
\end{tikzcd}$$

We should remark that $F/\mathrm{Im}\rho \circ f$ is finitely presented since $\mathrm{Im}\rho \circ f$ is so. By the construction, we have $H_M(M) = \underline{\mathrm{Hom}}_R(M, M)/\underline{\mathrm{rad}}_R(M, M) \cong k$. Moreover $\rho(f(M)) =$

$\rho(\underline{\mathrm{rad}}_R(M,M)) \subseteq \mathrm{rad}_R F(M)$, so that $F/\mathrm{Im}\rho \circ f(M) = F(M)/\mathfrak{m}F(M)$. This yields that $F/\mathrm{Im}\rho \circ f$ is a simple functor and we conclude that M admits an AR sequence. □

Corollary 13. *Let R be an isolated singularity. Then the topology of $\mathsf{Sp}(\mathcal{C})$ is discrete.*

The author thanks Tsutomu Nakamura for telling him the remark below.

Remark 14. Let $\mathrm{GProj}(R)$ be a category of Gorenstein-projective R-modules and $\mathrm{GProj}(R)^c$ the full subcategory consisting of compactly generated modules. It has been studied in [5] that the Ziegler spectrum is defined by using the functor category of the stable category of $\mathrm{GProj}(R)^c$. Suppose that R is Gorenstein. Then it is shown in [5, Theorem 2.33] that we have the triangulated equivalence $\underline{\mathcal{C}} \cong \underline{\mathrm{GProj}}(R)^c$. So if R is Gorenstein, the spectrum $\mathsf{Sp}(\mathcal{C})$ is nothing but the Ziegler spectrum which is considered in [5] restricted to finitely generated ones.

3. Cantor-Bendixson rank

In this section, we calculate a Cantor-Bendixson rank of $\mathsf{Sp}(\mathcal{C})$.

Definition 15 (Cantor-Bendixson rank)**.** Let $\mathcal{T}$ be a topological space. If $x \in \mathcal{T}$ is an isolated point, then $\mathrm{C}B(x) = 0$. Put $\mathcal{T}' \subset \mathcal{T}$ is a set of the **<u>non</u>**-isolated point. Define the induced topology on $\mathcal{T}'$. Set $\mathcal{T}^{(0)} = \mathcal{T}, \mathcal{T}^{(1)} = \mathcal{T}^{(0)\prime}, \cdots, \mathcal{T}^{(n+1)} = \mathcal{T}^{(n)\prime}$. We define $\mathrm{C}B(x) = n$ if $x \in \mathcal{T}^{(n)} \backslash \mathcal{T}^{(n+1)}$ If $\exists n$ such that $\mathcal{T}^{(n+1)} = \emptyset$ and $\mathcal{T}^{(n)} \neq \emptyset$, then $\mathrm{C}B(\mathcal{T}) = n$. Otherwise $\mathrm{C}B(\mathcal{T}) = \infty$.

Example 16. Let R be a DVR (e,g. $R = k[[x]]$). Then $\mathrm{C}B(\mathrm{Spec}R) = 1$ concerning the Zariski topology. Note that $\mathrm{Spec}R = \{(0), \mathfrak{m}\}$. (0) is an isolated point since $D(f) = \{(0)\}$ for some $f \in R\backslash\{0\}$. Thus $\mathrm{Spec}R' = \{\mathfrak{m}\} = \mathrm{Spec}R^{(1)}$, and $\mathfrak{m}$ is isolated in the induced topology. In the case $R = k[[x,y]]$, you can show that $\mathrm{C}B(\mathrm{Spec}R) = \infty$. Note that $\mathrm{Spec}R' = \mathrm{Spec}R$.

By the corollary, we know $\mathsf{Sp}(\mathcal{C})$ is a discrete topology if R is an isolated singularity.

Corollary 17. *Let R be an isolated singularity. Then* $\mathrm{C}B(\mathsf{Sp}(\mathcal{C})) = 0$.

The definition of CM_+-finite is introduced in [7].

Definition 18. We say that a Cohen–Macaulay local ring R is CM_+-finite if there exist only finitely many isomorphism classes of indecomposable MCM modules that are **<u>not</u>** locally free on the punctured spectrum.

Example 19. The following rings are CM_+-finite.

(1) A ring which is an isolated singularity. (Thus a ring which is of finite CM-representation type.)
(2) A hypersurface ring which is of countable CM-representation type.

Here we say that R is of finite (countable) CM-representation type if there exists only finitely (countably) many isomorphism classes of indecomposable MCM modules.

Theorem 20. *If R is* CM_+*-finite then* $\mathrm{C}B(\mathsf{Sp}(\mathcal{C})) \leq 1$.

Proof. We denote by $\mathcal{C}_0$ the subset of $\mathsf{Sp}(\mathcal{C})$ consisting of modules that are locally free on the punctured spectrum and put $\mathcal{C}_+$ as $\mathsf{Sp}(\mathcal{C})\backslash\mathcal{C}_0$. For all $M \in \mathcal{C}_0$, M is an isolated point since M admits an AR sequence. Thus $\mathrm{C}B(\mathcal{C}_0) = 0$.

On the other hand, for all $M \in \mathcal{C}_+$, M is not isolated. Since R is CM_+-finite, $\mathcal{C}_+$ is a finite set. Hence, for each $M \in \mathcal{C}_+$,

$$V_M := \bigcup_{X \neq M, X \in \mathcal{C}_+}^{finite} \gamma \circ \Sigma(X)$$

is closed in $\mathsf{Sp}(\mathcal{C})$. Thus $[\mathcal{C}_+] \bigcap [\mathsf{Sp}(\mathcal{C})\backslash V_M] = \{M\}$ is open in $\mathcal{C}_+ \cap \mathsf{Sp}(\mathcal{C})$. Therefore $\mathrm{C}B(\mathsf{Sp}(\mathcal{C})) \leq 1$. □

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General Education Program,
National Institute of Technology, Kure College
2-2-11, Agaminami, Kure Hiroshima, 737-8506 Japan
Email address: hiramatsu@kure-nct.ac.jp

QUANTUM PROJECTIVE PLANES AND BEILINSON ALGEBRAS OF 3-DIMENSIONAL QUANTUM POLYNOMIAL ALGEBRAS FOR TYPE S'

AYAKO ITABA

Abstract. Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quantum polynomial algebra where E is the projective plane $\mathbb{P}^2$ or a cubic divisor in $\mathbb{P}^2$, and $\sigma \in \mathrm{Aut}_k E$. In this report, we prove that, for a Type S' algebra $A = \mathcal{A}(E, \sigma)$, where $E \subset \mathbb{P}^2$ is a union of a line and a conic meeting at two points, and $\sigma \in \mathrm{Aut}_k E$, the following conditions are equivalent: (1) The noncommutative projective plane $\mathsf{Proj}_{\mathrm{nc}} A$ is finite over its center. (2) The Beilinson algebra ∇A of A is 2-representation tame. (3) The isomorphism classes of simple 2-regular modules over ∇A are parametrized by $\mathbb{P}^2$.

Key Words: Quantum polynomial algebras, Quantum projective planes, Calabi-Yau algebras, Beilinson algebras.

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1. Quantum polynomial algebras and quantum projective spaces

This report is based on [7]. Throughout this report, let k be an algebraically closed field of characteristic 0, and all algebras are defined over k. Unless otherwise described, let A be a connected graded k-algebra finitely generated in degree 1.

In noncommutative algebraic geometry, a quantum polynomial algebra defined by Artin and Schelter [2] is a basic and important research object, which is a noncommutative analogue of a commutative polynomial algebra.

Definition 1 ([2]). A right noetherian graded algebra A is called a *d-dimensional quantum polynomial algebra* if

(i) $\mathrm{gldim}\, A = d < \infty$,

(ii) $\mathrm{Ext}^i_A(k, A) \cong \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d, \end{cases}$ (*Gorenstein condition*)

(iii) $H_A(t) := \sum_{i=0}^{\infty} (\dim_k A_i) t^i = (1-t)^{-d}$ (*Hilbert series*).

A right noetherian graded algebra A is called a *d-dimensional AS-regular algebra* if the above conditions (i) and (ii) hold.

Artin and Schelter [2] gave the classifications of low dimensional quantum polynomial algebras as follows: For a 1-dimensional quantum polynomial algebra A, A is isomorphic to $k[x]$ as graded algebras up to isomorphism. For a 2-dimensional quantum polynomial algebra A, A is isomorphic to

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$$k\langle x,y\rangle/(-x^2+xy-yx), \text{ or } k_\lambda[x,y] := k\langle x,y\rangle/(xy-\lambda yx) \ \ (\lambda \in k\setminus\{0\})$$

as graded algebras up to isomorphism, where $k_\lambda[x,y] \cong k_{\lambda'}[x,y]$ if and only if $\lambda' = \lambda^{\pm 1}$. Moreover, Artin and Schelter [2] proved that every 3-dimensional quantum polynomial algebra is isomorphic to one of the following algebra as graded k-algebras:

$$A \cong k\langle x,y,z\rangle/(f_1,f_2,f_3), \text{ or } A \cong k\langle x,y\rangle/(g_1,g_2),$$

where, $f_i \in k\langle x,y,z\rangle_2$ and $g_i \in k\langle x,y\rangle_3$. Note that A is a 3-dimensional quantum polynomial algebra if and only if A is a 3-dimensional quadratic AS-regular algebra ([2]).

Artin, Tate and Van den Bergh [3] found a nice correspondence between 3-dimensional quantum polynomial algebras and geometric pair (E,σ), where E is the projective plane $\mathbb{P}^2$ or a cubic divisor in $\mathbb{P}^2$, and $\sigma \in \mathrm{Aut}_k E$. So, this result allows us to write a 3-dimensional quantum polynomial algebra A as the form $A = \mathcal{A}(E,\sigma)$. This result convinced us that algebraic geometry is very useful to study even noncommutative algebras.

Let A be a right noetherian graded algebra. The category of finitely generated graded right A-modules is denoted by $\mathsf{grmod}\, A$, and the full subcategory of $\mathsf{grmod}\, A$ consisting of finite dimensional modules over k id denoted by $\mathsf{tors}\, A$.

Definition 2 ([5]). (1) *The noncommutative projective scheme associated to A* is defined by $\mathsf{Proj}_{\mathrm{nc}}\, A = (\mathsf{tails}\, A, \pi A)$ where $\mathsf{tails}\, A := \mathsf{grmod}\, A/\mathsf{tors}\, A$ is the quotient category, $\pi : \mathsf{grmod}\, A \to \mathsf{tails}\, A$ is the quotient functor, and $A \in \mathsf{grmod}\, A$ is regular.

(2) If A is a d-dimensional quantum polynomial algebra. Then $\mathsf{Proj}_{\mathrm{nc}}\, A$ is called *a quantum $\mathbb{P}^{d-1}$*. In particular, if A is a 3-dimensional quantum polynomial algebra, then $\mathsf{Proj}_{\mathrm{nc}}\, A$ is called *a quantum projective plane.*

Note that, if A is commutative, then $\mathsf{Proj}_{\mathrm{nc}}\, A \cong (\mathsf{mod}\, X, \mathcal{O}_X)$, $X = \mathsf{Proj}\, A$. If A is a 2-dimensional quantum polynomial algebra, then $\mathsf{Proj}_{\mathrm{nc}}\, A \cong (\mathsf{coh}\, \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$.

2. Characterization of quantum projective planes finite their centers

For a 3-dimensional quantum polynomial algebra $A = \mathcal{A}(E,\sigma)$, Artin-Tate-Van den Bergh [4] gave the following geometric characterization when A is finite over its center $Z(A)$.

Theorem 3 ([4]). *For a 3-dimensional quantum polynomial algebra $A = \mathcal{A}(E,\sigma)$, then A is finite over its center $Z(A)$ if and only if the order $|\sigma|$ of σ is finite.*

To prove Theorem 3, "fat points of a quantum projective plane $\mathsf{Proj}_{\mathrm{nc}}\, A$"plays an essential role. By Artin [1], if A is finite over its center and $E \neq \mathbb{P}^2$, then $\mathsf{Proj}_{\mathrm{nc}}\, A$ has a fat point, however, the converse is not true.

Definition 4. Let A be a graded algebra.

(1) *A point of* $\mathsf{Proj}_{\mathrm{nc}}\, A$ is an isomorphism class of a simple object of the form $\pi M \in \mathsf{tails}\, A$ where $M \in \mathsf{grmod}\, A$ is a graded right A-module such that $\lim_{i\to\infty} \dim_k M_i < \infty$.

(2) A point πM is called *fat* if $\lim_{i\to\infty} \dim_k M_i > 1$ In this case, M is called *a fat point module over A.*

To check the existence of a fat point, the following was introduced by Mori [12].

Definition 5 ([12]). For a geometric pair (E, σ) where $E \subset \mathbb{P}^{n-1}$ and $\sigma \in \mathrm{Aut}_k E$,

$$\mathrm{Aut}_k(\mathbb{P}^{n-1}, E) := \{\phi|_E \in \mathrm{Aut}_k E \mid \phi \in \mathrm{Aut}_k \mathbb{P}^{n-1}\}$$

and $\|\sigma\| := \inf\{i \in \mathbb{N}^+ \mid \sigma^i \in \mathrm{Aut}_k(\mathbb{P}^{n-1}, E)\} = \tau|_E$ for some $\tau \in \mathrm{Aut}_k \mathbb{P}^{n-1}\}$, which is called *the norm of* σ.

For a geometric pair (E, σ), $\|\sigma\| \leq |\sigma|$ holds in general.

Lemma 6 ([12], [1]). *Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quantum polynomial algebra. Then the following hold*:

(1) $\|\sigma\| = 1$ *if and only if* $E = \mathbb{P}^2$.
(2) $1 < \|\sigma\| < \infty$ *if and only if* $\mathsf{Proj}_{\mathrm{nc}} A$ *has a fat point.*

For a d-dimensional quantum polynomial algebra, the following holds in general:

Lemma 7 ([13], [12]). *Let A and A' d-dimensional quantum polynomial algebras "satisfying the condition* (G1)*, where $\mathcal{P}(A) = (E, \sigma)$ and $\mathcal{P}(A') = (E', \sigma')$", respectively. Then the following hold*:

(1) *If $A \cong A'$, then $E \cong E'$ and $|\sigma| = |\sigma'|$.*
(2) *If $\mathsf{grmod}\, A \cong \mathsf{grmod}\, A'$, then $E \cong E'$, $\|\sigma\| = \|\sigma'\|$.*
In particular, when $d = 3$, if $\mathsf{Proj}_{\mathrm{nc}} A \cong \mathsf{Proj}_{\mathrm{nc}} A'$, then $E \cong E'$ and $\|\sigma\| = \|\sigma'\|$.

We remark that Lemma 7 (2) tells us that, for a 3-dimensional quantum polynomial algebra $A = \mathcal{A}(E, \sigma)$, the norm $\|\sigma\|$ of σ is a categorical invariant in $\mathsf{Proj}_{\mathrm{nc}} A$.

Definition 8 ([12], [10]). Let A be a d-dimensional quantum polynomial algebra. We say that $\mathsf{Proj}_{\mathrm{nc}} A$ is *finite over its center* if there exists a d-dimensional quantum polynomial algebra A' finite over its center such that $\mathsf{Proj}_{\mathrm{nc}} A \cong \mathsf{Proj}_{\mathrm{nc}} A'$.

For a 3-dimensional quantum polynomial algebra $A = \mathcal{A}(E, \sigma)$ the author and Mori [10] proved that the following results: This is a categorical analogue of Theorem 3.

Theorem 9 ([10]). *If $A = \mathcal{A}(E, \sigma)$ is a 3-dimensional Calabi-Yau quantum polynomial algebra, then $\|\sigma\| = |\sigma^3|$, so the following are equivalent*:

(1) $|\sigma| < \infty$.
(2) $||\sigma|| < \infty$.
(3) *A is finite over its center.*
(4) *$\mathsf{Proj}_{\mathrm{nc}} A$ is finite over its center.*

Theorem 10 ([10]). *If $A = \mathcal{A}(E, \sigma)$ is a 3-dimensional quantum polynomial algebra such that $E \neq \mathbb{P}^2$, and $\nu \in \mathrm{Aut}\, A$ the Nakayama automorphism of A. Then $\|\sigma\| = |\nu^* \sigma^3|$, so the following are equivalent:*

(1) $|\nu^* \sigma^3| < \infty$.
(2) $\|\sigma\| < \infty$.
(3) *$\mathsf{Proj}_{\mathrm{nc}} A$ is finite over its center.*
(4) *$\mathsf{Proj}_{\mathrm{nc}} A$ has a fat point.*

We apply the above results of the author and Mori [10] to representation theory of finite dimensional algebras.

Definition 11 ([6])**.** Let R be a finite dimensional algebra of gldim$R = d < \infty$. An auto-equivalence $\nu_d \in \operatorname{Aut} \mathsf{D}^{\mathrm{b}}(\mathsf{mod} R)$ is defined by $\nu_d(M) := M \otimes^{\mathrm{L}}_R DR[-d]$ where $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R)$ is the bounded derived category of $\mathsf{mod}\, R$ and $DR := \operatorname{Hom}_k(R, k)$. If $\nu_d^{-i}(R) \in \mathsf{mod}\, R$ for all $i \in \mathbb{N}$, then R is called *d-representation infinite*. In this case, we say that a module $M \in \mathsf{mod}\, R$ is *d-regular* if $\nu_d^i(M) \in \mathsf{mod}\, R$ for all $i \in \mathbb{Z}$.

In Minamoto-Mori [11], for a d-dimensional quantum polynomial algebra A, *the Beilinson algebra ∇A of A* is defined by

$$\nabla A := \begin{pmatrix} A_0 & A_1 & \cdots & A_{d-1} \\ 0 & A_0 & \cdots & A_{d-2} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}.$$

Theorem 12 ([11])**.** *If A is a d-dimensional quantum polynomial algebra A and the Beilinson algebra ∇A of A. Then ∇A is extremely Fano of global dimension of $d-1$, and there exists an equivalence of triangulated category* $\mathsf{D}^{\mathrm{b}}(\mathsf{tails\,A}) \cong \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\nabla\mathsf{A})$.

The Beilinson algebra is a typical example of $(d-1)$-representation infinite algebra in the sense of Herschend-Iyama-Oppermann [6] ([11]). To investigate representation theory of such an algebra, it is important to classify simple $(d-1)$-regular modules.

Remark 13. (1) If A is a 2-dimensional quantum polynomial algebra, then

$$\nabla A \cong \begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix} \cong k(\bullet \rightrightarrows \bullet),$$

that is, ∇A is isomorphic to a 2-Kronecker algebra, so ∇A is a finite dimensional hereditary algebra of tame representation type. It is known that the isomorphism classes of simple regular modules over ∇A are parameterized by $\mathbb{P}^1$ (cf. [12]).

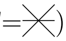
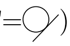
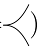
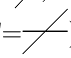
(2) For a 3-dimensional quantum polynomial algebra A, ∇A is a finite-dimensional algebra;

$$\nabla A \cong k\left(\bullet \,\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\, \bullet \,\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\, \bullet\right)\Big/ \text{(the same relations of } A\text{)}.$$

Corollary 14 ([10])**.** *Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quantum polynomial algebra with the Nakayama automorphism $\nu \in \operatorname{Aut} A$. Then the following are equivalent:*

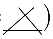
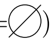
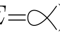
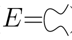
(1) $|\nu^*\sigma^3| (= \|\sigma\|) = 1$ *or* ∞.
(2) $\mathsf{Proj}_{\mathrm{nc}}\, A$ *has no fat point.*
(3) *The isomorphism classes of simple* 2*-regular modules over ∇A are parameterized by the set of closed points of $E \subset \mathbb{P}^2$.*

In particular, if $A = \mathcal{A}(E, \sigma)$ is one of the following types, then A satisfies all of the above conditions.

Type P ($E = \mathbb{P}^2$) Type T ($E=$) Type T' ($E=$)
Type CC ($E=$) Type TL ($E=$) Type WL ($E=$)

More precisely, if E is of Type P, then $\|\sigma\| = 1$ by Lemma 6, and if E is of Type T, Type T', Type CC, Type TL or Type WL, then $\|\sigma\|$ is infinite. The following types of

3-dimensional quantum polynomial algebras $A = \mathcal{A}(E, \sigma)$ have the case that $\|\sigma\|$ is finite.

Type S ($E=$) Type S' ($E=$) Type NC ($E=$) Type EC ($E=$)

In [10], for a 3-dimensional quantum polynomial algebra A, the author and Mori expect that the following are equivalent:

(1) $\mathsf{Proj}_{\mathrm{nc}}\, A$ is finite over its center.
(2) ∇A is 2-representation tame in the sense of Herschend-Iyama-Oppermann [6].
(3) The isomorphism classes of simple 2-regular modules over ∇A are parameterized by $\mathbb{P}^2$.

Note that these equivalences are shown for Type S in [12, Theorem 4.17, Theorem 4.21]. Do these equivalences in the above conjecture hold for Type S' in particular?

3. Main results

In this report, we prove the following results for Type S' algebra $A = \mathcal{A}(E, \sigma)$, where $E \subset \mathbb{P}^2$ is a union of a line and a conic meeting at two points, and $\sigma \in \mathrm{Aut}_k E$.

Let $A = \mathcal{A}(E, \sigma) = k\langle x, y, z\rangle/(f_1, f_2, f_3)$ be a 3-dimensional quantum polynomial algebra of Type S' where

$$\begin{cases} f_1 = yz - \alpha zy + x^2, \\ f_2 = zx - \beta xz, \\ f_3 = xy - \beta yx \quad (\alpha, \beta \in k,\ \alpha\beta^2 \neq 0, 1) \end{cases}$$

(see [8, Theorem 3.2], [9, Table 1 in Proposition 3.1]). For a 3-dimensional quantum polynomial algebra $A = \mathcal{A}(E, \sigma)$ of Type S', there exists the 3-dimensi onal Calabi-Yau quantum polynomial algebra A' of Type S' such that $\mathsf{grmod}\, A \cong \mathsf{grmod}\, A'$ so that $\mathsf{Proj}_{\mathrm{nc}}\, A \cong \mathsf{Proj}_{\mathrm{nc}}\, A'$ where $A' = \mathcal{A}(E, \sigma') = k\langle x, y, z\rangle/(g_1, g_2, g_3)$ is a 3-dimensional Calabi-Yau quantum polynomial algebra of Type S':

$$\begin{cases} g_1 = yz - \alpha zy + x^2, \\ g_2 = zx - \alpha xz, \\ g_3 = xy - \alpha yx \quad (\alpha^3 \neq 0, 1) \end{cases}$$

(see [9, Table 2 in Theorem 3.4]).

Proposition 15 ([7, Proposition 3.2])**.** *Let $A = \mathcal{A}(E, \sigma) = k\langle x, y, z\rangle/(g_1, g_2, g_3)$ be a 3-dimensional Calabi-Yau quantum polynomial algebra of Type S',*

where $\begin{cases} g_1 = yz - \alpha zy + x^2, \\ g_2 = zx - \alpha xz, \\ g_3 = xy - \alpha yx \quad (\alpha^3 \neq 0, 1). \end{cases}$ *Define* $g := xyz + (1 - \alpha^3)^{-1}x^3 \in A_3$.

(1) *If A is finite over its center $Z(A)$ (that is, $|\alpha|$ is finite), then $Z(A) = k[x^{|\alpha|}, y^{|\alpha|}, z^{|\alpha|}, g]$.*
(2) *If A is not finite over its center $Z(A)$ (that is, $|\alpha|$ is infinite), then $Z(A) = k[g]$.*

Theorem 16 ([12])**.** *Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quantum polynomial algebra. If the Beilinson algebra ∇A of A is not 2-representation tame, then the isomorphism classes of simple 2-regular modules over ∇A are parametrized by the set of points of $E \subsetneq \mathbb{P}^2$.*

Theorem 17 ([7, Theorem 4.3])**.** *Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quantum polynomial algebra of Type S'. If the Beilinson algebra ∇A of A is 2-representation tame, then the isomorphism classes of simple 2-regular modules over ∇A are parametrized by the set of points of $\mathbb{P}^2$.*

By using Proposition 15 and Theorems 16, 17, we have the following result:

Theorem 18 ([7, Theorem 4.4])**.** *For a 3-dimensional quantum polynomial algebra A of Type S', the following are equivalent*:

(1) *The noncommutative projective plane $\mathsf{Proj}_{\mathrm{nc}} A$ is finite over its center.*
(2) *The Beilinson algebra ∇A of A is 2-representation tame in the sense of Herschend, Iyama and Oppermann* [6]*.*
(3) *The isomorphism classes of simple 2-regular modules over ∇A are parameterized by $\mathbb{P}^2$.*

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Ayako Itaba
Katsushika Division
Institute of Arts and Sciences
Tokyo University of Science
6-3-1 Niijuku, Katsushika-ku,Tokyo, 125-8585, JAPAN
Email address: itaba@rs.tus.ac.jp

THE AUSLANDER–REITEN CONJECTURE FOR NORMAL RINGS

KAITO KIMURA

Abstract. In this article, we consider the Auslander–Reiten conjecture, which is a celebrated long-standing conjecture in ring theory. One of the main results of this article asserts that the conjecture holds for an arbitrary normal ring.

Key Words: Auslander–Reiten conjecture, Ext module, Serre's condition.

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1. Introduction

We refer the reader to [7] (`arXiv:2304.03956`) for details on the contents of this article. Throughout this article, we assume that R is a commutative noetherian ring and that M is a finitely generated R-module.

Auslander and Reiten [3] proposed the *generalized Nakayama conjecture*, which is rooted in the *Nakayama conjecture* [9] and asserts that for any artin algebra Λ, any indecomposable injective Λ-module appears as a direct summand in the minimal injective resolution of Λ. In addition, they proposed another conjecture, characterizing the projectivity of a module in terms of vanishing of Ext modules, which is called the *Auslander–Reiten conjecture*, and proved that this conjecture is true if and only if the generalized Nakayama conjecture is true.

The Auslander–Reiten conjecture remains meaningful for arbitrary commutative noetherian rings for formalization by Auslander, Ding, and Solberg [2]. The conjecture is known as follows: if $\operatorname{Ext}^i_R(M, M \oplus R) = 0$ for all $i \geq 1$, then M is projective. This conjecture is known to hold true if R is a complete intersection [2], or if R is a locally excellent Cohen-Macaulay normal ring containing the field of rational numbers $\mathbb{Q}$ [6], or if R is a Gorenstein normal ring [1], or if R is a Cohen–Macaulay normal ring and M is a maximal Cohen–Macaulay module of rank one [5], or if R is a Cohen–Macaulay normal ring and M is a maximal Cohen–Macaulay module such that $\operatorname{Hom}_R(M, M)$ is projective [4]. Recently, Kimura, Otake, and Takahashi [8] proved the conjecture for every Cohen-Macaulay normal ring. Even if R is not Cohen–Macaulay, it is known that R satisfies the conjecture if it is a normal ring and either $\operatorname{Ext}^i_R(\operatorname{Hom}_R(M, M), R) = 0$ for all $2 \leq i \leq \operatorname{depth} R$ or $\operatorname{Hom}_R(M, M)$ has finite G-dimension [10], or if it is a quotient of a regular ring and is a normal ring containing $\mathbb{Q}$ [4].

We give the following answer to this conjecture. We say that R satisfies *Serre's condition* (S_2) if $\operatorname{depth} R_{\mathfrak{p}} \geq \inf\{2, \operatorname{ht} \mathfrak{p}\}$ for all prime ideals $\mathfrak{p}$ of R.

The detailed version [7] of this article will be submitted for publication elsewhere.

Theorem 1. *Suppose that R satisfies (S_2). Then the Auslander–Reiten conjecture holds for R if it holds for $R_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of R such that $\operatorname{ht}\mathfrak{p} \le 1$. In particular, the Auslander–Reiten conjecture holds true for every normal ring.*

The above result is discussed in Section 2. It is worth noting that we shall prove the result of Kimura, Otake and Takahashi [8] without assuming Cohen–Macaulayness of the ring. We extend the method over Cohen–Macaulay rings to the general case, using the dualizing complex instead of the canonical module.

2. Comments on Theorem 1

In this section, we provide sufficient conditions for finitely generated modules over a commutative noetherian ring to be projective in terms of vanishing of Ext modules and prove the theorem stated in the Introduction. We prepare several lemmas to state Theorem 1. See [7] for proofs.

Lemma 2. *Let N be an R-module, and let I be an injective R-module. Then there is an isomorphism $\operatorname{Tor}_i^R(M, \operatorname{Hom}(N, I)) \cong \operatorname{Hom}(\operatorname{Ext}_R^i(M, N), I)$ for every integer $i \ge 0$.*

Lemma 3. *Let F be an R-linear functor on the category of R-modules. Let $\mathfrak{p}$ be a prime ideal of R and $E_R(R/\mathfrak{p})$ the injective hull of $R/\mathfrak{p}$. If $F(E_R(R/\mathfrak{p}))_\mathfrak{p}$ is the zero module, then so is $F(E_R(R/\mathfrak{p}))$.*

We denote by $(-)^*$ the R-dual $\operatorname{Hom}_R(-, R)$. Let R be a local ring, and let $F = (\cdots \to F_2 \to F_1 \xrightarrow{\alpha} F_0 \to 0)$ be a minimal free resolution of M. The (Auslander) transpose $\operatorname{Tr} M$ of M is defined as $\operatorname{Coker}(\alpha^*)$.

Lemma 4. *Let $(R, \mathfrak{m}, k)$ be a local ring, and let N be an R-module such that $k \otimes_R N$ is nonzero. Suppose that $\operatorname{Tor}_1^R(\operatorname{Tr} M, M \otimes_R N) = 0$. Then M is a free R-module.*

This Lemma 4 also played an important role in the proof of the main result of [8]. However, compared to [8, Proposition 3.3(1)], the assumption that N is finitely generated is removed by assuming $k \otimes_R N \ne 0$.

One of the main results of this article is the theorem below.

Theorem 5. *Let $(R, \mathfrak{m}, k)$ be a local ring of depth t. Suppose that $\operatorname{Ext}_R^i(M, R) = 0$ for all $1 \le i \le t$ and $\operatorname{Ext}_R^{t+1}(\operatorname{Tr} M, M^*) = 0$, and that M is locally free on the punctured spectrum of R. Then M is free.*

Proof. Put $d = \dim R$. We may assume that R admits a dualizing complex $D = (\cdots \to 0 \to D^0 \to \cdots \to D^{d-1} \to D^d \to 0 \to \cdots)$. Set $K = \operatorname{Ker}(D^{d-t} \to D^{d-t+1})$. It follows from Lemma 2 that $\operatorname{Tor}_{t+1}^R(\operatorname{Tr} M, M \otimes_R D^d) = 0$. Lemma 3 implies that for any $i \ne 0$ and $j \ne d$, $\operatorname{Tor}_i^R(\operatorname{Tr} M, M \otimes_R D^j) = 0$. From the above, we have $\operatorname{Tor}_1^R(\operatorname{Tr} M, M \otimes_R K) = 0$ by Lemma 2. Noting that $k \otimes_R K \ne 0$, we see that Lemma 4 concludes that M is free. $\square$

Below is a direct corollary of Theorem 5.

Corollary 6. *Let R be a local ring of depth $t \ge 2$. Suppose that M is locally free on the punctured spectrum of R. Then M is free in each of the two cases below.*
(1) $\operatorname{Ext}_R^i(M, R) = 0 = \operatorname{Ext}_R^{t-1}(M^*, M^*)$ *for all* $1 \le i \le t$.
(2) $\operatorname{Ext}_R^i(M, R) = 0 = \operatorname{Ext}_R^{t-1}(M, M)$ *for all* $1 \le i \le 2t + 1$.

We obtain Theorem 1 as a corollary of Corollary 6. Indeed, applying the case (2) of Corollary 6, we can prove by induction on $\operatorname{ht}\mathfrak{p}$ that $M_{\mathfrak{p}}$ is free for any prime ideal $\mathfrak{p}$ of R.

3. Comparison with previous studies

The results obtained in this article refine (or recover) a lot of results in the literature.

Remark 7. (1) Corollary 6(2) is a non-Gorenstein version of [1, Corollary 10]. Indeed, let R be a Gorenstein ring of dimension $d \geq 2$. It is seen that $\operatorname{Ext}^i_R(M, R) = 0$ for all $i > d$ and that M is maximal Cohen–Macaulay if and only if for all $1 \leq j \leq d$, $\operatorname{Ext}^j_R(M, R) = 0$.

(2) The Auslander–Reiten conjecture is known to hold for every Cohen–Macaulay normal ring by virtue of [10, Corollary 1.3]. Theorem 1 shows that the conjecture also holds for an arbitrary normal ring, i.e. it refines [10, Corollary 1.3].

(3) The Auslander–Reiten conjecture holds true if R is a quotient of a regular local ring and is a normal ring containing $\mathbb{Q}$ [4, Theorem 3.14]. In particular, every complete normal local ring of equicharacteristic zero satisfies the conjecture. Note that since the normality is not necessarily stable under completion, it is not easy to remove from these results the assumption that R is a quotient of a regular local ring or R is complete. Theorem 1, however, does make it happen.

(4) As mentioned in the introduction, besides (2) and (3) above, there are many results that show that the Auslander–Reiten conjecture holds in normal rings when some conditions are imposed; see [1, 4, 5, 6, 10] for instance. Theorem 1 improves all of them.

(5) The Auslander–Reiten conjecture is known to hold true if R is a complete intersection [2, Proposition 1.9]. Using this result, we see that if R is a complete intersection, then R satisfies (S_2) and the Auslander–Reiten conjecture holds for $R_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of R such that $\operatorname{ht}\mathfrak{p} \leq 1$. In this sense, Theorem 1 refines [2, Proposition 1.9].

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Graduate School of Mathematics
Nagoya University
Furocho, Chikusaku, Nagoya 464-8602, JAPAN
Email address: m21018b@math.nagoya-u.ac.jp

ON INDUCTIONS AND RESTRICTIONS OF SUPPORT τ-TILTING MODULES OVER GROUP ALGEBRAS

RYOTARO KOSHIO AND YUTA KOZAKAI

Abstract. Let G be a finite group, k an algebraically closed field of characteristic $p > 0$, and N a normal subgroup of G. Support τ-tilting modules over group algebras are under the one-to-one correspondences with many kinds of important objects for the representation theory. We will compare a certain subset of the support τ-tilting modules over kN and that of kG, and give a poset isomorphism between these two sets. Moreover, we introduce two applications of the results.

1. Motivation

Since τ-tilting theory was introduced by Adachi-Iyama-Reiten in [2], classifications and features of the support τ-tilting modules have been given for many kinds of algebras. In particular, for group algebras and their block algebras, the considerations of the support τ-tilting modules are equivalent to those of two-term tilting complexes which control derived equivalences, hence they are expected to help the solution of the Broué's Abelian Defect Group Conjecture. For that perspective, it is important to consider the support τ-tilting modules for group algebras and their block algebras.

Let k be an algebraically closed field of characteristic $p > 0$, G a finite group, N a normal subgroup of G, and X a support τ-tilting kN-module. In [4], the authors showed that if N has a cyclic Sylow p-subgroup and if the index of N in G is a power of p, then the induction functor $\mathrm{Ind}_N^G := kG \otimes_{kN} -$ gives a poset isomorphism between the set of support τ-tilting modules over kN and that over kG. Also, in [3], the first author showed that if X is G-invariant, then $\mathrm{Ind}_N^G X$ is a support τ-tilting module over kG.

Naturally, we consider the following two questions.

- For the restriction functor Res_N^G, when is $\mathrm{Res}_N^G M$ a support τ-tilting kG-module for support τ-tilting kG-module M?
- Without the assumption that N has a cyclic Sylow p-subgroup and that the index of N in G is a power of p, can we determine the image of G-invariant support τ-tilting modules over kN under the induction functor Ind_N^G?

In this report, we give some results as positive answers of the above questions. Moreover we give applications of the results.

2. Main results

In this section, let k be an algebraically closed field of characteristic $p > 0$, G a finite group and N a normal subgroup of G. Moreover Ind_N^G means the induction functor and

The detailed version of this paper will be submitted for publication elsewhere.

Res_N^G means the restriction functor. For $\Lambda \in \{kN, kG\}$ and Λ-module M, we denote by $\mathrm{add}\, M$ the set of all Λ-modules which are direct summand of $M^{\oplus r}$ for some integer $r \in \mathbb{Z}$.

First, we recall the definition of support τ-tilting modules introduced by Adachi-Iyama-Reiten [2]. For a finite dimensional algebra A and an A-module M, we denote by $|M|$ the number of pairwise non-isomorphic indecomposable direct summands of M and by $\mathrm{s}(M)$ the number of pairwise non-isomorphic composition factors of M.

Definition 1 ([2])**.** Let M be an A-module.

(1) We say that the A-module M is *τ-rigid* if $\mathrm{Hom}_A(M, \tau M) = 0$, here τ means the Auslander-Reiten translation.

(2) We say that the A-module M is a *support τ-tilting module* if M is τ-rigid and if $|M| = \mathrm{s}(M)$.

Here we remark that the above definition is different from the original one, but it is equivalent definition to the original one (see [1]).

Remark 2. Let M be a A-module. If A is a symmetric algebra, then τM is isomorphic to $\Omega^2 M$. In particular, if A is a group algebra or a block algebra of a group algebra, then the isomorphism holds.

2.1. **Restricted support τ-tilting modules.** As a answer to the first question in Section 1, we have one result. We recall that the relative projectivity of kG-modules.

Definition 3. Let G be a finite group, H a subgroup of G, and M a kG-module. We say that M is *relatively H-projective* if it holds that M is a direct summand of $\mathrm{Ind}_H^G \mathrm{Res}_H^G X$.

Now we state the first result.

Theorem 4. *Let k be an algebraically closed field of characteristic $p > 0$, G a finite group, N a normal subgroup of G, and M a support τ-tilting kG-module. If M is relatively N-projective and it holds that $\mathrm{Ind}_N^G \mathrm{Res}_N^G M \in \mathrm{add}\, M$, then the restricted module $\mathrm{Res}_N^G M$ is a support τ-tilting kN-module.*

Before stating the second result, we recall G-invariances of kN-modules.

Definition 5. Let G be a finite group, N a normal subgroup, and M a kN-module. For $g \in G$, we construct a kN-module gM by the following data.

- As a set $gM := \{gm \mid m \in M\}$.
- For $x \in N$ and $gm \in gM$, the action of x is given by $x(gm) := g(g^{-1}xgm)$.

We say that M is *G-invariant* if M is isomorphic to gM as kN-modules for any $g \in G$.

The next theorem explains how strong the assumption in Theorem 4 is in a sense.

Theorem 6. *Let k be an algebraically closed field of characteristic $p > 0$, G a finite group, N a normal subgroup of G, and M a support τ-tilting kG-module. The following conditions are equivalent:*

- *The support τ-tilting kG-module M is relatively N-projective and it holds that $\mathrm{Ind}_N^G \mathrm{Res}_N^G M \in \mathrm{add}\, M$.*
- $\mathrm{add}\, M = \mathrm{add}\, \mathrm{Ind}_N^G X$ *for some G-invariant support τ-tilting kN-module X.*
- *For each simple $k(G/N)$-module S, it holds that $S \otimes_k M \in \mathrm{add}\, M$.*

2.2. **The image of the induction functor** Ind_N^G**.** As a answer to the second question in Section 1, we introduce one result.

For $\Lambda \in \{kG, kN\}$ and Λ-modules X and Y, we say that X is add-*equivalent* to Y if $\mathrm{add}\, X = \mathrm{add}\, Y$, and we denote the set of add-equivalence classes of support τ-tilting Λ-modules by sτ-tiltΛ. Moreover we denote the set of add-equivalence classes of G-invariant support τ-tilting kN-modules by $(\mathrm{s}\tau\text{-tilt}kN)^G$.

We know that the induction functor Ind_N^G gives a well-defined map from $(\mathrm{s}\tau\text{-tilt}kN)^G$ to sτ-tiltkG by the following result.

Theorem 7 ([3, Theorem 3.2])**.** *For $M \in (\mathrm{s}\tau\text{-tilt}kN)^G$, the induced module Ind_N^G is a support τ-tilting kG-module.*

As we stated in Section 1, we wonder if we describe the image of $(\mathrm{s}\tau\text{-tilt}kN)^G$ by the induction functor explicitly. The following is one answer to this question.

Theorem 8. *Let $(\mathrm{s}\tau\text{-tilt}kG)^\star$ be the set of* add-*equivalence classes of support τ-tilting kG-modules satisfying the equivalent conditions in Theorem 6. Then the induction functor $\mathrm{Ind}_N^G M$ gives a poset isomorphism between $(\mathrm{s}\tau\text{-tilt}kN)^G$ and $(\mathrm{s}\tau\text{-tilt}kG)^\star$:*

$$\mathrm{Ind}_N^G : (\mathrm{s}\tau\text{-tilt}kN)^G \xrightarrow{\sim} (\mathrm{s}\tau\text{-tilt}kG)^\star \ \ (M \mapsto \mathrm{Ind}_N^G M).$$

In particular, the image of $(\mathrm{s}\tau\text{-tilt}kN)^G$ by the induction functor is $(\mathrm{s}\tau\text{-tilt}kG)^\star$

2.3. **Applications.** We consider the case that the quotient group G/N is a p-group. Then the only simple $k(G/N)$-module is the trivial $k(G/N)$-module up to isomorphism, here trivial $k(G/N)$-module means one dimensional vector space on which any element $\overline{g} \in G/N$ acts trivially. Moreover we can easily check that for any kG-module M and the trivial $k(G/N)$-module $k_{G/N}$, the isomorphism $k_{G/N} \otimes_k M \cong M$ holds. By using these facts and Theorem 6, we have the following.

Theorem 9. *Let G/N be a p-group. Then we have the following isomorphism of the partially ordered sets by the induction functor Ind_N^G:*

$$\mathrm{Ind}_N^G : (\mathrm{s}\tau\text{-tilt}kN)^G \xrightarrow{\sim} \mathrm{s}\tau\text{-tilt}kG \ \ (M \mapsto \mathrm{Ind}_G^{\tilde{G}} M).$$

As a further application, we consider the vertex of an indecomposable τ-rigid kG-module. We recall the definition of the vertices of the indecomposable kG-modules.

Definition 10. Let M be an indecomposable kG-module. We say that a subgroup H of G is a *vertex* of M if H is a minimal subgroup of G with the property that M is relatively H-projective.

It is known that a vertex is unique up to conjugacy, and a p-subgroup of G. Also, a vertex of the trivial kG-module is a Sylow p-subgroup. We consider vertices of indecomposable τ-rigid kG-modules, and we have the following result by using Theorem 9.

Theorem 11. *Let G be a finite group and k an algebraically closed field of characteristic $p > 0$. Then any indecomposable τ-rigid kG-module has a vertex contained in a Sylow p-subgroup properly if and only if G has a proper normal subgroup of p-power index.*

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Ryotaro Koshio
Deoartment of mathematics
Tokyo University of Science
Kagurazaka 1-3, Shinjuku, Tokyo 162-8601, Japan
Email address: `koshio@rs.tus.ac.jp`

Yuta Kozakai
Deoartment of mathematics
Tokyo University of Science
Kagurazaka 1-3, Shinjuku, Tokyo 162-8601, Japan
Email address: `Yuta Kozakai`

CLASSIFICATION OF TWISTED ALGEBRAS OF 3-DIMENSIONAL SKLYANIN ALGEBRAS

MASAKI MATSUNO

Abstract. A twisting system is one of the major tools to study graded algebras, however, it is often difficult to construct a (non-algebraic) twisting system if a graded algebra is given by generators and relations. In this paper, we show that a twisted algebra of a geometric algebra is determined by a certain automorphism of its point variety. As an application, we classify twisted algebras of 3-dimensional Sklyanin algebras up to graded algebra isomorphism.

1. Introduction

This paper is based on [5]. The notion of twisting system was introduced by Zhang in [8]. If there is a twisting system $\theta = \{\theta_n\}_{n\in\mathbb{Z}}$ for a graded algebra A, then we can define a new graded algebra A^θ, called a twisted algebra. Zhang gave a necessary and sufficient algebraic condition via a twisting system when two categories of graded right modules are equivalent ([8, Theorem 3.5]). Although a twisting system plays an important role to study a graded algebra, it is often difficult to construct a twisting system on a graded algebra if it is given by generators and relations.

Mori introduced the notion of geometric algebra $\mathcal{A}(E,\sigma)$ which is determined by a geometric data which consists of a projective variety E, called the point variety, and its automorphism σ. For these algebras, Mori gave a necessary and sufficient geometric condition when two categories of graded right modules are equivalent ([6, Theorem 4.7]). By using this geometric condition, we can easily construct a twisting system.

Cooney and Grabowski defined a groupoid whose objects are geometric noncommutative projective spaces and whose morphisms are isomorphisms between them. By studying a correspondence between the morphisms of this groupoid and a twisting system, they showed that the morphisms of this groupoid are parametrized by a set of certain automorphisms of the point variety ([1, Theorem 28]).

In this paper, we focus on studying a twisted algebra of a geometric algebra $\mathcal{A}(E,\sigma)$. For a twisting system θ on A, we set $\Phi(\theta) := \overline{(\theta_1|_{A_1})^*} \in \operatorname{Aut}_k \mathbb{P}(A_1^*)$ by dualization and projectivization. We find a subset $M(E,\sigma)$ of $\operatorname{Aut}_k \mathbb{P}(A_1^*)$ parametrizing twisted algebras of A up to isomorphism. We show that a twisted algebra of a geometric algebra is determined by a certain automorphism of its point variety. As an application, we classify twisted algebras of 3-dimensional Sklyanin algebras up to graded algebra isomorphism.

The detailed version of this paper is [5].

2. Twisting systems and twisted algebras

Throughout this paper, we fix an algebraically closed field k of characteristic zero and assume that a graded algebra is an $\mathbb{N}$-graded algebra $A = \bigoplus_{i\in\mathbb{N}} A_i$ over k. A graded algebra $A = \bigoplus_{i\in\mathbb{N}} A_i$ is called *connected* if $A_0 = k$. Let $\operatorname{GrAut}_k A$ denote the group of graded algebra automorphisms of A. We denote by $\operatorname{GrMod} A$ the category of graded right A-modules. We say that two graded algebras A and A' are *graded Morita equivalent* if two categories $\operatorname{GrMod} A$ and $\operatorname{GrMod} A'$ are equivalent.

Definition 1. Let A be a graded algebra. A set of graded k-linear automorphisms $\theta = \{\theta_n\}_{n\in\mathbb{Z}}$ of A is called a twisting system on A if

$$\theta_n(a\theta_m(b)) = \theta_n(a)\theta_{n+m}(b)$$

for any $l, m, n \in \mathbb{Z}$ and $a \in A_m$, $b \in A_l$. The twisted algebra of A by θ, denoted by A^θ, is a graded algebra A with a new multiplication $*$ defined by

$$a * b = a\theta_m(b)$$

for any $a \in A_m$, $b \in A_l$. A twisting system $\theta = \{\theta_n\}_{n\in\mathbb{Z}}$ is called *algebraic* if $\theta_{m+n} = \theta_m \circ \theta_n$ for every $m, n \in \mathbb{Z}$.

We denote by $\operatorname{TS}(A)$ the set of twisting systems on A. Zhang [8] found a necessary and sufficient algebraic condition for $\operatorname{GrMod} A \cong \operatorname{GrMod} A'$.

Theorem 2 ([8, Theorem 3.5])**.** *Let A and A' be graded algebras finitely generated in degree 1 over k. Then $\operatorname{GrMod} A \cong \operatorname{GrMod} A'$ if and only if A' is isomorphic to a twisted algebra A^θ by a twisting system $\theta \in \operatorname{TS}(A)$.*

Definition 3. For a graded algebra A, we define

$$\begin{aligned}
\operatorname{TS}_0(A) &:= \{\theta \in \operatorname{TS}(A) \mid \theta_0 = \mathrm{id}_A\} \\
\operatorname{TS}_{\mathrm{alg}}(A) &:= \{\theta \in \operatorname{TS}_0(A) \mid \theta \text{ is algebraic }\} \\
\operatorname{Twist}(A) &:= \{A^\theta \mid \theta \in \operatorname{TS}(A)\}/_{\cong} \\
\operatorname{Twist}_{\mathrm{alg}}(A) &:= \{A^\theta \mid \theta \in \operatorname{TS}_{\mathrm{alg}}(A)\}/_{\cong}.
\end{aligned}$$

Lemma 4 ([8, Proposition 2.4])**.** *Let A be a graded algebra. For every $\theta \in \operatorname{TS}(A)$, there exists $\theta' \in \operatorname{TS}_0(A)$ such that $A^\theta \cong A^{\theta'}$.*

It follows from Lemma 4 that

$$\operatorname{Twist}(A) = \{A^\theta \mid \theta \in \operatorname{TS}_0(A)\}/_{\cong},$$

so we may assume that $\theta \in \operatorname{TS}_0(A)$ to study $\operatorname{Twist}(A)$. By the definition of twisting system, it follows that $\theta \in \operatorname{TS}_{\mathrm{alg}}(A)$ if and only if $\theta_n = \theta_1^n$ for every $n \in \mathbb{Z}$ and $\theta_1 \in \operatorname{GrAut}_k A$, so

$$\operatorname{Twist}_{\mathrm{alg}}(A) = \{A^\phi \mid \phi \in \operatorname{GrAut}_k A\}/_{\cong}$$

where A^ϕ means the twisted algebra of A by $\{\phi^n\}_{n\in\mathbb{Z}}$.

3. Twisted algebras of geometric algebras

Let V be a finite dimensional k-vector space and $A = T(V)/(R)$ be a quadratic algebra where $T(V)$ is a tensor algebra over k and R is a subspace of $V \otimes V$. Since an element of R defines a multilinear function on $V^* \times V^*$, we can define a zero set associated to R by

$$\mathcal{V}(R) = \{(p, q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid g(p, q) = 0 \text{ for any } g \in R\}.$$

Definition 5. Let $A = T(V)/(R)$ be a quadratic algebra. A geometric pair (E, σ) consists of a projective variety $E \subset \mathbb{P}(V^*)$ and $\sigma \in \mathrm{Aut}_k E$. We say that A is a *geometric algebra* if there exists a geometric pair (E, σ) such that

(G1) $\mathcal{V}(R) = \{(p, \sigma(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in E\}$,

(G2) $R = \{g \in V \otimes V \mid g(p, \sigma(p)) = 0 \text{ for all } p \in E\}$.

In this case, we call E the *point variety* of A, and write $A = \mathcal{A}(E, \sigma)$.

We use the following notations introduced in [1]:

Definition 6. Let $E \subset \mathbb{P}(V^*)$ be a projective variety and $\sigma \in \mathrm{Aut}_k E$. We define

$$\mathrm{Aut}_k(E \uparrow \mathbb{P}(V^*)) := \{\tau \in \mathrm{Aut}_k E \mid \tau = \overline{\tau}|_E \text{ for some } \overline{\tau} \in \mathrm{Aut}_k \mathbb{P}(V^*)\},$$
$$\mathrm{Aut}_k(\mathbb{P}(V^*) \downarrow E) := \{\tau \in \mathrm{Aut}_k \mathbb{P}(V^*) \mid \tau|_E \in \mathrm{Aut}_k E\},$$
$$Z(E, \sigma) := \{\tau \in \mathrm{Aut}_k(\mathbb{P}(V^*) \downarrow E) \mid \sigma\tau|_E\sigma^{-1} = \tau|_E\},$$
$$M(E, \sigma) := \{\tau \in \mathrm{Aut}_k(\mathbb{P}(V^*) \downarrow E) \mid (\tau|_E\sigma)^i\sigma^{-i} \in \mathrm{Aut}_k(E \uparrow \mathbb{P}(V^*)) \ \forall i \in \mathbb{Z}\},$$
$$N(E, \sigma) := \{\tau \in \mathrm{Aut}_k(\mathbb{P}(V^*) \downarrow E) \mid \sigma\tau|_E\sigma^{-1} \in \mathrm{Aut}_k(E \uparrow \mathbb{P}(V^*))\}.$$

Note that $Z(E, \sigma) \subset M(E, \sigma) \subset N(E, \sigma) \subset \mathrm{Aut}_k(\mathbb{P}(V^*) \downarrow E)$, and $Z(E, \sigma)$, $N(E, \sigma)$ are subgroups of $\mathrm{Aut}_k(\mathbb{P}(V^*) \downarrow E)$.

Let $A = \mathcal{A}(E, \sigma)$ be a geometric algebra. The map $\Phi : \mathrm{TS}_0(A) \to \mathrm{Aut}_k \mathbb{P}(A_1^*)$ is defined by $\Phi(\theta) := \overline{(\theta_1|_{A_1})^*}$. This map plays an important role to study twisted algebras of geometric algebras.

Lemma 7 ([5, Lemma 3.3 and Lemma 3.4])**.** *Let $A = \mathcal{A}(E, \sigma)$ be a geometric algebra.*

(1) *$\Phi(\mathrm{TS}_0(A)) = M(E, \sigma)$.*

(2) *$\Phi(\mathrm{TS}_{\mathrm{alg}}(A)) = Z(E, \sigma)$.*

The following is one of the main results.

Theorem 8 ([5, Theorem 3.5])**.** *Let $A = \mathcal{A}(E, \sigma)$ be a geometric algebra.*

(1) *$\mathrm{Twist}(A) = \{\mathcal{A}(E, \tau|_E\sigma) \mid \tau \in M(E, \sigma)\}/_{\cong}$.*

(2) *$\mathrm{Twist}_{\mathrm{alg}}(A) = \{\mathcal{A}(E, \tau|_E\sigma) \mid \tau \in Z(E, \sigma)\}/_{\cong}$.*

4. Twisted algebras of 3-dimensional Sklyanin algebras

In this section, we classify twisted algebras of 3-dimensional Sklyanin algebras. A 3-dimensional Sklyanin algebra is a typical example of 3-dimensional quadratic AS-regular algebras. It is known that every 3-dimensional Sklyanin algebra is a geometric algebra $\mathcal{A}(E, \sigma)$ where E is an elliptic curve in $\mathbb{P}^2$ and σ is a translation by some point $p \in E$.

First, we recall some properties of elliptic curves in $\mathbb{P}^2$. Let E be an elliptic curve in $\mathbb{P}^2$. We use a *Hesse form*

$$E = \mathcal{V}(x^3 + y^3 + z^3 - 3\lambda xyz)$$

where $\lambda \in k$ with $\lambda^3 \neq 1$. It is known that every elliptic curve in $\mathbb{P}^2$ can be written in this form (see [2, Corollary 2.18]). The j-invariant of a Hesse form E is given by

$$j(E) = \frac{27\lambda^3(\lambda^3+8)^3}{(\lambda^3-1)^3}$$

(see [2, Proposition 2.16]). The j-invariant $j(E)$ classifies elliptic curves in $\mathbb{P}^2$ up to projective equivalence (see [3, Theorem IV 4.1 (b)]). We fix the group structure on E with the zero element $o := (1,-1,0) \in E$ (see [2, Theorem 2.11]). For a point $p \in E$, a *translation* by p, denoted by σ_p, is an automorphism of E defined by $\sigma_p(q) = p + q$ for every $q \in E$. We define $\mathrm{Aut}_k(E, o) := \{\sigma \in \mathrm{Aut}_k E \mid \sigma(o) = o\}$. It is known that $\mathrm{Aut}_k(E, o)$ is a finite cyclic subgroup of $\mathrm{Aut}_k E$ (see [3, Corollary IV 4.7]).

Lemma 9 ([4, Theorem 4.6])**.** *A generator of* $\mathrm{Aut}_k(E, o)$ *is given by*

(1) $\tau_E(a, b, c) := (b, a, c)$ *if* $j(E) \neq 0, 12^3$,

(2) $\tau_E(a, b, c) := (b, a, \varepsilon c)$ *if* $\lambda = 0$ *(so that* $j(E) = 0$*)*,

(3) $\tau_E(a, b, c) := (\varepsilon^2 a+\varepsilon b+c, \varepsilon a+\varepsilon^2 b+c, a+b+c)$ *if* $\lambda = 1+\sqrt{3}$ *(so that* $j(E) = 12^3$*)*

where ε *is a primitive* 3*rd root of unity. In particular,* $\mathrm{Aut}_k(E, o)$ *is a subgroup of* $\mathrm{Aut}_k(E \uparrow \mathbb{P}^2) = \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$.

Remark 10. When $j(E) = 0, 12^3$, we may fix $\lambda = 0, 1 + \sqrt{3}$ respectively as in Lemma 9, because if two elliptic curves E and E' in $\mathbb{P}^2$ are projectively equivalent, then for every $\mathcal{A}(E, \sigma)$, there exists an automorphism $\sigma' \in \mathrm{Aut}_k E'$ such that $\mathcal{A}(E, \sigma) \cong \mathcal{A}(E', \sigma')$ (see [7, Lemma 2.6]).

It follows from [4, Proposition 4.5] that every automorphism $\sigma \in \mathrm{Aut}_k E$ can be written as $\sigma = \sigma_p\tau_E^i$ where σ_p is a translation by a point $p \in E$, τ_E is a generator of $\mathrm{Aut}_k(E, o)$ and $i \in \mathbb{Z}_{|\tau_E|}$. For any $n \geq 1$, we call a point $p \in E$ *n-torsion* if $np = o$. We set $E[n] := \{p \in E \mid np = o\}$ and $T[n] := \{\sigma_p \in \mathrm{Aut}_k E \mid p \in E[n]\}$. It follows from [4, Theorem 4.12 (3)] that every automorphism $\sigma \in \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$ can be written as $\sigma = \sigma_q\tau_E^i$ where $q \in E[3]$ and $i \in \mathbb{Z}_{|\tau_E|}$.

Let $E = \mathcal{V}(x^3 + y^3 + z^3 - 3\lambda xyz)$ be an elliptic curve in $\mathbb{P}^2$ and $p = (a, b, c) \in E \setminus E[3]$. Then $\mathcal{A}(E, \sigma_p)$ is called a 3*-dimensional Sklyanin algebra*, and

$$\mathcal{A}(E, \sigma_p) = k\langle x, y, z\rangle/(ayz + bzy + cx^2, azx + bxz + cy^2, axy + byx + cz^2).$$

Lemma 11 ([5, Lemma 4.10])**.** *Let* $A = \mathcal{A}(E, \sigma_p)$ *be a* 3*-dimensional Sklyanin algebra where* $p \in E \setminus E[3]$.

(1) *For* $\sigma_q\tau_E^i \in \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$, $\sigma_q\tau_E^i \in Z(E, \sigma_p)$ *if and only if* $p - \tau_E^i(p) = o$.

(2) *For* $\sigma_q\tau_E^i \in \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$, $\sigma_q\tau_E^i \in N(E, \sigma_p)$ *if and only if* $p - \tau_E^i(p) \in E[3]$.

(3) $M(E, \sigma_p) = N(E, \sigma_p)$.

By Theorem 8, to classify twisted algebras of 3-dimensional Sklyanin algebras $\mathcal{A}(E, \sigma_p)$ up to isomorphism of graded algebras, it is enough to classify subsets $Z(E, \sigma_p)$ and $M(E, \sigma_p)$ of $\mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$.

Theorem 12 ([5, Theorem 4.11])**.** *Let* $A = \mathcal{A}(E, \sigma_p)$ *be a* 3*-dimensional Sklyanin algebra. Then the following table gives* $Z(E, \sigma_p)$ *and* $M(E, \sigma_p)$*;*

Type	$j(E)$	$Z(E, \sigma_p)$	$M(E, \sigma_p)$
	$j(E) \neq 0, 12^3$	$\begin{cases} T[3] & \text{if } p \notin E[2] \\ \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E) & \text{if } p \in E[2] \end{cases}$	$\begin{cases} T[3] & \text{if } p \notin E[6] \\ \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E) & \text{if } p \in E[6] \end{cases}$
EC	$j(E) = 0$	$\begin{cases} T[3] & \text{if } p \notin E[2] \\ T[3] \rtimes \langle \tau_E^3 \rangle & \text{if } p \in E[2] \end{cases}$	$\begin{cases} T[3] & \text{if } p \notin \mathcal{E} \cup E[6] \\ T[3] \rtimes \langle \tau_E^2 \rangle & \text{if } p \in \mathcal{E} \\ T[3] \rtimes \langle \tau_E^3 \rangle & \text{if } p \in E[6] \end{cases}$
	$j(E) = 12^3$	$\begin{cases} T[3] & \text{if } p \notin E[2] \\ T[3] \rtimes \langle \tau_E^2 \rangle & \text{if } p \in E[2] \setminus \langle (1,1,\lambda) \rangle \\ \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E) & \text{if } p = (1,1,\lambda) \end{cases}$	$\begin{cases} T[3] & \text{if } p \notin E[6] \\ T[3] \rtimes \langle \tau_E^2 \rangle & \text{if } p \in E[6] \setminus \mathcal{F} \\ \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E) & \text{if } p \in \mathcal{F} \end{cases}$

where $\mathcal{E} := \{(a, b, c) \in E \mid a^9 = b^9 = c^9\} \subset E[9] \setminus E[6]$ *and* $\mathcal{F} := \langle (1, 1, \lambda) \rangle \oplus E[3]$.

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Katsushika division
Institute of Arts and Sciences
Tokyo University of Science
6-3-1 Niijuku, Katsushika-Ku, Tokyo, 125-8585, Japan
Email address: masaki.matsuno@rs.tus.ac.jp

THE CLASSIFICATION OF 3-DIMENSIONAL CUBIC AS-REGULAR ALGEBRAS OF TYPE P, S, T AND WL

MASAKI MATSUNO AND YU SAITO

Abstract. Classification of AS-regular algebras is one of the most important projects in noncommutative algebraic geometry. In this paper, we extend the notion of geometric algebra to cubic algebras, and give a geometric condition for isomorphism and graded Morita equivalence. One of the main results is a complete list of defining relations of 3-dimensional cubic AS-regular algebras corresponding to $\mathbb{P}^1 \times \mathbb{P}^1$ or a union of irreducible divisors of bidegree $(1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, we classify them up to isomorphism and up to graded Morita equivalence in terms of their defining relations.

1. Artin-Schelter regular algebras

Throughout this report, let k be an algebraically closed field of characteristic 0, A a graded algebra finitely generated in degree 1 over k. That is, $A = k\langle x_1, \cdots, x_n\rangle/I$ where $\deg x_i = 1$ for any $i = 1, \cdots, n$, and I is a homogeneous two-sided ideal of $k\langle x_1, \cdots, x_n\rangle$ with $I_0 = I_1 = 0$. We call $A = \langle x_1, \cdots, x_n\rangle/I$ a *cubic algebra* if I is an two-sided ideal of $k\langle x_1, \cdots, x_n\rangle$ generated by homogeneous polynomials of degree three. We denote by $\operatorname{GrMod} A$ the category of graded right A-modules and graded right A-module homomorphisms preserving degrees. We say that two graded algebras A and A' are *graded Morita equivalent* if the categories $\operatorname{GrMod} A$ and $\operatorname{GrMod} B$ are equivalent.

Let A be a graded algebra. We recall that

$$\operatorname{GKdim} A := \inf\{\alpha \in \mathbb{R} \mid \dim(\textstyle\sum_{i=0}^{n} A_i) \leq n^\alpha \text{ for all } n \gg 0\}$$

is called the *Gelfand-Kirillov dimension* of A. In noncommutative algebraic geometry, Artin-Schelter regular algebras are main objects to study.

Definition 1 ([1]). A graded algebra A is called a *d-dimensional Artin-Schelter regular* (*simply AS-regular*) *algebra* if A satisfies the following conditions:

(1) $\operatorname{gldim} A = d < \infty$,
(2) $\operatorname{GKdim} A < \infty$,
(3) $\operatorname{Ext}^i_A(k, A) = \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

It follows from [1, Theorem 1.5 (i)] that a 3-dimensional AS-regular algebra A finitely generated in degree 1 over k is one of the following forms:

$$A = k\langle x, y, z\rangle/(f_1, f_2, f_3)$$

where f_i are homogeneous polynomials of degree 2 (quadratic case), or

$$A = k\langle x, y\rangle/(g_1, g_2)$$

The detailed version of this paper has been submitted for publication elsewhere.

where g_j are homogeneous polynomials of degree 3 (cubic case). In this report, we focus on studying 3-dimensional cubic AS-regular algebras.

2. 3-GEOMETRIC ALGEBRAS

Artin, Tate and Van den Bergh [2] found a nice one-to-one correspondence between 3-dimensional AS-regular algebras and pairs (E, σ) where E is a scheme and σ is an automorphism of E. Focusing on pairs (E, σ), Mori introduced the notion of geometric algebra which determines and is determined by a pair (E, σ) (see [3, Definition 4.3]). In this report, we extend the notion of geometric algebra to cubic algebras.

Let $A = k\langle x_1, \cdots, x_n\rangle/(R)$ be a cubic algebra where R is a subspace of $k\langle x_1, \cdots, x_n\rangle_3$. We denote by $\mathbb{P}^{n-1}$ the projective space of dimension $n-1$ over k. We define the zero set of R by

$$\mathcal{V}(R) := \{(p, q, r) \in (\mathbb{P}^{n-1})^{\times 3} \mid f(p, q, r) = 0 \ \ \forall f \in R\}.$$

Let $E \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ be a projective variety and $\pi_i : \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ i-th projections where $i = 1, 2$. We set the following notation:

$$\operatorname{Aut}_k^G E := \{\sigma \in \operatorname{Aut}_k E \mid \pi_1\sigma(p, q) = \pi_2(p, q) \ \ \forall (p, q) \in E\}.$$

We say that a pair (E, σ) is a 3-*geometric pair* if $\sigma \in \operatorname{Aut}_k^G E$.

Definition 2. Let $A = k\langle x_1, \cdots, x_n\rangle/(R)$ be a cubic algebra. We say that A is a 3-*geometric algebra* if there exists 3-geometric pair (E, σ) such that

(G1) $\mathcal{V}(R) = \{(p, q, \pi_2\sigma(p, q)) \mid (p, q) \in E\}$,

(G2) $R = \{f \in k\langle x_1, \cdots, x_n\rangle_3 \mid f(p, q, \pi_2\sigma(p, q)) = 0 \ \ \forall (p, q) \in E\}$.

In this case, we write $A = \mathcal{A}(E, \sigma)$.

The following theorem tells us that classifying geometric algebras is equivalent to classifying 3-geometric pairs.

Theorem 3 (cf. [4, Lemma 2.5]). *Let $A = \mathcal{A}(E, \sigma)$ and $A' = \mathcal{A}(E', \sigma')$ be 3-geometric algebras.*

(1) *$A \cong A'$ as graded algebras if and only if there exists an automorphism τ of $\mathbb{P}^{n-1}$ such that $(\tau \times \tau)(E) = E'$ and the diagram*

$$\begin{tikzcd} E \arrow[r, "\tau\times\tau"] \arrow[d, "\sigma"'] & E' \arrow[d, "\sigma'"] \\ E \arrow[r, "\tau\times\tau"'] & E' \end{tikzcd}$$

commutes.

(2) $\operatorname{GrMod} A \cong \operatorname{GrMod} A'$ *if and only if there exists a sequence $\{\tau_i\}_{i\in\mathbb{Z}}$ of automorphisms of $\mathbb{P}^{n-1}$ such that $(\tau_i \times \tau_{i+1})(E) = E'$ and the diagram*

$$\begin{tikzcd} E \arrow[r, "\tau_i\times\tau_{i+1}"] \arrow[d, "\sigma"'] & E' \arrow[d, "\sigma'"] \\ E \arrow[r, "\tau_{i+1}\times\tau_{i+2}"'] & E' \end{tikzcd}$$

commutes for all $i \in \mathbb{Z}$.

Definition 4. Let E and E' be projective varieties in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$.

(1) We say that E and E' are *equivalent*, denoted by $E \sim E'$, if $E' = (\tau_1 \times \tau_2)(E)$ for some $\tau_1, \tau_2 \in \mathrm{Aut}_k \mathbb{P}^{n-1}$.

(2) We say that E and E' are 2-*equivalent*, denoted by $E \sim_2 E'$, if $E' = (\tau \times \tau)(E)$ for some $\tau \in \mathrm{Aut}_k \mathbb{P}^{n-1}$.

It is clear that if E and E' are 2-equivalent, then they are equivalent. Let $A = \mathcal{A}(E, \sigma)$ and $A' = \mathcal{A}(E', \sigma')$ be 3-geometric algebras. If $A \cong A'$ (resp.GrMod $A \cong$ GrMod A'), then E and E' are 2-equivalent (resp.equivalent) by Theorem 3. As the first step of classification of geometric algebras up to graded algebra isomorphism (resp.graded Morita equivalence), we need to classify projective varieties up to 2-equivalence (resp.equivalence).

3. Main results

In [2], Artin-Tate-Van den Bergh found a nice geometric characterization of 3-dimensional AS-regular algebras finitely generated in degree 1 over k. In this report, we focus on the cubic case.

Theorem 5 ([3])**.** *Every* 3*-dimensional cubic AS-reguar algebra A is a* 3*-geometric algebra* $A = \mathcal{A}(E, \sigma)$. *Moreover, E is $\mathbb{P}^1 \times \mathbb{P}^1$ or a divisor of bidegree* $(2, 2)$ *in* $\mathbb{P}^1 \times \mathbb{P}^1$.

In this report, we study two cases when $E = \mathbb{P}^1 \times \mathbb{P}^1$ and E is a union of two irreducible divisors of bidegree $(1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. For each case, we

(I) give a complete list of defining relations of 3-dimensional cubic AS-regular algebras,

(II) classify them up to isomorphism as graded algebras in terms of their defining relations, and

(III) classify them up to graded Morita equivalence in terms of their defining relations.

We first treat the case $E = \mathbb{P}^1 \times \mathbb{P}^1$. We denote by ν an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by $\nu(p, q) = (q, p)$ for $(p, q) \in \mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 6. $\mathrm{Aut}_k^G(\mathbb{P}^1 \times \mathbb{P}^1) = \{(\mathrm{id} \times \rho)\nu \mid \rho \in \mathrm{Aut}_k \mathbb{P}^1\}$.

Example 7. For $\rho \in \mathrm{Aut}_k \mathbb{P}^1 (\cong \mathrm{PGL}_2(k))$, we set
$$A_\rho := \mathcal{A}(\mathbb{P}^1 \times \mathbb{P}^1, (\mathrm{id} \times \rho)\nu).$$
By Theorem 3 (1), $A_\rho \cong A_{\rho'}$ if and only if there exists $\tau \in \mathrm{Aut}_k \mathbb{P}^1$ such that the diagram

$$\begin{tikzcd}
\mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "\tau\times\tau"] \arrow[d, "(\mathrm{id}\times\rho)\nu"'] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "(\mathrm{id}\times\rho')\nu"] \\
\mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "\tau\times\tau"'] & \mathbb{P}^1 \times \mathbb{P}^1
\end{tikzcd}$$

commutes if and only if there exists $\tau \in \mathrm{Aut}_k \mathbb{P}^1$ such that $\rho'\tau = \tau\rho$. Hence, A_ρ is isomorphic to A_{ρ_λ} or A_{ρ_J} where $\lambda \in k \setminus \{0\}$, $\rho_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ and $\rho_J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Moreover, $A_{\rho_\lambda} \cong A_{\rho_{\lambda'}}$ if and only if $\lambda' = \lambda^{\pm 1}$.

We next treat the case when E is a union of two irreducible divisors of bidegree $(1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 8. *Let $C = \mathcal{V}(f) \subset \mathbb{P}^1 \times \mathbb{P}^1$ where $f \in k[x_1, y_1] \circ k[x_2, y_2]$. Assume that C is irreducible. Then C is a divisor of bidegree $(1,1)$ if and only if there exists $\tau \in \mathrm{Aut}_k \mathbb{P}^1$ such that $C = C_\tau := \{(p, \tau(p)) \mid p \in \mathbb{P}^1\}$.*

By Lemma 8, if E is a union of two irreducible divisors of bidegree $(1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, then $E = C_{\tau_1} \cup C_{\tau_2}$ for some $\tau_i \in \mathrm{Aut}_k \mathbb{P}^1$ $(i = 1, 2)$. The following result is one of our main results.

Theorem 9. *Let $E = C_{\tau_1} \cup C_{\tau_2}$ be a union of two irreducible divisors of bidegree $(1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Then one of the following statements holds:*

(1) $|C_{\tau_1} \cap C_{\tau_2}| = 2$ (*if and only if* $\tau_2^{-1}\tau_1 \sim \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ *for some* $\lambda \in k \setminus \{0, 1\}$),

(2) $|C_{\tau_1} \cap C_{\tau_2}| = 1$ (*if and only if* $\tau_2^{-1}\tau_1 \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$),

(3) $|C_{\tau_1} \cap C_{\tau_2}| = \infty$ (*if and only if* $\tau_2^{-1}\tau_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$).

In this report, we define the types of 3-geometric pairs (E, σ) as follows:

(1) Type P: $E = \mathbb{P}^1 \times \mathbb{P}^1$ and $\sigma = (\mathrm{id} \times \tau)\nu \in \mathrm{Aut}_k^G(\mathbb{P}^1 \times \mathbb{P}^1)$ (Type P is divided into Type P_i $(i = 1, 2)$ in terms of the Jordan canonical form of τ).

(2) Type S: $E = C_{\tau_1} \cup C_{\tau_2}$ is a union of two irreducible divisors of bidegree $(1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ such that $|C_{\tau_1} \cap C_{\tau_2}| = 2$. Type S is divided into Type S_i $(i = 1, 2)$; Type S_1: σ fixes each components and Type S_2: σ switches each components.

(3) Type T: $E = C_{\tau_1} \cup C_{\tau_2}$ is a union of two irreducible divisors of bidegree $(1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ such that $|C_{\tau_1} \cap C_{\tau_2}| = 1$. Type T is divided into Type T_i $(i = 1, 2)$; Type T_1: σ fixes each components and Type T_2: σ switches each components.

(4) Type WL: $E = C_{\tau_1} \cup C_{\tau_2}$ is a union of two irreducible divisors of bidegree $(1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ such that $|C_{\tau_1} \cap C_{\tau_2}| = \infty$. Type WL is divided into Type WL_i $(i = 1, 2)$ in terms of the Jordan canonical form of $\tau_1 (= \tau_2)$.

The following theorem lists all possible defining relations of algebras in each type up to isomorphism of graded algebra.

Theorem 10. *Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional cubic AS-regular algebra. For each type the following table describes*

(I) the defining relations of A, and

(II) the conditions to be isomorphic in terms of their defining relations.

Moreover, every algebra listed in the following table is AS-regular. In the following table, if $X \neq Y$ or $i \neq j$, then Type X_i algebra is not isomorphic to any Type Y_j algebra.

Type	*(I) defining relations* $(\alpha, \beta \in k)$	*(II) condition to be graded algebra isomorphic*
P_1	$\begin{cases} x^2y - \alpha yx^2, \\ xy^2 - \alpha y^2x \quad (\alpha \neq 0) \end{cases}$	$\alpha' = \alpha^{\pm 1}$
P_2	$\begin{cases} x^2y - yx^2 + yxy, \\ xy^2 - y^2x + y^3 \end{cases}$	————————

S_1	$\begin{cases}\alpha\beta x^2y+(\alpha+\beta)xyx+yx^2,\\ \alpha\beta xy^2+(\alpha+\beta)yxy+y^2x\end{cases}$ $(\alpha\beta\neq 0,\ \alpha^2\neq\beta^2)$	$\{\alpha',\beta'\}=\{\alpha,\beta\},\{\alpha^{-1},\beta^{-1}\}$
S_2	$\begin{cases}xy^2+y^2x+(\alpha+\beta)x^3,\\ x^2y+yx^2+(\alpha^{-1}+\beta^{-1})y^3\end{cases}$ $(\alpha\beta\neq 0,\ \alpha^2\neq\beta^2)$	$\dfrac{\alpha'}{\beta'}=\left(\dfrac{\alpha}{\beta}\right)^{\pm 1}$
T_1	$\begin{cases}x^2y-2xyx+yx^2-2(2\beta-1)yxy\\ +2(2\beta-1)xy^2+2\beta(\beta-1)y^3,\\ xy^2-2yxy+y^2x\end{cases}$	$\beta'=\beta,1-\beta$
T_2	$\begin{cases}x^2y+2xyx+yx^2+2y^3,\\ xy^2+2yxy+y^2x\end{cases}$	————————
WL_1	$\begin{cases}\alpha^2xy^2+y^2x-2\alpha yxy,\\ yx^2+\alpha^2x^2y-2\alpha xyx\end{cases}$ $(\alpha\neq 0)$	$\alpha'=\alpha^{\pm 1}$
WL_2	$\begin{cases}xy^2+y^2x-2yxy,\\ 4xy^2+2y^3+yx^2+x^2y\\ \qquad -4yxy-2xyx\end{cases}$	————————

The following theorem lists all possible defining relations of algebras in each type up to graded Morita equivalence.

Theorem 11. *Let $A=\mathcal{A}(E,\sigma)$ be a 3-dimensional cubic AS-regular algebra. For each type the following table describes*
(I) the defining relations of A, and
(III) the conditions to be graded Morita equivalent in terms of their defining relations. Moreover, every algebra listed in the following table is AS-regular. In the following table, if $X\neq Y$, then Type X algebra is not graded Morita equivalent to any Type Y algebra.

Type	*(I) defining relations* $(\alpha,\beta\in k)$	*(III) condition to be graded Morita equivalent*
P	$\begin{cases}x^2y-yx^2,\\ xy^2-y^2x\end{cases}$	————————
S	$\begin{cases}\alpha\beta x^2y+(\alpha+\beta)xyx+yx^2,\\ \alpha\beta xy^2+(\alpha+\beta)yxy+y^2x\end{cases}$ $(\alpha\beta\neq 0,\ \alpha^2\neq\beta^2)$	$\dfrac{\alpha'}{\beta'}=\left(\dfrac{\alpha}{\beta}\right)^{\pm 1}$
T	$\begin{cases}x^2y+yx^2+2xy^2\\ \quad -2xyx-2yxy,\\ xy^2+y^2x-2yxy\end{cases}$	————————

WL	$\begin{cases} xy^2 + y^2x - 2yxy \\ yx^2 + x^2y - 2xyx \end{cases}$	———————

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MASAKI MATSUNO
KATSUSHIKA DIVISION
INSTITUTE OF ARTS AND SCIENCES
TOKYO UNIVERSITY OF SCIENCE
6-3-1, NIIJUKU, KATSUSHIKA-KU, TOKYO 125-8585 JAPAN
Email address: masaki.matsuno@rs.tus.ac.jp

YU SAITO
GRADUATE SCHOOL OF INTEGRADED SCIENCE AND TECHNOLOGY
SHIZUOKA UNIVERSITY
OHYA 836, SHIZUOKA 422-8529, JAPAN
Email address: saito.yu.18@shizuoka.ac.jp

QUIVER HEISENBERG ALGEBRAS AND THE ALGEBRA $B(Q)$

HIROYUKI MINAMOTO

Abstract. This is a report on ongoing joint work with Martin Herschend about quiver Heisenberg algebras (QHA) and the algebra ${}^{v}B(Q)$. In this note, we mainly investigate QHA of Dynkin type. The first main result tells that QHA ${}^{v}\Lambda(Q)$ of Dynkin type is finite dimensional if and only if the weight $v \in \mathbf{k}Q_0$ is regular (see Definition 1), and moreover that if this is the case, ${}^{v}\Lambda(Q)$ is a symmetric algebra. In the case $\mathrm{char}\mathbf{k} = 0$, the "if" part of the first statement is proved by Etingof and Rains [10], and the second is verified for a generic weight by Etingof, Latour and Rains [11].

Compare to the preprojective algebras $\Pi(Q)$ which are only Frobenius in general, QHA ${}^{v}\Lambda(Q)$ can be said to be well-behaved, since they are always symmetric. Making use of this, we investigate silting theory of QHA of Dynkin type. We obtain results which are analogous to the results for $\Pi(Q)$ by Aihara-Mizuno [3].

1. Introduction

Throughout this note $\mathbf{k}$ is an algebraically closed field and Q is a finite acyclic quiver. For $\mathbf{k}Q$-module M, the dimension vector $\underline{\dim}M$ is regarded as an element of $\mathbf{k}Q_0 = \mathbf{k} \times \cdots \times \mathbf{k}$ (not of $\mathbb{Z}Q_0$).

For an element $v \in \mathbf{k}Q_0$, which we call *weight*, we define the *weighted dimension* of M to be

$$ {}^{v}\dim M := \sum_{i \in Q_0} v_i \dim e_i M. $$

Definition 1. A weight $v \in \mathbf{k}Q_0$ is called *regular* if

$$ {}^{v}\dim M \neq 0 \quad (\forall M \in \mathrm{ind}\ Q) $$

Remark 2. In the case Q is Dynkin and $\mathrm{char}\ \mathbf{k} = 0$, the vector space $\mathbf{k}Q_0$ may be identified with the Cartan subalgebra $\mathfrak{h}$ of the semi-simple Lie algebra $\mathfrak{g}$ corresponding to Q. By Gabriel's theorem the dimension vectors of indecomposable $\mathbf{k}Q$-modules are precisely the roots of $\mathfrak{g}$, so the regularity given here coincides with that are used by Etingof-Rains [10].

Example 3. Let Q be a directed A_3-quiver.

$$ Q: \ 1 \xrightarrow{\ \alpha\ } 2 \xrightarrow{\ \beta\ } 3 . $$

The dimension vectors of indecomposable modules are

$$ \begin{smallmatrix}1\\0\\0\end{smallmatrix},\ \begin{smallmatrix}0\\1\\0\end{smallmatrix},\ \begin{smallmatrix}0\\0\\1\end{smallmatrix},\ \begin{smallmatrix}1\\1\\0\end{smallmatrix},\ \begin{smallmatrix}0\\1\\1\end{smallmatrix},\ \begin{smallmatrix}1\\1\\1\end{smallmatrix}. $$

The detailed version of this paper will be submitted for publication elsewhere.

Thus, regularity of a weight $v = (v_1, v_2, v_3)^t$ is

$$\begin{aligned} &v_1 \neq 0,\ v_2 \neq 0,\ v_3 \neq 0, \\ &v_1 + v_2 \neq 0,\ v_2 + v_3 \neq 0,\ v_1 + v_2 + v_3 \neq 0. \end{aligned}$$

Looking the weighted dimension of simple modules S_i $(i \in Q_0)$, we obtain

Lemma 4. *A regular weight v is sincere i.e., $v_i \neq 0$ $(\forall i \in Q_0)$.*

Let $\overline{Q}$ be the double of Q.

$$\boxed{Q} \qquad i \xrightarrow{\alpha} j \qquad\qquad \begin{tikzcd} i \arrow[r, bend left, "\alpha"] & j \arrow[l, bend left, "\alpha^*"] \end{tikzcd} \qquad \boxed{\overline{Q}}$$

For $i \in Q_0$, ρ_i denotes the mesh relation at i

$$\rho_i := \sum_{\alpha \in Q_1 : t(\alpha) = i} \alpha\alpha^* - \sum_{\alpha \in Q_1 : h(\alpha) = i} \alpha^*\alpha.$$

Definition 5. The quiver Heisenberg algebra ${}^v\Lambda(Q)$ with the weight $v \in \mathbf{k}Q_0$ is defined to be

$${}^v\Lambda(Q) := \frac{\mathbf{k}[z]\overline{Q}}{(\rho_i - v_i z e_i \mid i \in Q_0)}.$$

Remark 6. This algebra is a special case of algebras studied in [6, 7, 10].

Remark 7. If v is sincere, then ${}^v\Lambda(Q)$ is isomorphic to the algebra which was given in previous talks of QHA, via the isomorphism

$${}^v\Lambda(Q) \cong \frac{\mathbf{k}\overline{Q}}{([a, {}^v\rho] \mid a \in \overline{Q}_1)}, \quad z \mapsto {}^v\rho$$

where ${}^v\rho := \sum_i v_i^{-1}\rho_i$ the "weighted mesh relation" and $[a, {}^v\rho] = a{}^v\rho - {}^v\rho a$ is the commutator.

We recall an indecomposable decomposition of ${}^v\Lambda(Q)$ as $\mathbf{k}Q$-module.

Theorem 8 ([12])**.** *If v is regular, then as $\mathbf{k}Q$-modules*

$${}^v\Lambda(Q) \cong \bigoplus_M M^{\dim M}.$$

where M runs over representatives of isomorphism class of indecomposable preprojective modules.

In particular, in the case Q is Dynkin, if v is regular, ${}^v\Lambda(Q)$ is finite dimensional. One of our main result asserts that the converse holds and moreover, if this is the case, ${}^v\Lambda(Q)$ is symmetric.

Theorem 9 ((1) [12], (2) [13])**.** *Let Q be a Dynkin quiver. The followings hold.*

(1) *A weight v is regular if and only if ${}^v\Lambda(Q)$ is finite dimensional.*

(2) *If a weight v is regular, then ${}^{v}\Lambda(Q)$ is symmetric.*

Remark 10. In the case char$\mathbf{k} = 0$, Etingof-Latour-Rains [11] showed that ${}^{v}\Lambda(Q)$ is symmetric for a generic weight v.

In the next section, we explain keys of proofs. In the third section, we discuss silting theory of ${}^{v}\Lambda(Q)$.

2. Proof of Theorem 9

2.1. **Proof of Theorem 9(1).** We only have to prove "if" direction. We do this by proving the contraposition. Namely, we show that if v is not regular, then ${}^{v}\Lambda(Q)$ is infinite dimensional. In the case v is not sincere, using an explicit presentation of ${}^{v}\Lambda(Q)$ by a quiver with relations, we can directly check that $\dim {}^{v}\Lambda(Q) = \infty$. Thus we may assume that v is sincere (and not regular). In that case, we conclude $\dim {}^{v}\Lambda(Q) = \infty$ by the following proposition.

To state the proposition, we recall that ${}^{v}\Lambda(Q)$ acquires a grading that counts the number of extra arrows α^*, which we call the $*$-*grading*. Let ${}^{v}\Lambda(Q)_n$ denote the $*$-degree n-part of ${}^{v}\Lambda(Q)$. It is clear that ${}^{v}\Lambda(Q)_0 = \mathbf{k}Q$ and ${}^{v}\Lambda(Q)_n$ has a canonical structure of $\mathbf{k}Q$-bimodule.

Proposition 11 ([12])**.** *Assume that v is sincere but not regular. Let M be an indecomposable $\mathbf{k}Q$-module such that ${}^{v}\dim M = 0$. Then for any $n \geq 0$, M is a direct summand of ${}^{v}\Lambda(Q)_n \otimes_{\mathbf{k}Q} M$ as $\mathbf{k}Q$-module.*

In particular ${}^{v}\Lambda(Q)_n \neq 0$ for all $n \geq 0$.

The case $n = 0$ is clear. For the case $n = 1$, we recall that there is a canonical exact triangle which is obtained from analysis of QHA and preprojective algebra

$$M \to {}^{v}\widetilde{\Lambda}(Q)_1 \otimes^{\mathbb{L}}_{\mathbf{k}Q} M \to \nu_1^{-1} M \to,$$

in the derived category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}Q\mathrm{mod})$ where ${}^{v}\widetilde{\Lambda}(Q)$ is the derived quiver Heisenberg algebra given in the next section. We can show that ${}^{v}\dim M = 0$ if and only if the above exact triangle splits. We note that in the case ${}^{v}\dim M \neq 0$, the exact triangle is an almost split exact triangle.

The case $n \geq 2$ uses the following exact triangle

$$\widetilde{\Pi}_1 \otimes^{\mathbb{L}} {}^{v}\widetilde{\Lambda}_{n-2} \otimes^{\mathbb{L}} M \to {}^{v}\widetilde{\Lambda}_1 \otimes^{\mathbb{L}} {}^{v}\widetilde{\Lambda}_{n-1} \otimes^{\mathbb{L}} M \to {}^{v}\widetilde{\Lambda}_n \otimes^{\mathbb{L}} M$$

Please see [12] for details.

2.2. **Proof of Theorem 9(2).** Main ingredients of our proof is the followings:

(i) A general result about derived preprojective algebra of d-representation finite algebra.

(ii) The algebra ${}^{v}B(Q)$.

(iii) A direct computation of the cohomology algebra of derived QHA.

2.2.1. Let A be a d-representation finite algebra. Iyama-Oppermann [15] showed that the $d+1$-preprojective algebra $\Pi := \Pi_{d+1}(A)$ is Frobenius. Let ν be the Nakayama automorphism of Π, i.e., $\Pi \cong {}_{\nu}\mathrm{D}(\Pi)$ as Π-bimodules.

Theorem 12 ([13]). *Let $\widetilde{\Pi}$ be the $d+1$-derived preprojective algebra of A. Then, the cohomology algebra $\mathrm{H}(\widetilde{\Pi})$ of the derived $d+1$-preprojective algebra $\widetilde{\Pi}$ is isomorphic to the skew polynomial algebra $\Pi[u;\nu]$*

$$\mathrm{H}(\widetilde{\Pi}) \cong \Pi[u;\nu]$$

as cohomologically graded algebras, where u is a formal variable of cohomological degree $-d$ and

$$au = u\nu(a) \quad (\forall a \in \Pi).$$

This theorem connects the Nakayama automorphism ν to the algebra structure of $\mathrm{H}(\widetilde{\Pi})$.

2.2.2. We introduce a finite dimensional algebra ${}^{v}B(Q)$.

Definition 13. For a quiver Q and a regular weight v, we define

$${}^{v}B(Q) := \begin{pmatrix} \mathbf{k}Q & {}^{v}\Lambda(Q)_1 \\ 0 & \mathbf{k}Q \end{pmatrix}$$

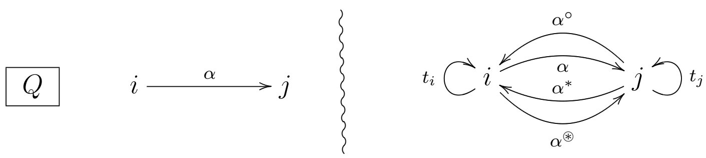
the bypath algebra (a.k.a., 2-path algebra) of Q.

The algebra ${}^{v}B(Q)$ has various properties that are 1-dimension higher version of that of the path algebra $\mathbf{k}Q$. Among other things, we have a 1-dimension higher version of Gabriel's dichotomy of representation types.

Theorem 14 ([13]). *The followings hold.*

(1) *${}^{v}B(Q)$ is 2-representation finite if and only if Q is Dynkin.*
(2) *${}^{v}B(Q)$ is 2-representation infinite if and only if Q is non-Dynkin.*

Recall that the derived QHA ${}^{v}\widetilde{\Lambda}(Q)$ is a DGA explicitly defined by the quiver



the differential is defined by

$$d(\alpha) := 0, d(\alpha^*) := 0, d(\alpha^{\circ}) := -[\alpha^*, {}^{v}\rho], d(\alpha^{\circledast}) := [\alpha, {}^{v}\rho],$$
$$d(t_i) := \sum_{\alpha \in Q_1} e_i[\alpha, \alpha^{\circ}]e_i + \sum_{\alpha \in Q_1} e_i[\alpha^*, \alpha^{\circledast}]e_i.$$

If $\mathrm{char}\mathbf{k} \neq 2$, ${}^{v}\widetilde{\Lambda}(Q)$ is the Ginzburg dg-algebra $\mathcal{G}(\overline{Q}, W)$ where

$$W := -\frac{1}{2}{}^{v}\rho\rho = -\frac{1}{2}\sum_{i \in Q_0} v_i^{-1}\rho_i^2.$$

Lemma 15 ([13]). *The 3-derived preprojective algebra of ${}^{v}B(Q)$ and the 2-ed quasi-Veronese algebra of ${}^{v}\widetilde{\Lambda}(Q)$ are isomorphic*

$$\widetilde{\Pi}_3({}^{v}B(Q)) \cong {}^{v}\widetilde{\Lambda}(Q)^{[2]}.$$

2.2.3. By (more or less) direct computation we have

Theorem 16 ([12]). *Assume that Q is Dynkin and v is regular. Then,*

$$\mathrm{H}({}^{v}\widetilde{\Lambda}(Q)) \cong {}^{v}\Lambda(Q)[u]$$

where u is a formal variable of cohomological degree -2.

Comparing the right hand sides of the isomorphisms given in Theorem 12 for ${}^{v}B(Q)$ and Theorem 16 via Lemma 15, we conclude that $\nu_\Lambda = \mathrm{id}_\Lambda$ up to inner automorphisms.

3. Silting theory of QHA of Dynkin type

Compare to the preprojective algebras $\Pi(Q)$ which are only Frobenius in general, QHA ${}^{v}\Lambda(Q)$ can be said to be well-behaved, since they are always symmetric. Making use of this, we investigate silting theory of QHA of Dynkin type. Before doing this, first we introduce a general construction of a tilting complex.

3.1. In this subsection, Q denote a quiver which is not necessarily Dynkin.

Let $i \in Q_0$. We define a complex $T^{(i)}$ over ${}^{v}\Lambda$ to be

$$T^{(i)} := {}^{v}\Lambda(1-e_i) \oplus \left[{}^{v}\Lambda e_i \xrightarrow{(\pm a^*)_{a\in h^{-1}(i)}} \bigoplus_{a\in h^{-1}(i)} {}^{v}\Lambda e_{t(a)} \right]$$

where the right factor is a complex placed in $-1, 0$-th cohomological. degree.

This complex is a "family version" of the tilting complex of Crawley-Boevey-Kimura [8]. The reduction $\Pi \otimes_{{}^{v}\Lambda} T^{(i)}$ is the tilting complex introduced by Baumann-Kamniter [4] and Buan-Iyama-Reiten-Scott [5].

Let $r : W_Q \curvearrowright \mathbf{k}Q_0$ be the dual action. Let r_i be the action of the Coxeter generator s_i.

Theorem 17 ([13]). *The complex $T^{(i)}$ is a tilting complex and*

$$\mathrm{End}_{{}^{v}\Lambda}(T^{(i)})^{\mathrm{op}} \cong {}^{r_i(v)}\Lambda.$$

3.2. From now we assume that Q is Dynkin. We note $\mathrm{silt}{}^{v}\Lambda = \mathrm{tilt}{}^{v}\Lambda$ by Theorem 5.

Then, it is straightforward to check that $T^{(i)}$ is the left silting mutation of ${}^{v}\Lambda$:

$$T^{(i)} = \mu_i^-({}^{v}\Lambda).$$

Thus, taking iterated mutations

$${}^{w(v)}\Lambda \cong \mathrm{End}_{{}^{v}\Lambda}(\mu_{i_n}^- \cdots \mu_{i_1}^-({}^{v}\Lambda))^{\mathrm{op}}$$

where $w = s_{i_n} \cdots s_{i_1}$.

There are following bijections,

$$W_Q \xrightarrow{1:1} \mathrm{sttilt}\Pi(Q) \xrightarrow{1:1} 2\mathrm{silt}\Pi(Q).$$

the first is established by Mizuno [14], the second is a consequence of a general result due to Adachi-Iyama-Reiten [1]

The weighted mesh relation ${}^v\rho$ is central in ${}^v\Lambda(Q)$ and we have a canonical isomorphism ${}^v\Lambda(Q)/({}^v\rho) \cong \Pi(Q)$. Applying a general result by Eisele-Janssens-Raedschelders [9], we obtain bijections

$$W_Q \xrightarrow{1:1} \mathrm{sttilt}\,{}^v\Lambda(Q) \xrightarrow{1:1} 2\mathrm{silt}\,{}^v\Lambda(Q)$$

which is given by

$$w = s_{i_n} \cdots s_{i_1} \mapsto \mu^+_{i_n} \cdots \mu^+_{i_1}({}^v\Lambda)$$

By general criteria due to Aihara-Mizuno [3], we conclude that ${}^v\Lambda(Q)$ is silting discrete.

As a consequence of the preceding consideration, we obtain the following results which are analogous to the results for $\Pi(Q)$ by Aihara-Mizuno [3].

Theorem 18 ([13]). *Assume that Q is Dynkin and v is regular.*

(1) *The algebra ${}^v\Lambda(Q)$ is silting discrete.*

(2) *A silting complex T is a tilting complex and*

$$\mathrm{End}_{{}^v\Lambda(Q)}(T)^{\mathrm{op}} \cong {}^{w(v)}\Lambda(Q)$$

for some $w \in W_Q$.

Theorem 19 ([13]). *Let B_Q be the braid group of Q. There is a bijection*

$$B_Q \xrightarrow{1:1} \mathrm{silt}\,{}^v\Lambda(Q), \ \ b \mapsto \mu_b({}^v\Lambda(Q)).$$

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Department of Mathematics
Osaka Metropolitan University
Sakai, Osaka 599-8531 JAPAN
Email address: minamoto@omu.ac.jp

WALL-AND-CHAMBER STRUCTURES OF STABILITY PARAMETERS FOR SOME DIMER QUIVERS

YUSUKE NAKAJIMA

ABSTRACT. It is known that any projective crepant resolution of a three-dimensional Gorenstein toric singularity can be described as the moduli space of representations of a quiver associated to a dimer model for some stability parameter. The space of stability parameters has the wall-and-chamber structure and we can track the variations of projective crepant resolutions by observing such a structure. In this article, we consider dimer models giving rise to projective crepant resolutions of a toric compound Du Val singularity. We show that sequences of zigzag paths, which are special paths on a dimer model, determine the wall-and-chamber structure of the space of stability parameters.

1. INTRODUCTION

The moduli space of representations of a quiver, introduced in [10], is defined as the GIT quotient associated to a stability parameter. For some nice singularities, resolutions of singularities can be described as moduli spaces of representations of a quiver. For example, any projective crepant resolution of a three-dimensional Gorenstein quotient singularity $\mathbb{C}^3/G$ defined by the action of a finite subgroup $G \subset \mathrm{SL}(3,\mathbb{C})$ on $\mathbb{C}^3$ can be described as the moduli space of representations of the McKay quiver of G (see [2, 14]). Also, any projective crepant resolution of a three-dimensional Gorenstein toric singularity can be described as the moduli space of representations of the quiver associated to a dimer model (see [9]). It is known that the space of stability parameters associated to a quiver has the *wall-and-chamber structure*, that is, it is decomposed into chambers separated by walls. The moduli spaces associated to stability parameters contained in the same chamber are isomorphic, but a stability parameter contained in another chamber would give a different moduli space. Thus, it is important to detect the wall-and-chamber structure of the space of stability parameters to understand the relationships among projective crepant resolutions of the above singularities. The purpose of this article is to detect the wall-and-chamber structure for a particular class of three-dimensional Gorenstein toric singularities called toric compound Du Val (cDV) singularities. In particular, we will see that the combinatorics of a dimer model associated to a toric cDV singularity control the wall-and-chamber structure.

2. PRELIMINARIES ON DIMER MODELS AND ASSOCIATED QUIVERS

2.1. **Dimer models.** We first introduce dimer models and related notions which are originally derived from theoretical physics (e.g., [4, 6]).

The detailed version of this paper will be submitted for publication elsewhere.

A *dimer model* Γ on the real two-torus $\mathbb{T} := \mathbb{R}^2/\mathbb{Z}^2$ is a finite bipartite graph on $\mathbb{T}$ inducing a polygonal cell decomposition of $\mathbb{T}$. Since Γ is a bipartite graph, the set Γ_0 of nodes of Γ is divided into two subsets Γ_0^+, Γ_0^-, and edges of Γ connect nodes in Γ_0^+ with those in Γ_0^-. We denote by Γ_1 the set of edges. We color the nodes in Γ_0^+ white, and those in Γ_0^- black throughout this article. A *face* of Γ is a connected component of $\mathbb{T} \backslash \Gamma_1$. We denote by Γ_2 the set of faces. In the rest of this article, we assume that any dimer model satisfies a certain nice condition called the *consistency condition*, see e.g., [8, Section 6] for more details. For example, Figure 1 is a consistent dimer model on $\mathbb{T}$, where the outer frame is a fundamental domain of $\mathbb{T}$.

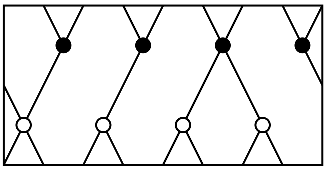

FIGURE 1. An example of a dimer model

We say that a path on a dimer model is a *zigzag path* if it makes a maximum turn to the right on a black node and a maximum turn to the left on a white node. For example, the paths (displayed in thick lines) in Figure 2 are all zigzag paths on the dimer model given in Figure 1.

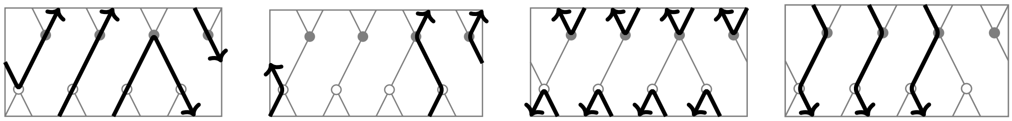

FIGURE 2. Zigzag paths on the dimer model given in Figure 1

We fix two 1-cycles on $\mathbb{T}$ generating the homology group $\mathrm{H}_1(\mathbb{T})$, and take a fundamental domain of $\mathbb{T}$ along such two cycles. Since we can consider a zigzag path z on Γ as a 1-cycle on $\mathbb{T}$, we have the homology class $[z] \in \mathrm{H}_1(\mathbb{T}) \cong \mathbb{Z}^2$, which is called the *slope* of z. Note that for a consistent dimer model Γ, any edge of Γ is contained in exactly two zigzag paths and any slope is a primitive element. Then, for a consistent dimer model Γ, we assign the lattice polygon called the zigzag polygon (cf. [8, Section 12]). Let $[z]$ be the slope of a zigzag path z on Γ. By normalizing $[z] \in \mathbb{Z}^2$, we consider it as an element of the unit circle S^1. Then, the set of slopes has a natural cyclic order along S^1. We consider the sequence $\big([z_i]\big)_{i=1}^k$ of slopes of zigzag paths on Γ such that they are cyclically ordered starting from $[z_1]$, where k is the number of zigzag paths. We note that some slopes may coincide in general. We set another sequence $(w_i)_{i=1}^k$ in $\mathbb{Z}^2$ defined as $w_0 = (0,0)$ and

$$w_{i+1} = w_i + [z_{i+1}]' \quad (i = 0, 1, \dots, k-1).$$

Here, $[z_{i+1}]' \in \mathbb{Z}^2$ is the element obtained from $[z_{i+1}]$ by rotating 90 degrees in the anti-clockwise direction. One can see that $w_k = (0,0)$ since the sum of all slopes is equal to zero. We call the convex hull of $\{w_i\}_{i=1}^k$ the *zigzag polygon* of Γ and denote it by Δ_Γ. Note that there are several choices of an initial zigzag path z_1, but the zigzag polygon is determined uniquely up to unimodular transformations. By definition, we see that the slope of a zigzag path is an outer normal vector of some side of Δ_Γ, and the number of zigzag paths having the same slope $v \in \mathbb{Z}^2$ coincides with the number of primitive segments of the side of Δ_Γ whose outer normal vector is v.

Example 1. We consider the dimer model in Figure 1 and its zigzag paths as in Figure 2. Then, we have the cyclically ordered sequence of slopes

$$\big((0,-1),\ (0,-1),\ (0,-1),\ (1,1),\ (0,1),\ (0,1),\ (-1,0)\big),$$

where we take a $\mathbb{Z}$-basis of $\mathrm{H}_1(\mathbb{T}) \cong \mathbb{Z}^2$ along the vertical and horizontal lines of the fundamental domain of $\mathbb{T}$. Thus, we have the zigzag polygon as in Figure 3.

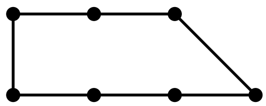

FIGURE 3. The zigzag polygon of the dimer model given in Figure 1

On the other hand, any lattice polygon can be described as the zigzag polygon of a consistent dimer model as follows.

Theorem 2 (see e.g., [5, 8])**.** *For any lattice polygon Δ, there exists a consistent dimer model Γ such that $\Delta = \Delta_\Gamma$.*

2.2. **Toric rings associated to dimer models.** Let Γ be a consistent dimer model. We next consider the cone σ_Γ over the zigzag polygon Δ_Γ, that is, σ_Γ is the cone whose section on the hyperplane at height one is Δ_Γ.

Let $\mathsf{N} := \mathbb{Z}^3$ be a lattice and $\mathsf{M} := \mathrm{Hom}_\mathbb{Z}(\mathsf{N}, \mathbb{Z})$ be the dual lattice of N. We set $\mathsf{N}_\mathbb{R} := \mathsf{N} \otimes_\mathbb{Z} \mathbb{R}$ and $\mathsf{M}_\mathbb{R} := \mathsf{M} \otimes_\mathbb{Z} \mathbb{R}$. We denote the standard inner product by $\langle\ ,\ \rangle : \mathsf{M}_\mathbb{R} \times \mathsf{N}_\mathbb{R} \to \mathbb{R}$. For the vertices $\widetilde{v}_1, \dots, \widetilde{v}_n \in \mathbb{Z}^2$ of Δ_Γ, we let $v_i := (\widetilde{v}_i, 1) \in \mathsf{N}$ $(i = 1, \dots, n)$. The cone σ_Γ over Δ_Γ is defined as

$$\sigma_\Gamma := \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_n \subset \mathsf{N}_\mathbb{R}.$$

Then, we consider the dual cone

$$\sigma_\Gamma^\vee := \{x \in \mathsf{M}_\mathbb{R} \mid \langle x, v_i\rangle \geq 0 \text{ for any } i = 1, \dots, n\}.$$

Using this cone, we can define the *toric ring* (*toric singularity*) R_Γ *associated to* Γ as

$$R_\Gamma := \mathbb{C}[\sigma_\Gamma^\vee \cap \mathsf{M}] = \mathbb{C}[t_1^{a_1} t_2^{a_2} t_3^{a_3} \mid (a_1, a_2, a_3) \in \sigma_\Gamma^\vee \cap \mathsf{M}],$$

which is Gorenstein in dimension three. We note that any three-dimensional Gorenstein toric ring can be described with this form. Precisely, let σ be a strongly convex rational polyhedral cone in $\mathsf{N}_\mathbb{R}$ which defines a three-dimensional Gorenstein toric ring R. Then, it is known that, after applying an appropriate unimodular transformation (which does not change the associated toric ring up to isomorphism) to σ, the cone σ can be described as the cone over a certain lattice polygon Δ_R. We call the lattice polygon Δ_R the *toric* *diagram* of R. By Theorem 2, there exists a consistent dimer model Γ such that $\Delta_\Gamma = \Delta_R$ for any three-dimensional Gorenstein toric ring R, in which case we have $R = R_\Gamma$.

2.3. **Quivers associated to dimer models.** Let Γ be a dimer model. As the dual of Γ, we obtain the quiver Q_Γ associated to Γ, which is embedded in $\mathbb{T}$, as follows. We assign a vertex dual to each face in Γ_2 and an arrow dual to each edge in Γ_1. We fix the orientation of any arrow so that the white node is on the right of the arrow. For example, Figure 4 is the quiver associated to the dimer model in Figure 1. We simply denote the quiver Q_Γ by Q unless it causes any confusion. Let $Q = (Q_0, Q_1)$ be the quiver associated to a dimer model, where Q_0 is the set of vertices and Q_1 is the set of arrows. Let $\mathsf{hd}(a), \mathsf{tl}(a) \in Q_0$ be respectively the head and tail of an arrow $a \in Q_1$. A *path* of length $r \geq 1$ is a finite sequence of arrows $\gamma = a_1 \cdots a_r$ with $\mathsf{hd}(a_i) = \mathsf{tl}(a_{i+1})$ for $i = 1, \dots, r-1$. We define $\mathsf{tl}(a) = \mathsf{tl}(a_1), \mathsf{hd}(a) = \mathsf{hd}(a_r)$ for a path $\gamma = a_1 \cdots a_r$. A *relation* in Q is a $\mathbb{C}$-linear combination of paths of length at least two having the same head and tail. We especially consider relations in Q defined as follows. For each arrow $a \in Q_1$, there exist two paths γ_a^+, γ_a^- such that $\mathsf{hd}(\gamma_a^\pm) = \mathsf{tl}(a)$, $\mathsf{tl}(\gamma_a^\pm) = \mathsf{hd}(a)$ and γ_a^+ (resp. γ_a^-) goes around the white (resp. black) node incident to the edge dual to a clockwise (resp. counterclockwise), see e.g., [12, Figure 6]. We define the set of relations $\mathcal{J}_Q := \{\gamma_a^+ - \gamma_a^- \mid a \in Q_1\}$ and call the pair $(Q, \mathcal{J}_Q)$ the *quiver with relations* associated to Γ.

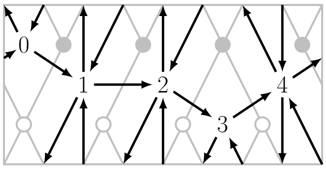

FIGURE 4. The quiver associated to the dimer model given in Figure 1

We then introduce representations of a quiver with relations. A *representation* of $(Q, \mathcal{J}_Q)$ consists of a set of $\mathbb{C}$-vector spaces $\{M_v \mid v \in Q_0\}$ together with $\mathbb{C}$-linear maps $\varphi_a : M_{\mathsf{tl}(a)} \to M_{\mathsf{hd}(a)}$ satisfying the relations $\mathcal{J}_Q$, that is, $\varphi_{\gamma_a^+} = \varphi_{\gamma_a^-}$ for any $a \in Q_1$. Here, for a path $\gamma = a_1 \cdots a_r$, the map φ_γ is defined as the composite $\varphi_{a_1} \cdots \varphi_{a_r}$ of $\mathbb{C}$-linear maps. (Note that in this article, a composite fg of morphisms means we first apply f then g.) In the rest of this article, we assume that the *dimension vector* of any representation $M = ((M_v)_{v \in Q_0}, (\varphi_a)_{a \in Q_1})$ of $(Q, \mathcal{J}_Q)$ is $\underline{1} := (1, \dots, 1)$, that is, $\underline{1} = (\dim_{\mathbb{C}} M_v)_{v \in Q_0}$. For representations M, M' of $(Q, \mathcal{J}_Q)$, a morphism from M to M' is a family of $\mathbb{C}$-linear maps $\{f_v : M_v \to M'_v\}_{v \in Q_0}$ such that $\varphi_a f_{\mathsf{hd}(a)} = f_{\mathsf{tl}(a)} \varphi'_a$ for any arrow $a \in Q_1$. We say that representations M and M' are *isomorphic*, if f_v is an isomorphism of vector spaces for all $v \in Q_0$. A representation N of $(Q, \mathcal{J}_Q)$ is called a *subrepresentation* of M if there is an injective morphism $N \to M$.

Next, we introduce moduli spaces parametrizing quiver representations satisfying a certain stability condition. We consider the weight space

$$\Theta(Q) := \left\{ \theta = (\theta_v)_{v \in Q_0} \in \mathbb{Z}^{Q_0} \mid \sum_{v \in Q_0} \theta_v = 0 \right\}$$

and let $\Theta(Q)_{\mathbb{R}} := \Theta(Q) \otimes_{\mathbb{Z}} \mathbb{R}$. We call an element $\theta \in \Theta(Q)_{\mathbb{R}}$ a *stability parameter*.

Let M be a representation of $(Q, \mathcal{J}_Q)$ of dimension vector $\underline{1}$. For a subrepresentation N of M, we define $\theta(N) := \sum_{v \in Q_0} \theta_v(\dim_{\mathbb{C}} N_v)$, and hence $\theta(M) = 0$ in particular. For a stability parameter $\theta \in \Theta(Q)_{\mathbb{R}}$, we introduce θ-stable representations as follows.

Definition 3 (see [10])**.** Let $\theta \in \Theta(Q)_{\mathbb{R}}$. We say that a representation M is *θ-semistable* (resp. *θ-stable*) if $\theta(N) \geq 0$ (resp. $\theta(N) > 0$) for any non-zero proper subrepresentation N of M. We say that θ is *generic* if every θ-semistable representation is θ-stable.

By [10, Proposition 5.3], for a generic parameter $\theta \in \Theta(Q)_{\mathbb{R}}$, one can construct the fine moduli space $\mathcal{M}_\theta(Q, \mathcal{J}_Q, \underline{1})$ parametrizing isomorphism classes of θ-stable representations of $(Q, \mathcal{J}_Q)$ with dimension vector $\underline{1}$ as the GIT (geometric invariant theory) quotient. In the following, we let $\mathcal{M}_\theta = \mathcal{M}_\theta(Q, \mathcal{J}_Q, \underline{1})$ for simplicity. This moduli space gives a projective crepant resolution of a three-dimensional Gorenstein toric singlarity as follows.

Theorem 4 (see [7, Theorem 6.3 and 6.4], [9, Corollary 1.2])**.** *Let Γ be a consistent dimer model, and Q be the associated quiver. Let R_Γ be the three-dimensional Gorenstein toric ring associated to Γ. Then, for a generic parameter $\theta \in \Theta(Q)_{\mathbb{R}}$, the moduli space $\mathcal{M}_\theta$ is a projective crepant resolution of* $\operatorname{Spec} R_\Gamma$. *Moreover, any projective crepant resolution of* $\operatorname{Spec} R_\Gamma$ *can be obtained as the moduli space $\mathcal{M}_\theta$ for some generic parameter $\theta \in \Theta(Q)_{\mathbb{R}}$.*

It is known that the space $\Theta(Q)_{\mathbb{R}}$ of stability parameters has a *wall-and-chamber structure*. Namely, we define an equivalence relation on the set of generic parameters so that $\theta \sim \theta'$ if and only if any θ-stable representation of $(Q, \mathcal{J}_Q)$ is also θ'-stable and vice versa, and this relation gives rise to the decomposition of stability parameters into finitely many chambers which are separated by walls. Here, a *chamber* is an open cone in $\Theta(Q)_{\mathbb{R}}$ consisting of equivalent generic parameters and a *wall* is a codimension one face of the closure of a chamber. Note that any generic parameter lies on some chamber (see [9, Lemma 6.1]), and $\mathcal{M}_\theta$ is unchanged unless a parameter θ moves in the same chamber of $\Theta(Q)_{\mathbb{R}}$.

3. Wall-and-chamber structures for toric cDV singularities

In the following, we detect the wall-and-chamber structure of $\Theta(Q)_{\mathbb{R}}$ for the quiver Q associated to a dimer model giving rise to projective crepant resolutions of a toric compound Du Val singularity. Compound Du Val (cDV) singularities, which are fundamental pieces in the minimal model program, are singularities giving rise to Du Val (or Kleinian, ADE) singularities as hyperplane sections. It is known that toric cDV singularities can be classified into the following two types (e.g., see [3, footnote (18)]):

$$(cA_{a+b-1}) : \mathbb{C}[x, y, z, w]/(xy - z^a w^b),$$
$$(cD_4) : \mathbb{C}[x, y, z, w]/(xyz - w^2),$$

where a, b are integers with $a \geq 1$ and $a \geq b \geq 0$. Note that the former one is a cDV singularity of type cA_{a+b-1} and the latter one is of type cD_4. Since these are three dimensional Gorenstein toric rings, they can also be described as the form explained in Subsection 2.2. In particular, we can take the toric diagram of the toric cDV singularities of type cA_{a+b-1} as the trapezoid, which will be denoted by $\Delta(a, b)$, whose vertices are $(0, 0)$, $(a, 0)$, $(b, 1)$, and $(0, 1)$. For example, Figure 3 shows $\Delta(3, 2)$. By Theorem 2, there exists a consistent dimer model whose zigzag polygon is $\Delta(a,b)$, see [11, Subsection 1.2], [12, Section 5] for the precise construction. In general, such a dimer model is not unique, thus we choose one of them and denote the chosen one by $\Gamma_{a,b}$. By construction, the dimer model $\Gamma_{a,b}$ has $n := a+b$ faces. We label one of the faces with 0, and label the face right next to k with $k+1 \pmod n$ for $k = 0,1,\dots,n-1$. Also, we will use these labels as the names of vertices of the associated quiver Q.

We here focus on the toric cA_{n-1} singularity $R_{a,b} := \mathbb{C}[x,y,z,w]/(xy - z^a w^b)$ where $n := a+b$, and the associated dimer model $\Gamma_{a,b}$. Let Q be the quiver obtained as the dual graph of $\Gamma_{a,b}$. By Theorem 4, the quiver Q gives rise to projective crepant resolutions of $\operatorname{Spec} R_{a,b}$ as moduli spaces. By the definition of the zigzag polygon, we have the set $\{u_1,\dots,u_n\}$ of zigzag paths on $\Gamma_{a,b}$ such that $[u_k]$ is either $(0,-1)$ or $(0,1)$ for $k = 1,\dots,n$, and $a = \#\{k \mid [u_k] = (0,-1)\}$, $b = \#\{k \mid [u_k] = (0,1)\}$. We rearrange $u_1,\dots,u_n$ if necessary, and construct the sequence $(u_1,\dots,u_n)$ of the zigzag paths so that u_k consists of the edges shared by the faces $k-1$ and $k \pmod n$ for any $k = 1,\dots,n$. Also, we define a total order $<$ on $\{u_1,\dots,u_n\}$ as $u_n < u_{n-1} < \cdots < u_2 < u_1$.

By [12, Lemma 5.2], we see that any pair of zigzag paths (u_i,u_j) on $\Gamma_{a,b}$ divide the two-torus $\mathbb{T}$ into two parts (see Figure 5). We denote the region containing the face 0 by $\mathcal{R}^-(u_i,u_j)$, and the other region by $\mathcal{R}^+(u_i,u_j)$. By abuse of notation, we also use the notation $\mathcal{R}^\pm(u_i,u_j)$ for the set of vertices of Q contained in $\mathcal{R}^\pm(u_i,u_j)$. Since we essentially use one of $\mathcal{R}^\pm(u_i,u_j)$, we let $\mathcal{R}(u_i,u_j) := \mathcal{R}^+(u_i,u_j)$.

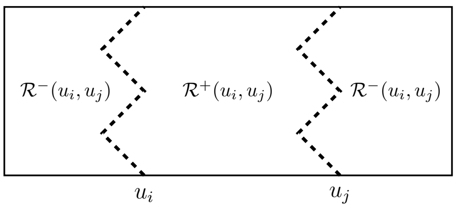

FIGURE 5

For the quiver Q associated to $\Gamma_{a,b}$, any $\theta \in \Theta(Q)_\mathbb{R}$ satisfies $\theta_0 = -\sum_{v\neq 0}\theta_v$. When we consider $\Theta(Q)_\mathbb{R}$, we employ the coordinates θ_v with $v \neq 0$. Then, the wall-and-chamber structure of $\Theta(Q)_\mathbb{R}$ can be determined by zigzag paths of the dimer model $\Gamma_{a,b}$ as follows.

Theorem 5 (see [12, Theorems 6.11, 6.12, and Corollary 6.13])**.** *Let the notation be the same as above. Then, there exists a one-to-one correspondence between the following sets:*

(a) *the set of chambers in* $\Theta(Q)_\mathbb{R}$*,*

(b) *the set* $\left\{\mathcal{Z}_\omega = (u_{\omega(1)},\dots,u_{\omega(n)}) \mid \omega \in \mathfrak{S}_n\right\}$ *of sequences of zigzag paths,*

such that under this correspondence, if a chamber $C \subset \Theta(Q)_\mathbb{R}$ *corresponds to a sequence* $\mathcal{Z}_\omega$*, then for any* $k = 1,\dots,n-1$*, we have the following:*

(1) *We see that* $W_k := \{\theta \in \Theta(Q)_\mathbb{R} \mid \sum_{v\in\mathcal{R}_k}\theta_v = 0\}$ *is a wall of* C*, where* $\mathcal{R}_k := \mathcal{R}(u_{\omega(k)}, u_{\omega(k+1)})$ *is the region determined by the zigzag paths* $u_{\omega(k)}, u_{\omega(k+1)}$ *(see Figure 5).*

(2) *Any parameter* $\theta \in C$ *satisfies* $\sum_{v\in\mathcal{R}_k}\theta_v > 0$ *(resp.* $\sum_{v\in\mathcal{R}_k}\theta_v < 0$*) if* $u_{\omega(k)} < u_{\omega(k+1)}$ *(resp.* $u_{\omega(k+1)} < u_{\omega(k)}$*).*
(3) *Suppose that* C' *is the chamber separated from* C *by the wall* W_k*. Let* $\theta \in C$ *and* $\theta' \in C'$*. If* $[u_{\omega(k)}] = -[u_{\omega(k+1)}]$*, then* $\mathcal{M}_\theta$ *and* $\mathcal{M}_{\theta'}$ *are related by a flop. If* $[u_{\omega(k)}] = [u_{\omega(k+1)}]$*, then we have* $\mathcal{M}_\theta \cong \mathcal{M}_{\theta'}$*.*
(4) *The action of the adjacent transposition* $s_k \in \mathfrak{S}_n$ *swapping* k *and* $k+1$ *on* $\mathcal{Z}_\omega$ *induces a crossing of the wall* W_k *in* $\Theta(Q)_\mathbb{R}$*. In particular, the chambers in* $\Theta_\mathbb{R}(Q)$ *can be identified with the Weyl chambers of type* A_{n-1}*.*

For the case cD_4, we have similar results as shown in [12, Theorem 8.1], although some modifications are required. Note that the homological minimal model program [13] also detects the wall-and-chamber structure of $\Theta(Q)_\mathbb{R}$, whereas our method provides a more combinatorial way to observe it.

In addition, it is known that the projective crepant resolution $\mathcal{M}_\theta$ can also be described as the toric variety associated to the toric fan induced from a triangulation of $\Delta(a,b)$ (see e.g., [1, Chapter 11]). For the sequence $\mathcal{Z}_\omega$ corresponding to a chamber $C \subset \Theta(Q)_\mathbb{R}$, there is a certain way to obtain the triangulation of $\Delta(a,b)$ giving rise to the projective crepant resolution $\mathcal{M}_\theta$ with $\theta \in C$, see [12, Subsection 6.1] for more details.

Example 6 (The suspended pinch point (cf. [9, Example 12.5]))**.** We consider the dimer model Γ shown in the left of Figure 6. We can see that the zigzag polygon of Γ is $\Delta(2,1)$. We also consider the zigzag paths u_1, u_2, u_3 shown in the right of Figure 6. In particular, the slopes of these zigzag paths are $[u_1] = [u_2] = (0,-1)$, and $[u_3] = (0,1)$. We fix a total order $u_3 < u_2 < u_1$.

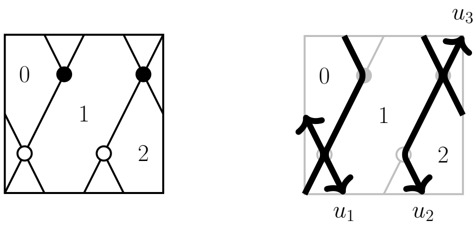

FIGURE 6. The dimer model Γ whose zigzag polygon is $\Delta(2,1)$ (left), the zigzag paths u_1, u_2, u_3 on Γ (right).

Let Q be the quiver associated to Γ. Then the space of stability parameters is

$$\Theta(Q)_\mathbb{R} = \{\theta = (\theta_0, \theta_1, \theta_2) \mid \theta_0 + \theta_1 + \theta_2 = 0\}.$$

By Theorem 5, we have the wall-and-chamber structure of $\Theta(Q)_\mathbb{R}$ as shown in Figure 7. For example, the sequence (u_3, u_2, u_1) corresponds to the chamber C described as

$$C = \{\theta \in \Theta(Q)_\mathbb{R} \mid \theta_1 > 0,\ \theta_2 > 0\}.$$

Indeed, since $\mathcal{R}(u_3, u_2) = \{2\}$ and $u_3 < u_2$, any parameter in C satisfies the inequality $\theta_2 > 0$. Also, since $\mathcal{R}(u_2, u_1) = \{1\}$ and $u_2 < u_1$, any parameter in C also satisfies the inequality $\theta_1 > 0$. A crossing of the wall $\theta_2 = 0$ of C corresponds to a swapping of u_3 and u_2. Also, a crossing of the wall $\theta_1 = 0$ of C corresponds to a swapping of u_2 and u_1.

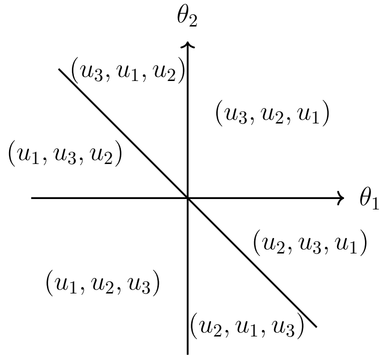

Figure 7

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Department of Mathematics
Kyoto Sangyo University
Motoyama, Kamigamo, Kita-Ku
Kyoto, 603-8555, Japan

Email address: ynakaji@cc.kyoto-su.ac.jp

THE MODULI OF 4-DIMENSIONAL SUBALGEBRAS OF THE FULL MATRIX RING OF DEGREE 3

KAZUNORI NAKAMOTO AND TAKESHI TORII

ABSTRACT. We describe the moduli $\mathrm{Mold}_{3,4}$ of 4-dimensional subalgebras of the full matrix ring of degree 3. We show that $\mathrm{Mold}_{3,4}$ has three irreducible components, whose relative dimensions over $\mathbb{Z}$ are 5, 2, 2, respectively.

Key Words: moduli of subalgebras, full matrix ring.
2020 *Mathematics Subject Classification*: Primary 14D22; Secondary 16S80, 16S50.

1. INTRODUCTION

Let k be a field. We say that k-subalgebras A and B of $\mathrm{M}_3(k)$ are equivalent (or $A \sim B$) if $P^{-1}AP = B$ for some $P \in \mathrm{GL}_3(k)$. If k is an algebraically closed field, then there are 26 equivalence classes of k-subalgebras of $\mathrm{M}_3(k)$ over k ([4]).

Definition 1 ([2, Definition 1.1], [3, Definition 3.1])**.** We say that a subsheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras of $\mathrm{M}_n(\mathcal{O}_X)$ is a *mold* of degree n on a scheme X if $\mathrm{M}_n(\mathcal{O}_X)/\mathcal{A}$ is a locally free sheaf. We denote by $\mathrm{rank}\mathcal{A}$ the rank of $\mathcal{A}$ as a locally free sheaf.

Proposition 2 ([2, Definition and Proposition 1.1], [3, Definition and Proposition 3.5])**.** *The following contravariant functor is representable by a closed subscheme of the Grassmann scheme* $\mathrm{Grass}(d, n^2)$:

$$\begin{array}{rcccl} \mathrm{Mold}_{n,d} & : & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \mathcal{A} \mid \mathcal{A} \textit{ is a rank } d \textit{ mold of degree } n \textit{ on } X \right\}. \end{array}$$

We consider the moduli $\mathrm{Mold}_{3,d}$ of rank d molds of degree 3 over $\mathbb{Z}$. For $d = 1, 2, 3, 6, 7, 8, 9$, we have the following theorem:

Theorem 3 ([4])**.** *Let $n = 3$. If $d \leq 3$ or $d \geq 6$, then*

$$\begin{array}{rcl} \mathrm{Mold}_{3,1} & = & \mathrm{Spec}\mathbb{Z}, \\ \mathrm{Mold}_{3,2} & \cong & \mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}, \\ \mathrm{Mold}_{3,3} & = & \overline{\mathrm{Mold}^{\mathrm{reg}}_{3,3}} \cup \overline{\mathrm{Mold}^{\mathrm{S}_2}_{3,3}} \cup \overline{\mathrm{Mold}^{\mathrm{S}_3}_{3,3}}, \textit{ where the relative dimensions of} \\ & & \overline{\mathrm{Mold}^{\mathrm{reg}}_{3,3}}, \overline{\mathrm{Mold}^{\mathrm{S}_2}_{3,3}}, \textit{ and } \overline{\mathrm{Mold}^{\mathrm{S}_3}_{3,3}} \textit{ over } \mathbb{Z} \textit{ are } 6, 4, \textit{ and } 4, \textit{ respectively,} \\ \mathrm{Mold}_{3,6} & \cong & \mathrm{Flag}_3 := \mathrm{GL}_3/\{(a_{ij}) \in \mathrm{GL}_3 \mid a_{ij} = 0 \textit{ for } i > j\}, \\ \mathrm{Mold}_{3,7} & \cong & \mathbb{P}^2_{\mathbb{Z}} \coprod \mathbb{P}^2_{\mathbb{Z}}, \\ \mathrm{Mold}_{3,8} & = & \emptyset, \\ \mathrm{Mold}_{3,9} & = & \mathrm{Spec}\mathbb{Z}. \end{array}$$

The detailed version of this paper will be submitted for publication elsewhere.

The cases $d = 4, 5$ remain. In this paper, we describe the moduli $\mathrm{Mold}_{3,4}$ of rank 4 molds of degree 3. We introduce several rank 4 molds of degree 3 on a commutative ring R.

Definition 4 ([4])**.** For a commutative ring R, we define

(1) $(\mathrm{B}_2 \times \mathrm{D}_1)(R) = \left\{ \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(R) \right\}$,

(2) $\mathrm{N}_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c, d \in R \right\}$,

(3) $\mathrm{S}_6(R) = \left\{ \begin{pmatrix} a & c & d \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c, d \in R \right\}$,

(4) $\mathrm{S}_7(R) = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & a & d \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c, d \in R \right\}$,

(5) $\mathrm{S}_8(R) = \left\{ \begin{pmatrix} a & c & d \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c, d \in R \right\}$,

(6) $\mathrm{S}_9(R) = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & b & d \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c, d \in R \right\}$.

There are 6 equivalence classes of 4-dimensional subalgebras of $\mathrm{M}_3(k)$ over an algebraically closed field k: $(\mathrm{B}_2 \times \mathrm{D}_1)(k)$, $\mathrm{N}_3(k)$, $\mathrm{S}_6(k)$, $\mathrm{S}_7(k)$, $\mathrm{S}_8(k)$, and $\mathrm{S}_9(k)$.

The following theorem is our main result in this paper.

Theorem 5 (Theorem 19, [4])**.** *When $d = 4$, we have an irreducible decomposition*

$$\mathrm{Mold}_{3,4} = \overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}} \coprod \mathrm{Mold}_{3,4}^{\mathrm{S}_7} \coprod \mathrm{Mold}_{3,4}^{\mathrm{S}_8}$$

such that irreducible components are all connected components. The relative dimensions of $\overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}}$, $\mathrm{Mold}_{3,4}^{\mathrm{S}_7}$*, and* $\mathrm{Mold}_{3,4}^{\mathrm{S}_8}$ *over* $\mathbb{Z}$ *are* 5, 2*, and* 2*, respectively. Moreover, both* $\mathrm{Mold}_{3,4}^{\mathrm{S}_7}$ *and* $\mathrm{Mold}_{3,4}^{\mathrm{S}_8}$ *are isomorphic to* $\mathbb{P}^2_{\mathbb{Z}}$*, and*

$$\overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}} = \mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1} \cup \mathrm{Mold}_{3,4}^{\mathrm{S}_6} \cup \mathrm{Mold}_{3,4}^{\mathrm{S}_9} \cup \mathrm{Mold}_{3,4}^{\mathrm{N}_3}$$

is isomorphic to $\mathrm{Flag}_3 \times_{\mathbb{P}^2_{\mathbb{Z}}} \mathrm{Flag}_3 \times_{\mathbb{P}^2_{\mathbb{Z}}} \mathrm{Flag}_3 = \{(L_1 \subset W_2, L_1 \subset W_1, L_2 \subset W_1) \in \mathrm{Flag}_3 \times \mathrm{Flag}_3 \times \mathrm{Flag}_3\}$*. In particular,* $\mathrm{Mold}_{3,4}$ *is smooth over* $\mathbb{Z}$*.*

Remark 6 ([1]). We need to say the relation between $\mathrm{Mold}_{d,d}$ and the variety Alg_d of algebras defined by Gabriel in [1]. Let $V = ke_1 \oplus ke_2 \oplus \cdots \oplus ke_d$ be a d-dimensional vector space over a field k. For $\varphi \in \mathrm{Hom}_k(V \otimes_k V, V)$, put $\varphi(e_i \otimes e_j) = \sum_{l=1}^n c_{ij}^l e_l$. We say that φ *determines an algebra structure* on V with 1 if the multiplication $e_i \cdot e_j = c_{ij}^l e_l$ defines an algebra V over k with 1. Then we define the variety Alg_d of d-dimensional algebras in the sense of Gabriel by

$$\mathrm{Alg}_d = \left\{ \varphi \in \mathrm{Hom}_k(V \otimes_k V, V) \;\middle|\; \begin{array}{c} \varphi \textit{ determines an} \\ \text{algebra structure} \\ \text{on } V \text{ with } 1 \end{array} \right\} \subset \mathbb{A}_k^{d^3}.$$

Then we can define a morphism $\Psi_d : \mathrm{Alg}_d \to \mathrm{Mold}_{d,d}$ by

$$\varphi \mapsto \{\varphi(v \otimes -) \in \mathrm{End}_k(V) \cong \mathrm{M}_d(k) \mid v \in V\}.$$

If we could prove that $U_d = \{A \subset \mathrm{M}_d(k) \mid A \text{ is a } d\text{-dimensional tame algebra }\}$ is open in $\mathrm{Mold}_{d,d}$ for any d, then $\Psi_d^{-1}(U_d) = \{A \mid \ d\text{-dimensional tame algebra }\}$ would also be open in Alg_d, which gives an affirmative answer to "Tame type is open conjecture". Hence, we believe that $\mathrm{Mold}_{n,d}$ is an important geometric object. This is one of our motivations to investigate $\mathrm{Mold}_{n,d}$.

2. Several Tools

In this section, we introduce several tools for describing $\mathrm{Mold}_{3,4}$. Let A be an associative algebra over a commutative ring R. Assume that A is projective over R. Let $A^e = A \otimes_R A^{op}$ be the enveloping algebra of A. For an A-bimodule M over R, we can regard it as an A^e-module. We define the i-th Hochschild cohomology group $\mathrm{HH}^i(A, M)$ of A with coefficients in M as $\mathrm{Ext}^i_{A^e}(A, M)$.

Let $\mathcal{A}$ be the universal mold on $\mathrm{Mold}_{n,d}$. For $x \in \mathrm{Mold}_{n,d}$, denote by $\mathcal{A}(x) = \mathcal{A} \otimes_{\mathcal{O}_{\mathrm{Mold}_{n,d}}} k(x) \subset \mathrm{M}_n(k(x))$ the mold corresponding to x, where $k(x)$ is the residue field of x. As applications of Hochschild cohomology to the moduli $\mathrm{Mold}_{n,d}$, we have the following tools.

Theorem 7 ([3, Theorem 1.1])**.** *For each point $x \in \mathrm{Mold}_{n,d}$,*

$$\dim_{k(x)} T_{\mathrm{Mold}_{n,d}/\mathbb{Z},x} = \dim_{k(x)} \mathrm{HH}^1(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x)) + n^2 - \dim_{k(x)} N(\mathcal{A}(x)),$$

where $N(\mathcal{A}(x)) = \{b \in \mathrm{M}_n(k(x)) \mid [b,a] = ba - ab \in \mathcal{A}(x) \text{ for any } a \in \mathcal{A}(x)\}$.

Theorem 8 ([3, Theorem 1.2])**.** *Let $x \in \mathrm{Mold}_{n,d}$. If $\mathrm{HH}^2(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x)) = 0$, then the canonical morphism $\mathrm{Mold}_{n,d} \to \mathbb{Z}$ is smooth at x.*

For a rank d mold A of degree n on a locally noetherian scheme S, we can consider a $\mathrm{PGL}_{n,S}$-orbit $\{P^{-1}AP \mid P \in \mathrm{PGL}_{n,S}\}$ in $\mathrm{Mold}_{n,d} \otimes_{\mathbb{Z}} S$, where $\mathrm{PGL}_{n,S} = \mathrm{PGL}_n \otimes_{\mathbb{Z}} S$. For $x \in S$, put $A(x) = A \otimes_{\mathcal{O}_S} k(x)$, where $k(x)$ is the residue field of x. By using $\mathrm{HH}^1(A(x), \mathrm{M}_n(k(x))/A(x))$, we have:

Theorem 9 ([3, Theorem 1.3])**.** *Assume that $\mathrm{HH}^1(A(x), \mathrm{M}_n(k(x))/A(x)) = 0$ for each $x \in S$. Then the $\mathrm{PGL}_{n,S}$-orbit $\{P^{-1}AP \mid P \in \mathrm{PGL}_{n,S}\}$ is open in $\mathrm{Mold}_{n,d} \otimes_{\mathbb{Z}} S$.*

These tools are useful for investigating $\mathrm{Mold}_{3,4}$. For each rank 4 molds of $\mathrm{M}_3(R)$ over a commutative ring R, we obtained the following table:

TABLE 1. Hochschild cohomology $\mathrm{HH}^*(A, \mathrm{M}_3(R)/A)$ for R-subalgebras A of $\mathrm{M}_3(R)$ (*cf.* [3, Table 2])

A	$d = \mathrm{rank} A$	$H^* = \mathrm{HH}^*(A, \mathrm{M}_3(R)/A)$	${}^t A$	$N(A)$	$\dim T_{\mathrm{Mold}_{3,d}/\mathbb{Z},A}$
$(\mathrm{B}_2 \times \mathrm{D}_1)(R) = \left\{ \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\}$	4	$H^i = 0$ for $i \geq 0$	$(\mathrm{B}_2 \times \mathrm{D}_1)(R)$	$(\mathrm{B}_2 \times \mathrm{D}_1)(R)$	5
$\mathrm{N}_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \right\}$	4	$H^i = \begin{cases} R^2 & (i = 0) \\ R^{i+1} & (i \geq 1) \end{cases}$	$\mathrm{N}_3(R)$	$\mathrm{B}_3(R)$	5
$\mathrm{S}_6(R) = \left\{ \begin{pmatrix} a & c & d \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \right\}$	4	$H^i = R$ for $i \geq 0$	$\mathrm{S}_9(R)$	$\mathrm{S}_{13}(R)$	5
$\mathrm{S}_7(R) = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & a & d \\ 0 & 0 & b \end{pmatrix} \right\}$	4	$H^i = \begin{cases} R^3 & (i = 0) \\ 0 & (i \geq 1) \end{cases}$	$\mathrm{S}_8(R)$	$\mathrm{P}_{2,1}(R)$	2
$\mathrm{S}_8(R) = \left\{ \begin{pmatrix} a & c & d \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \right\}$	4	$H^i = \begin{cases} R^3 & (i = 0) \\ 0 & (i \geq 1) \end{cases}$	$\mathrm{S}_7(R)$	$\mathrm{P}_{1,2}(R)$	2
$\mathrm{S}_9(R) = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & b & d \\ 0 & 0 & b \end{pmatrix} \right\}$	4	$H^i = R$ for $i \geq 0$	$\mathrm{S}_6(R)$	$\mathrm{S}_{14}(R)$	5

3. DESCRIPTION OF $\mathrm{Mold}_{3,4}$

In this section, we describe $\mathrm{Mold}_{3,4}$. Let V be a free module of rank 3 over $\mathbb{Z}$. Fix a canonical basis $\{e_1, e_2, e_3\}$ of V over $\mathbb{Z}$. We define schemes $\mathbb{P}^*(V)$, $\mathbb{P}_*(V)$, and $\mathrm{Flag}(V)$ over $\mathbb{Z}$ as contravariant functors from the category of schemes to the category of sets in the following way:

$$\begin{aligned} \mathbb{P}^*(V)(X) &= \left\{ \, W \mid \; W \text{ is a rank 2 subbundle of } \mathcal{O}_X \otimes_{\mathbb{Z}} V \text{ on } X \, \right\}, \\ \mathbb{P}_*(V)(X) &= \left\{ \, L \mid \; L \text{ is a rank 1 subbundle of } \mathcal{O}_X \otimes_{\mathbb{Z}} V \text{ on } X \, \right\}, \\ \mathrm{Flag}(V)(X) &= \left\{ \, (L, W) \in (\mathbb{P}_*(V) \times \mathbb{P}^*(V))(X) \mid L \subset W \, \right\} \end{aligned}$$

for a scheme X.

Remark 10. If we consider the case over a field k, then $\mathbb{P}^*(V)$, $\mathbb{P}_*(V)$, and Flag over k are regarded as

$$\begin{aligned} \mathbb{P}^*(V) &= \{W \subset V \mid W \text{ is a 2-dimensional subspace of } V\}, \\ \mathbb{P}_*(V) &= \{L \subset V \mid L \text{ is a 1-dimensional subspace of } V\}, \\ \mathrm{Flag}(V) &= \{(L, W) \in \mathbb{P}_*(V) \times \mathbb{P}^*(V) \mid 0 \subset L \subset W \subset V\}, \end{aligned}$$

respectively.

Let us consider rank 4 molds

$$(\mathrm{B}_2 \times \mathrm{D}_1)(\mathbb{Z}) = \left\{ \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(\mathbb{Z}) \right\},$$

$$\begin{aligned} \mathrm{S}_7(\mathbb{Z}) &= \left\{ \begin{pmatrix} a & 0 & c \\ 0 & a & d \\ 0 & 0 & b \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{Z} \right\}, \\ \mathrm{S}_8(\mathbb{Z}) &= \left\{ \begin{pmatrix} a & c & d \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{Z} \right\} \end{aligned}$$

over $\mathbb{Z}$. Let $A = \mathrm{B}_2 \times \mathrm{D}_1$, S_7, or S_8. Then $\mathrm{HH}^1(A(k), \mathrm{M}_3(k)/A(k)) = 0$ for any field k by Table 1. The image of the morphism $\phi_A : \mathrm{PGL}_3 \to \mathrm{Mold}_{3,4}$ defined by $P \mapsto P^{-1}A(\mathbb{Z})P$ is open by Theorem 9.

Definition 11 ([4])**.** We define open subschemes of $\mathrm{Mold}_{3,4}$ by

$$\begin{aligned} \mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1} &= \operatorname{Im}\phi_{\mathrm{B}_2\times\mathrm{D}_1}, \\ \mathrm{Mold}_{3,4}^{\mathrm{S}_7} &= \operatorname{Im}\phi_{\mathrm{S}_7}, \\ \mathrm{Mold}_{3,4}^{\mathrm{S}_8} &= \operatorname{Im}\phi_{\mathrm{S}_8}. \end{aligned}$$

Remark 12. Let $A = \mathrm{B}_2 \times \mathrm{D}_1$, S_7, or S_8. Then $\mathrm{HH}^2(A(k), \mathrm{M}_3(k)/A(k)) = 0$ for any field k by Table 1. By [3], the canonical morphism $\mathrm{Mold}_{3,4}^{A} \to \mathbb{Z}$ is smooth.

Theorem 13 ([4])**.** *The subschemes* $\mathrm{Mold}_{3,4}^{\mathrm{S}_7}$ *and* $\mathrm{Mold}_{3,4}^{\mathrm{S}_8}$ *are open and closed in* $\mathrm{Mold}_{3,4}$. *Moreover,* $\mathrm{Mold}_{3,4}^{\mathrm{S}_7} \cong \mathbb{P}^*(V)$ *and* $\mathrm{Mold}_{3,4}^{\mathrm{S}_8} \cong \mathbb{P}_*(V)$.

Outline of proof. For simplicity, here we only consider the case over a field k. For $W \in \mathbb{P}^*(V)$, set

$$A_W = \{ f \in \mathrm{End}_k(V) \cong \mathrm{M}_3(k) \mid f(W) \subseteq W \text{ and } f|W \text{ is scalar } \} \subset \mathrm{M}_3(k).$$

Let us define a morphism

$$\begin{array}{rccc} \psi_{\mathrm{S}_7} : & \mathbb{P}^*(V) & \to & \mathrm{Mold}_{3,4}^{\mathrm{S}_7} \\ & W & \mapsto & A_W. \end{array}$$

We can verify that ψ_{S_7} is an isomorphism.

For $L \in \mathbb{P}_*(V)$, set

$$A_L = \{ f \in \mathrm{End}_k(V) \cong \mathrm{M}_3(k) \mid f(L) \subseteq L \text{ and } f : V/L \to V/L \text{ is scalar } \}.$$

Let us define a morphism

$$\begin{array}{rccc} \psi_{\mathrm{S}_8} : & \mathbb{P}_*(V) & \to & \mathrm{Mold}_{3,4}^{\mathrm{S}_8} \\ & L & \mapsto & A_L. \end{array}$$

We can verify that ψ_{S_8} is an isomorphism. □

Definition 14. We define

$$\begin{aligned} \mathrm{Q}(V) &= \mathrm{Flag}(V) \times_{\mathbb{P}_*(V)} \mathrm{Flag}(V) \times_{\mathbb{P}^*(V)} \mathrm{Flag}(V) \\ &= \{(L_1, W_2; L_1, W_1; L_2, W_1) \mid \dim_k L_i = 1, \dim_k W_i = 2\} \\ &= \{(L_1, L_2, W_1, W_2) \mid L_1 \subset W_1, L_1 \subset W_2, L_2 \subset W_1\}. \end{aligned}$$

Let us define the projection $\pi : \mathrm{Q}(V) \to \mathrm{Flag}(V)$ by

$$(L_1, L_2, W_1, W_2) \mapsto (L_1, W_1).$$

We can verify that π is a fiber bundle with fiber $\mathbb{P}^1 \times \mathbb{P}^1$.

For $(L_1, L_2, W_1, W_2) \in \mathrm{Q}(V)$, set

$$A_{(L_1,L_2,W_1,W_2)} = \left\{ f \in \mathrm{M}_3(k) \;\middle|\; \begin{array}{c} f(L_i) \subset L_i, f(W_i) \subset W_i \ (i = 1, 2), \text{ and} \\ L_2 \cong W_1/L_1 \cong V/W_2 \text{ as } k[f]\text{-modules} \end{array} \right\}.$$

Let us define a morphism

$$\begin{array}{cccc} \psi_{\mathrm{B}_2\times\mathrm{D}_1} : & \mathrm{Q}(V) & \to & \mathrm{Mold}_{3,4} \\ & (L_1, L_2, W_1, W_2) & \mapsto & A_{(L_1,L_2,W_1,W_2)}. \end{array}$$

Theorem 15 ([4])**.** *The image of* $\psi_{\mathrm{B}_2\times\mathrm{D}_1}$ *is open and closed in* $\mathrm{Mold}_{3,4}$*. Moreover,* $\psi_{\mathrm{B}_2\times\mathrm{D}_1}$ *gives an isomorphism between* $\mathrm{Q}(V)$ *and the closure* $\overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}}$ *of* $\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}$*.*

Outline of proof. It can be verified that $\psi_{\mathrm{B}_2\times\mathrm{D}_1}$ is a monomorphism. By a long discussion, we can also prove that $\psi_{\mathrm{B}_2\times\mathrm{D}_1}$ is formally étale. Hence, $\psi_{\mathrm{B}_2\times\mathrm{D}_1}$ gives an isomorphism between $\mathrm{Q}(V)$ and an open subscheme of $\mathrm{Mold}_{3,4}$ which coincides with $\overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}}$. □

Definition 16 ([4])**.** Let $A = \mathrm{N}_3$, S_6, or S_9. We define

$$\mathrm{Mold}_{3,4}^{A} \;=\; \{x \in \mathrm{Mold}_{3,4} \mid \mathcal{A}(x) \otimes_{k(x)} \overline{k(x)} \sim A(\overline{k(x)})\},$$

where $\overline{k(x)}$ is an algebraic closure of $k(x)$.

We can also prove the following theorems.

Theorem 17 ([4])**.** *For the closure* $\overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}}$ *of* $\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}$*, we obtain*

$$\overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}} = \mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1} \coprod \mathrm{Mold}_{3,4}^{\mathrm{S}_6} \coprod \mathrm{Mold}_{3,4}^{\mathrm{S}_9} \coprod \mathrm{Mold}_{3,4}^{\mathrm{N}_3}.$$

Theorem 18 ([4])**.** *By the isomorphism* $\overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}} \cong \mathrm{Q}(V)$*, we have*

$$\begin{array}{rcl} \mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1} & = & \{(L_1, L_2, W_1, W_2) \in \mathrm{Q}(V) \mid L_1 \neq L_2, W_1 \neq W_2\}, \\ \mathrm{Mold}_{3,4}^{\mathrm{S}_6} & = & \{(L_1, L_2, W_1, W_2) \in \mathrm{Q}(V) \mid L_1 = L_2, W_1 \neq W_2\}, \\ \mathrm{Mold}_{3,4}^{\mathrm{S}_9} & = & \{(L_1, L_2, W_1, W_2) \in \mathrm{Q}(V) \mid L_1 \neq L_2, W_1 = W_2\}, \\ \mathrm{Mold}_{3,4}^{\mathrm{N}_3} & = & \{(L_1, L_2, W_1, W_2) \in \mathrm{Q}(V) \mid L_1 = L_2, W_1 = W_2\}. \end{array}$$

By using Theorem 18, let us describe a deformation of 4-dimensional subalgebras of M_3. We define a 2-dimensional closed subscheme $\mathrm{Q}^{st}(V)$ of $\mathrm{Q}(V) \cong \overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}}$.

For simplicity, let us consider the case over a field k. Set $L_1^{st} = ke_1$ and $W_1^{st} = ke_1 \oplus ke_2$. Put $* = (L_1^{st}, W_1^{st}) \in \mathrm{Flag}(V)$. Then we have the following fiber product:

$$\begin{array}{ccc} \mathrm{Q}^{st}(V) & \to & \mathrm{Q}(V) \\ \downarrow & & \downarrow \\ * & \to & \mathrm{Flag}(V). \end{array}$$

Note that $\mathrm{Q}^{st}(V) \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$.

Let $L_2(s_1) = \left\langle \begin{bmatrix} 1 \\ -s_1 \\ 0 \end{bmatrix} \right\rangle$ and $W_2(s_2) = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ s_2 \end{bmatrix} \right\rangle$. Then

$$\{(s_1, s_2) \in \mathbb{A}^2_k\} \cong (\mathbb{P}^1_k \setminus \{\infty\}) \times (\mathbb{P}^1_k \setminus \{\infty\})$$

gives an affine open subscheme of $\mathrm{Q}^{st}(V)$ by considering $(L_1^{st}, L_2(s_1), W_1^{st}, W_2(s_2))$. We write

$$A(s_1, s_2) = \left\{ \begin{bmatrix} a + s_1 b & b & c \\ 0 & a & d \\ 0 & 0 & a + s_2 d \end{bmatrix} \middle| \, a, b, c, d \in k \right\}$$

for $\psi_{\mathrm{B}_2 \times \mathrm{D}_1}(s_1, s_2) \in \overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2 \times \mathrm{D}_1}}$.

Note that

$$\begin{aligned} A(s_1, s_2): &\quad \mathrm{B}_2 \times \mathrm{D}_1 \text{ type} &\quad \text{if } s_1 \neq 0, s_2 \neq 0, \\ A(0, s_2): &\quad \mathrm{S}_6 \text{ type} &\quad \text{if } s_2 \neq 0, \\ A(s_1, 0): &\quad \mathrm{S}_9 \text{ type} &\quad \text{if } s_1 \neq 0, \\ A(0,0): &\quad \mathrm{N}_3 \text{ type.} \end{aligned}$$

Summarizing the discussions above, we obtain the main theorem.

Theorem 19 ([4])**.** *We have an irreducible decomposition*

$$\mathrm{Mold}_{3,4} = \overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2 \times \mathrm{D}_1}} \coprod \mathrm{Mold}_{3,4}^{\mathrm{S}_7} \coprod \mathrm{Mold}_{3,4}^{\mathrm{S}_8},$$

whose irreducible components are all connected components. Moreover, $\overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2 \times \mathrm{D}_1}} \cong \mathrm{Q}(V)$, $\mathrm{Mold}_{3,4}^{\mathrm{S}_7} \cong \mathbb{P}^2_{\mathbb{Z}}$, *and* $\mathrm{Mold}_{3,4}^{\mathrm{S}_8} \cong \mathbb{P}^2_{\mathbb{Z}}$ *over* $\mathbb{Z}$.

By considering the PGL_3-orbits in $\mathrm{Mold}_{3,4}$ over a field k, we have:

Corollary 20 ([4])**.** *Let k be an arbitrary field. Then there exist* 6 *equivalence classes of* 4*-dimensional subalgebras of* $\mathrm{M}_3(k)$ *over k:* $(\mathrm{B}_2 \times \mathrm{D}_1)(k)$, $\mathrm{N}_3(k)$, $\mathrm{S}_6(k)$, $\mathrm{S}_7(k)$, $\mathrm{S}_8(k)$, *and* $\mathrm{S}_9(k)$.

Remark 21. Let S be a 4-dimensional subalgebra of $\mathrm{M}_3(k)$ over a field k. Let A be one of $(\mathrm{B}_2 \times \mathrm{D}_1)(k)$, $\mathrm{N}_3(k)$, $\mathrm{S}_6(k)$, $\mathrm{S}_7(k)$, $\mathrm{S}_8(k)$, or $\mathrm{S}_9(k)$. If $S \otimes_k K$ is equivalent to $A \otimes_k K$ for an extension field K of k, then S is equivalent to A over k by Corollary 20.

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Center for Medical Education and Sciences
Faculty of Medicine
University of Yamanashi
1110 Shimokato, Chuo, Yamanashi 409-3898, Japan
Email address: nakamoto@yamanashi.ac.jp

Department of Mathematics
Okayama University
Okayama 700-8530 JAPAN
Email address: torii@math.okayama-u.ac.jp

GOVOROV–LAZARD TYPE THEOREMS, BIG COHEN–MACAULAY MODULES, AND COHEN–MACAULAY HEARTS

TSUTOMU NAKAMURA

Abstract. Let R be a Cohen–Macaulay local ring with a canonical module and let A be an R-order. We report that a Govorov–Lazard type theorem holds for the category of weak (balanced) big Cohen–Macaulay modules over A. This theorem, which is a generalization of a result due to Holm for the case $R = A$, enables us to show that every complete pure-injective big Cohen–Macaulay A-module is a direct summand of a direct product of finitely generated CM A-modules, provided that R is complete. This fact is well known if R is artinian. We also study big Cohen–Macaulay modules over a non-Cohen–Macaulay local ring R, using the Cohen–Macaulay heart of R.

1. Introduction

Let A be a ring and denote by $\mathsf{Mod}\,A$ (resp. $\mathsf{mod}\,A$) the category of (right) A-modules (resp. finitely presented A-modules). Let $\mathcal{C}$ be an additive subcategory of $\mathsf{Mod}\,A$ closed under direct limits. It is a delicate problem in general whether every module in $\mathcal{C}$ can be presented as a direct limit of modules in $\mathcal{C} \cap \mathsf{mod}\,A$. If this is possible, we say that a *Govorov–Lazard type theorem* holds for $\mathcal{C}$, and write

$$\varinjlim(\mathcal{C} \cap \mathsf{mod}\,A) = \mathcal{C}.$$

For example, it is well known that $\varinjlim \mathsf{mod}\,A = \mathsf{Mod}\,A$. Govorov [3] and Lazard [4] independently proved that $\varinjlim \mathsf{proj}\,A = \mathsf{Flat}\,A$, where $\mathsf{Flat}\,A$ (resp. $\mathsf{proj}\,A$) denotes the category of flat (resp. finitely generated projective) A-modules. Moreover, for an Iwanaga–Gorenstein ring A, Enochs and Jenda [2] showed that a Govorov–Lazard type theorem holds for the category $\mathsf{GFlat}\,A$ of Gorenstein-flat A-modules, where $(\mathsf{GFlat}\,A) \cap \mathsf{mod}\,A$ coincides with the category of finitely generated Gorenstein-projective A-modules. If A is not Iwanaga–Gorenstein, a Govorov–Lazard type theorem may not hold for $\mathsf{GFlat}\,A$; this is due to Holm and Jørgensen [6].

2. Results

Let R be a commutative noetherian local ring. An R-module M is called *(balanced) big CM* (=Cohen–Macaulay) if every system of parameters of R is an M-regular sequence. We call an R-module M a *weak* big CM if every system of parameters of R is a weak M-regular sequence (cf. [5]). We denote by $\mathsf{WCM}\,R$ the category of weak big CM modules. Then $\mathsf{WCM}\,R \cap \mathsf{mod}\,R = \mathsf{CM}\,R$, where the right-hand side denotes the category of (maximal) CM modules. Holm [5] showed that $\varinjlim \mathsf{CM}\,R = \mathsf{WCM}\,R$ holds for any CM local ring R with a canonical module. Our first result extends this to orders over a CM local ring R

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with a canonical module. Recall that a (possibly noncommutative) R-algebra A is said to be an *R-order* if A is CM as an R-module. We denote by $\mathsf{CM}\,A$ (resp. $\mathsf{WCM}\,A$) the category of A-modules being CM (resp. weak big CM) as R-modules.

Theorem 1. *Let R be a CM local ring with a canonical module and let A be an R-order. Then we have*

$$\varinjlim \mathsf{CM}\,A = \mathsf{WCM}\,A.$$

If R is artinian, then A is an Artin R-algebra, and then It is well known that every pure-injective module over a A is a direct summand of a direct product of finitely generated A-modules. Using the above theorem, we can extend this fact to an order A over a complete CM local ring R. Note that a *CM* (resp. *big CM*) A-module means an A-module which is CM (resp. big CM) as an R-module.

Corollary 2. *Let R be a complete CM local ring and let A be an R-order. Then every pure-injective complete big CM module is a direct summand of a direct product of CM A-modules.*

By André' noble work [1], every commutative noetherian local ring R admits a big Cohen–Macaulay module. On the other hand, it is still an open question if every complete noetherian local ring R admits a (finitely generated) CM R-module. This question is known as the *small CM conjecture*. Then there might be little hope that we could have $\varinjlim \mathsf{CM}\,R = \mathsf{WCM}\,R$ in general. So we would like to give another formulation.

Assume that R is a homomorphic image of a CM local ring. We use the *Cohen–Macaulay heart* $\mathcal{H}_{\mathsf{CM}}$ of R introduced in [7]. This is the heart of some compactly generated generated t-structure in the (unbounded) derived category $\mathsf{D}\,(R)$. There are several remarkable facts: $\mathcal{H}_{\mathsf{CM}}$ is a locally coherent Grothendieck category and derived equivalent to $\mathsf{Mod}\,R$. Furthermore, we have

$$\mathcal{H}_{\mathsf{CM}} \cap \mathsf{Mod}\,R = \mathsf{WCM}\,R.$$

Denote by $\mathsf{fp}(\mathcal{H}_{\mathsf{CM}})$ the subcategory of finitely presented objects in $\mathcal{H}_{\mathsf{CM}}$. The locally coherence of $\mathcal{H}_{\mathsf{CM}}$ implies that a Govorov–Lazard type theorem holds for $\mathcal{H}_{\mathsf{CM}}$, that is, each object in $\mathcal{H}_{\mathsf{CM}}$ is a direct limit of objects in $\mathsf{fp}(\mathcal{H}_{\mathsf{CM}})$:

$$\varinjlim \mathsf{fp}(\mathcal{H}_{\mathsf{CM}}) = \mathcal{H}_{\mathsf{CM}}.$$

Hence we have:

Proposition 3. *Let R be a homomorphic image of a CM local ring. Then every weak big CM module is a direct limit of finitely presented objects in $\mathcal{H}_{\mathsf{CM}}$.*

Remark 4. When R admits a dualizing complex D (such that $\inf\{i \mid H^i(D) \neq 0\} = 0$), there is an equivalence

$$\mathrm{RHom}_R(-, D) : (\mathsf{mod}\,R)^{\mathsf{op}} \xrightarrow{\sim} \mathsf{fp}(\mathcal{H}_{\mathsf{CM}}).$$

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Department of Mathematics
Faculty of Education
Mie University
1577 Kurimamachiya-cho, Tsu city, Mie, 514-8507, JAPAN
Email address: `t.nakamura.math@gmail.com`

K_0 OF WEAK WALDHAUSEN EXTRIANGULATED CATEGORIES

YASUAKI OGAWA

Abstract. We modify the axiom of the Waldhausen structure so that it matches better with extriangulated categories. It enables us to define an abelian group $K_0(\mathcal{C})$ of a weak Waldhausen category $\mathcal{C}$ which generalizes that of an extriangulated category. As one might expect, it behaves nicely in the context of Quillen's localization and resolution theorems. We obtain two applications: the first one generalizes exact sequences of the Grothendieck groups associated with the Serre/Verdier localization to some types of "one-sided" exact localizations; the second one reveals close relations between Quillen's theorems and Palu's index.

1. Introduction

The higher algebraic K-theory for an exact category $\mathcal{C}$ was introduced by Quillen, which is now called Quillen's Q-construction [18]. Such a construction makes $\mathcal{C}$ to be the simplicial category $B\mathcal{C}$ by inverting certain morphisms and the K-theory is defined via its geometric realization $|B\mathcal{C}|$. The first foundational result in [18] is the *localization theorem* which extracts a long exact sequence of K-groups from the Serre quotient. The second one is the *resolution theorem* which shows that if we can identify a suitable subcategory $\mathcal{X}$ of an exact category $\mathcal{C}$, then $K(\mathcal{C}) \cong K(\mathcal{X})$. However, not all K-groups can be recovered as those of some abelian/exact categories. It turned out that Quillen's K-theory for exact categories does not possess satisfactory generality that K-theorists had in mind, where triangulated categories come in. To tackle this problem, Waldhausen introduced a generalization of exact categories, now called the Waldhausen category, in which K-theory still exists [21]. As applications of his abstract localization theorem, Thomason-Trobaugh established a K-theory of the derived categories [20] and Schlichting generalized it to any algebraic triangulated category [19].

On one hand, the notion of extriangulated category was introduced by Nakaoka-Palu [13] as a simultaneous generalization of exact categories and triangulated categories. A localization theory of them was also developed in [12] which contains many quotient processes in algebraic contexts as well as the Serre/Verdier quotient. In this article, focusing only on the Grothendieck groups, we generalize a part of the Waldhausen theory on exact categories to the extriangulated case, more specifically, we define the weak Waldhausen extriangulated category $(\mathcal{C}, \mathsf{C}, \mathsf{W})$ together with its Grothendieck group $K_0(\mathcal{C}, \mathsf{C}, \mathsf{W})$.

First, as a benefit of introducing the weak Waldhausen structure, we obtain an exact sequence of Grothendieck groups associated with some localizations such as the Serre/Vedier

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quotient (Theorem 12), which contains an extriangulated counter part of Quillen's localization theorem. The above assertion for the Serre/Verdier quotient goes back to Heller and Grothendieck, respectively. Furthermore, it can apply to abelian localizations of triangulated categories which can be traced back to hearts of t-structures in the sense of [2]. Since then, abelian localizations have been found using cluster tilting subcategories [10]. These constructions were unified in [1] and placed in an extriangulated context in [11]. A generalization from cluster tilting to rigid subcategories was initiated in [3, 4], and has been further developed in the literature.

Our second aim is to reveal a close relation between the resolution theorem and abelian localization. To this end, we establish the extriangulated version (Theorem 14) and it provides a slight generalization and a better understanding for Palu's index which was introduced in connection with the Caldero-Chapoton map [16]. Let triangulated category $\mathcal{C}$ and a 2-cluster tilting subcategory $\mathcal{X} \subseteq \mathcal{C}$ be given. For each object $C \in \mathcal{C}$, Palu's *index* $\mathsf{ind}_{\mathcal{X}}(C)$ *of C with respect to* $\mathcal{X}$ is defined as an element of the split Grothendieck group $K_0^{\mathrm{sp}}(\mathcal{X})$. Recently, it is interpreted and generalized via a certain relative extriangulated structure of $\mathcal{C}$ naturally defined by a given subcategory $\mathcal{X}$ [17, 9]. We prove that such results indeed come from the resolution theorem.

Notation and convention. All categories and functors in this article are always assumed to be additive, and subcategories will always be full. For a category $\mathcal{C}$, we denote the class of all morphisms in $\mathcal{C}$ by $\mathsf{Mor}\,\mathcal{C}$, and $\mathsf{mod}\,\mathcal{C}$ is the category of finitely presented contravariant functors from $\mathcal{C}$ to the abelian category Ab of abelian groups.

2. Localization of extriangulated categories

This section is devoted to recall the localization theory of extriangulated category by a suitable thick subcategory, which was introduced in the pursuit of unifying the Serre/Verdier quotient [12]. We also recall a specific case, namely, a localization of triangulated category by an extension-closed subcategory [14].

Nakaoka-Palu's *extriangulated category* is defined to be an additive category $\mathcal{C}$ equipped with

- a biadditive functor $\mathbb{E}\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Ab}$, where Ab is the category of abelian groups, and
- a correspondence $\mathfrak{s}$ that associates an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{g} C]$ of a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$ to each element $\delta \in \mathbb{E}(C, A)$ for any $A, C \in \mathcal{C}$,

where the triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies some axioms. We refer the reader to [13] for an in-depth treatment, see also [15, §2,3]. It turns out that an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is equipped with the class of sequences of the form $A \xrightarrow{f} B \xrightarrow{g} C$ which is called an $\mathfrak{s}$-*conflation.* The pair of an $\mathfrak{s}$-conflation and the corresponding element $\delta \in \mathbb{E}(C, A)$ is called an $\mathfrak{s}$-*triangle* and denoted by $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$. In contrast to triangulated/exact categories, if we state the axiom for extriangulated category, the *realization* $\mathfrak{s}$ is indispensable.

Let us introduce an exact sequence of extriangulated categories as a generalization of the Serre/Verdier quotient. We denote by ET the category of extriangulated categories and exact functors.

Definition 1. A sequence $(\mathcal{N},\mathbb{E}',\mathfrak{s}') \xrightarrow{(F,\phi)} (\mathcal{C},\mathbb{E},\mathfrak{s}) \xrightarrow{(Q,\mu)} (\mathcal{D},\mathbb{F},\mathfrak{t})$ in ET is called an *exact sequence* of extriangulated categories, if the following conditions are fulfilled.

(1) F is fully faithful.
(2) $\operatorname{Im} F = \operatorname{Ker} Q$ holds.
(3) For any map $(G,\psi)\colon (\mathcal{C},\mathbb{E},\mathfrak{s}) \to (\mathcal{D}',\mathbb{F}',\mathfrak{t}')$ in ET with $G\circ F = 0$, there uniquely exists an exact functor $(G',\psi')\colon (\mathcal{D},\mathbb{F},\mathfrak{t}) \to (\mathcal{D}',\mathbb{F}',\mathfrak{t}')$ such that $(G,\psi) = (G',\psi')\circ(Q,\mu)$.

Let us remind a construction of the Verdier quotient: given a triangulated category $\mathcal{C}$ and a thick subcategory $\mathcal{N}\subseteq\mathcal{C}$, we associate the class $\mathcal{S}_{\mathcal{N}}$ of morphisms in $\mathcal{C}$ to $\mathcal{N}$, namely, $\mathcal{S}_{\mathcal{N}} := \{s\in \mathsf{Mor}\,\mathcal{C} \mid {}^{\exists}A \xrightarrow{s} B \to N \to A[1] \text{ with } \mathcal{N}\in\mathcal{N}\}$. Then the Verdier quotient $\mathcal{C}/\mathcal{N}$ is defined to be the Gabriel-Zisman localization $\mathcal{C}[\mathcal{S}_{\mathcal{N}}^{-1}]$ and it gives rise to an exact sequence $\mathcal{N}\to\mathcal{C}\to\mathcal{C}/\mathcal{N}$ in the category of triangulated categories and exact functors.

Similarly to the case of the Verdier quotient, we associate the class $\mathcal{S}_{\mathcal{N}}$ to the pair $(\mathcal{C},\mathcal{N})$ of an extriangulated category $\mathcal{C}$ and a thick subcategory $\mathcal{N}\subseteq\mathcal{C}$. The following is a basic machinery to establish an exact sequence in ET, see [12, Thm. 3.5] for a detailed setup.

Theorem 2. *Let* $(\mathcal{C},\mathbb{E},\mathfrak{s})$ *be an extriangulated category with a thick subcategory* $\mathcal{N}$. *Suppose* $\mathcal{S}_{\mathcal{N}}$ *satisfies conditions* (MR1)–(MR4) *in* [12, Thm. 3.5]. *Then there is an extriangulated category* $(\mathcal{C}/\mathcal{N},\widetilde{\mathbb{E}},\widetilde{\mathfrak{s}})$ *together with an exact functor* $(Q,\mu)\colon (\mathcal{C},\mathbb{E},\mathfrak{s}) \to (\mathcal{C}/\mathcal{N},\widetilde{\mathbb{E}},\widetilde{\mathfrak{s}})$. *Furthermore, the following natural sequence forms an exact sequence in* ET.

$$(\mathcal{N},\mathbb{E}|_{\mathcal{N}},\mathfrak{s}|_{\mathcal{N}}) \xrightarrow{\ \mathsf{inc}\ } (\mathcal{C},\mathbb{E},\mathfrak{s}) \xrightarrow{\ (Q,\mu)\ } (\mathcal{C}/\mathcal{N},\widetilde{\mathbb{E}},\widetilde{\mathfrak{s}}) \tag{2.1}$$

Unfortunately, it is not easy to check the conditions (MR1)–(MR4). Except for the Verdier/Serre quotient, just a few examples of subcategories which yields (2.1) are know, e.g. *biresolving* subcategories [12, §§4.3] and *percolating* subcategories [12, §§4.4].

We now specialize to the case when $(\mathcal{C},\mathbb{E},\mathfrak{s})$ corresponds to a triangulated category and recall the localization theory from [14] that we need.

Setup 3. We fix a triangulated category $\mathcal{C}$ (with suspension [1]) and an extension-closed subcategory $\mathcal{N}\subseteq\mathcal{C}$ that is closed under direct summands. We denote by $(\mathcal{C},\mathbb{E},\mathfrak{s})$ the extriangulated category corresponding to the triangulated category $\mathcal{C}$.

As an application of the relative theory for extriangulated categories [8], we know any extension-closed subcategory $\mathcal{N}$ determines relative structures on $\mathcal{C}$. As pointed out in [5, Prop. A.4], these relative structures are natural from the viewpoint of constructing exact substructures of an exact category.

Proposition 4. [14, Prop. 2.1] *For* $A,C\in\mathcal{C}$, *define subsets of* $\mathbb{E}(C,A) = \mathcal{C}(C,A[1])$ *as follows.*

$$\mathbb{E}^L_{\mathcal{N}}(C,A) := \{h\colon C\to A[1] \mid \forall x\colon N\to C \text{ with } N\in\mathcal{N}, \text{ we have } hx\in[\mathcal{N}[1]]\}$$
$$\mathbb{E}^R_{\mathcal{N}}(C,A) := \{h\colon C\to A[1] \mid \forall y\colon A\to N \text{ with } N\in\mathcal{N}, \text{ we have } y\circ h[-1]\in[\mathcal{N}[-1]]\}$$

Then both $\mathbb{E}^L_{\mathcal{N}}$ *and* $\mathbb{E}^R_{\mathcal{N}}$ *give rise to closed subfunctors of* $\mathbb{E}$. *In particular, putting* $\mathbb{E}_{\mathcal{N}} := \mathbb{E}^L_{\mathcal{N}}\cap\mathbb{E}^R_{\mathcal{N}}$, *we obtain extriangulated structures*

$$\mathcal{C}^L_{\mathcal{N}} := (\mathcal{C},\mathbb{E}^L_{\mathcal{N}},\mathfrak{s}^L_{\mathcal{N}}),\quad \mathcal{C}^R_{\mathcal{N}} := (\mathcal{C},\mathbb{E}^R_{\mathcal{N}},\mathfrak{s}^R_{\mathcal{N}}),\quad \mathcal{C}_{\mathcal{N}} := (\mathcal{C},\mathbb{E}_{\mathcal{N}},\mathfrak{s}_{\mathcal{N}}),$$

all relative to the triangulated structure $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$.

With respect to the relative structure $\mathcal{C}_{\mathcal{N}}$, the pair $(\mathcal{C}, \mathcal{N})$ yields a class $\mathcal{S}_{\mathcal{N}}$ of morphisms in $\mathcal{C}$ satisfying the needed conditions to obtain an exact sequence in ET.

Theorem 5. [14, Thm. A, Lem. 2.4, Cor. 2.11] *We have an exact sequence* $(\mathcal{N}, \mathbb{E}, \mathfrak{s}) \xrightarrow{\mathsf{inc}} (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}) \xrightarrow{(Q,\mu)} (\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ *in* ET*. Furthermore, if* $\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$ *holds in the triangulated category* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$*, the following are true.*

1. *The quotient category* $\mathcal{C}/\mathcal{N} := (\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ *is abelian.*
2. *The quotient functor* (Q, μ) *induces a right exact functor* $Q\colon (\mathcal{C}, \mathbb{E}^R_{\mathcal{N}}, \mathfrak{s}^R_{\mathcal{N}}) \to \mathcal{C}/\mathcal{N}$ *and a left exact functor* $Q\colon (\mathcal{C}, \mathbb{E}^L_{\mathcal{N}}, \mathfrak{s}^L_{\mathcal{N}}) \to \mathcal{C}/\mathcal{N}$*. In addition, it induces a cohomological functor* $Q\colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to \mathcal{C}/\mathcal{N}$*.*

We call the case $\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$ *in which we have the resulting abelian category* $\mathcal{C}/\mathcal{N}$ *the* abelian localization *of* $\mathcal{C}$ *by* $\mathcal{N}$*.*

We can think of hearts of t-structures in the sense of [2] as a prototypical example of the abelian localization. Since then, it has been found and generalized via cluster tilting subcategories [10], rigid subcategories [4, 3] and cotorsion pairs [1]. In turn, Theorem 5 can apply to these phenomenon. To clarify our point of focus, we record the following immediate result.

Example 6. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a triangulated category and $\mathcal{X} \subseteq \mathcal{C}$ be a contravariantly finite rigid subcategory. We consider an extension closed subcategory $\mathcal{N} := \mathcal{X}^{\perp_0} = \{C \in \mathcal{C} \mid (\mathcal{X}, C) = 0\}$. Since $\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$ is true, Theorem 5 provides a right exact functor $Q\colon \mathcal{C}^R_{\mathcal{N}} \to \mathcal{C}/\mathcal{N}$. Furthermore, we can verify that there exists a natural exact equivalence $\mathcal{C}/\mathcal{N} \cong \mathsf{mod}\,\mathcal{X}$. Thus we have a right exact functor $Q \cong (\mathcal{X}, -)$ with the kernel $\mathcal{N}$ as below.

$$(\mathcal{N}, \mathbb{E}|_{\mathcal{N}}, \mathfrak{s}|_{\mathcal{N}}) \xrightarrow{\quad \mathsf{inc} \quad} (\mathcal{C}, \mathbb{E}^R_{\mathcal{N}}, \mathfrak{s}^R_{\mathcal{N}}) \xrightarrow{\quad Q \quad} \mathsf{mod}\,\mathcal{X} \tag{2.2}$$

Note that this sequence does not sit in ET any more.

3. Weak Waldhausen categories

We introduce the notion of weak Waldhausen category. This is a simultaneous generalization of the (classical) Waldhausen category and extriangulated category. Also, we define its Grothendieck group.

Definition 7. Let $\mathcal{C}$ be an additive category equipped with a class Seq of *distinguished sequences* of the form

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \tag{3.1}$$

in $\mathcal{C}$, and a class W of morphisms in $\mathcal{C}$. Denote by C (resp. F) the class of morphisms f (resp. g) appearing in a distinguished sequence (3.1). The morphisms in C (resp. F) are called *cofibrations* (resp. *fibrations*) and denoted by $\rightarrowtail$ (resp. $\twoheadrightarrow$). The morphisms in W are called *weak equivalences* and are denoted by $\xrightarrow{\sim}$.

1. The triplet $(\mathcal{C}, \mathsf{Seq}, \mathsf{W})$ is called a *weak Waldhausen (additive) category* if the following axioms are satisfied.

(WC0) The class C is closed under composition and contains all isomorphisms.

(WC1) Seq contains all split exact sequences and is closed under isomorphism. Any distinguished sequence (3.1) is a weak cokernel sequence.

(WC2) Any pair (f, c) of a cofibration $A \overset{f}{\rightarrowtail} B$ and a morphism $A \xrightarrow{c} C$ yields a cofibration $A \xrightarrow{\binom{f}{-c}} B \oplus C$. Furthermore, the associated distinguished sequences of the form $A \xrightarrow{\binom{f}{-c}} B \oplus C \xrightarrow{(c'\ f')} D$ satisfy that f' belongs to C.

(WW0) The class W is closed under composition and contains all isomorphisms.

(WW1) (Gluing axiom) Consider a commutative diagram of the form

$$\begin{tikzcd} C \arrow[d, "\sim"] & A \arrow[l, "c"'] \arrow[r, tail, "f"] \arrow[d, "\sim"] & B \arrow[d, "\sim"] \\ C' & A' \arrow[l, "c'"'] \arrow[r, tail, "f'"] & B' \end{tikzcd} \tag{3.2}$$

in which all vertical arrows are weak equivalences and the feathered arrows are cofibrations. Then from a distinguished weak cokernel of $\binom{f}{-c}$ to a distinguished weak cokernel of $\binom{f'}{-c'}$, there is an induced morphism that is also a weak equivalence.

(2) The triplet $(\mathcal{C}, \mathsf{Seq}, \mathsf{W})$ is called a *weak coWaldhausen category* if the triplet $(\mathcal{C}^{\mathsf{op}}, \mathsf{Seq}^{\mathsf{op}}, \mathsf{W}^{\mathsf{op}})$ is a weak Waldhausen additive category.

(3) The triplet $(\mathcal{C}, \mathsf{Seq}, \mathsf{W})$ is called a *weak biWaldhausen category* if $(\mathcal{C}, \mathsf{Seq}, \mathsf{W})$ is both weak Waldhausen and weak coWaldhausen.

Example 8. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Define $\mathsf{Seq}_{\mathfrak{s}}$ to be the class of all $\mathfrak{s}$-conflations, and $\mathsf{W}_{\mathfrak{s}}$ to be the class of all isomorphisms in $\mathcal{C}$. Then $(\mathcal{C}, \mathsf{Seq}_{\mathfrak{s}}, \mathsf{W}_{\mathfrak{s}})$ is a weak biWaldhausen category.

We introduce some concepts for weak Waldhausen categories by analogy to the classical theory.

Definition 9. Let $(\mathcal{C}, \mathsf{Seq}, \mathsf{W})$ and $(\mathcal{C}', \mathsf{Seq}', \mathsf{W}')$ be weak Waldhausen categories.

(1) An additive functor $F : \mathcal{C} \to \mathcal{C}'$ is called an *exact functor* if it preserves distinguished sequences and weak equivalences, namely, $F(\mathsf{Seq}) \subseteq \mathsf{Seq}'$ and $F(\mathsf{W}) \subseteq \mathsf{W}'$ hold.

(2) Suppose $(\mathcal{C}, \mathsf{Seq}, \mathsf{V})$ is a weak Waldhausen category with $\mathsf{V} \subseteq \mathsf{W}$. Then the identity functor $\mathsf{id}_{\mathcal{C}} \colon (\mathcal{C}, \mathsf{Seq}, \mathsf{V}) \to (\mathcal{C}, \mathsf{Seq}, \mathsf{W})$ is exact. An object $C \in \mathcal{C}$ is W-*acyclic* if the zero map $0 \rightarrowtail C$ belongs to W. We denote by $\mathcal{N}^{\mathsf{W}}$ the full subcategory of all W-acyclic objects in $(\mathcal{C}, \mathsf{Seq}, \mathsf{V})$. In this case, the subcategory admits a natural weak Waldhausen structure $(\mathcal{N}^{\mathsf{W}}, \mathsf{Seq}', \mathsf{V}')$ which is a restriction of $(\mathcal{C}, \mathsf{Seq}, \mathsf{V})$.

We denote by wWald the category of weak Waldhausen categories and exact functors. Analogously to the case of extriangulated category, we introduce their exact sequence.

Definition 10. The natural sequence

$$(\mathcal{N}^{\mathsf{W}}, \mathsf{Seq}', \mathsf{V}') \xrightarrow{\mathsf{inc}} (\mathcal{C}, \mathsf{Seq}, \mathsf{V}) \xrightarrow{\mathsf{id}} (\mathcal{C}, \mathsf{Seq}, \mathsf{W}) \tag{3.3}$$

in Definition 9(2) is called a *localization sequence.* Moreover it is called an *exact sequence* in wWald if the functor $(\mathcal{C}, \mathsf{Seq}, \mathsf{V}) \xrightarrow{\mathsf{id}_{\mathcal{C}}} (\mathcal{C}, \mathsf{Seq}, \mathsf{W})$ is universal among exact functors $F\colon (\mathcal{C}, \mathsf{Seq}, \mathsf{V}) \to (\mathcal{D}, \mathsf{Seq}', \mathsf{W}')$ with $F|_{\mathcal{N}^{\mathsf{W}}} = 0$, where $(\mathcal{D}, \mathsf{Seq}', \mathsf{W}')$ is a weak Waldhausen category satisfying the *saturation* and *extension axioms* (see [21, p. 327]).

The Grothendieck group for weak Waldhausen categories is defined as follows.

Definition 11. Assume that $(\mathcal{C}, \mathsf{C}, \mathsf{W})$ is a weak Waldhausen category. The *Grothendieck group* $K_0(\mathcal{C}) := K_0(\mathcal{C}, \mathsf{C}, \mathsf{W})$ is defined to be the abelian group freely generated by the set of isomorphism classes $[C]$ of each object $C \in \mathcal{C}$, modulo to the relations:

- $[C] = [C']$ for each weak equivalence $C \xrightarrow{\sim} C'$; and
- $[B] = [A] + [C]$ for each distinguished sequence $A \rightarrowtail B \twoheadrightarrow C$.

To state our abstract localization theorem we define subclasses of $\mathsf{Mor}\,\mathcal{C}$:

- $\mathcal{L}^{\mathsf{ac}} := \mathsf{C} \cap \mathsf{W}$; $\mathcal{R}^{\mathsf{ac}} := \mathsf{F} \cap \mathsf{W}$; and
- $\mathcal{R}^{\mathsf{ac}}_{\mathsf{ret}} := \{g \in \mathsf{Mor}\,\mathcal{C} | g \text{ is a retraction and } \operatorname{Ker} g \in \mathcal{N}\}$.

The first result can be regarded as a version of Shclichting's theorem [19, Thm. 11].

Theorem 12 (Localization Theorem)**.** *Consider a localization sequence of weak Waldhausen categories as* (3.3)*. If we assume that*

(1) W *consists of finite compositions of morphisms from* $\mathcal{L}^{\mathsf{ac}} \cup \mathcal{R}^{\mathsf{ac}}_{\mathsf{ret}} \cup \mathsf{V}$*; or*
(2) W *consists of finite compositions of morphisms from* $\mathcal{L}^{\mathsf{ac}} \cup \mathcal{R}^{\mathsf{ac}} \cup \mathsf{V}$ *and* $\mathcal{C}$ *is a weak biWaldhausen,*

then it becomes an exact sequence in wWald *which induces a right exact sequence in* Ab *as follows.*

$$K_0(\mathcal{N}^{\mathsf{W}}, \mathsf{Seq}', \mathsf{V}') \xrightarrow{K_0(\mathsf{inc})} K_0(\mathcal{C}, \mathsf{Seq}, \mathsf{V}) \xrightarrow{K_0(\mathsf{id})} K_0(\mathcal{C}, \mathsf{Seq}, \mathsf{W}) \longrightarrow 0 \tag{3.4}$$

The second one is an extriangulated version of Quillen's resolution theorem at the level of K_0, see [15, Thm. 4.5] for more details.

Definition 13. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, let $\mathcal{X} \subseteq \mathcal{C}$ be a subcategory and fix an object $C \in \mathcal{C}$. A *finite* $\mathcal{X}$*-resolution (in* $\mathcal{C}$*)* of C is defined to be a complex

$$X_n \xrightarrow{f_{n-1}} \cdots \xrightarrow{g_2 f_1} X_1 \xrightarrow{g_1 f_0} X_0 \xrightarrow{g_0} C, \tag{3.5}$$

where $X_i \in \mathcal{X}$ for each $0 \leq i \leq n$, and $C_{i+1} \xrightarrow{f_i} X_i \xrightarrow{g_i} C_i$ is an $\mathfrak{s}$-conflation for each $0 \leq i \leq n-1$ with $(C_0, C_n) := (C, X_n)$. In this case, we say that the $\mathcal{X}$-resolution is of *length* n.

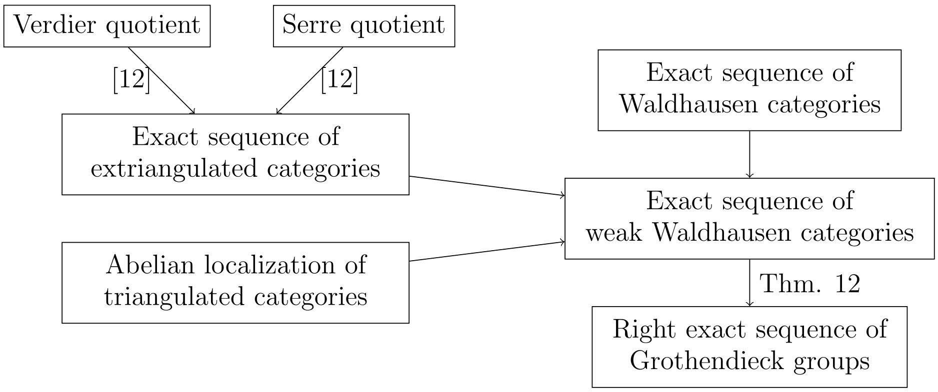
Theorem 14 (Resolution Theorem)**.** *Let* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *be an extriangulated category. Suppose* $\mathcal{X}$ *is an extension-closed subcategory of* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$*, such that* $\mathcal{X}$ *is closed under taking cocones of* $\mathfrak{s}$*-deflations in* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$*. If any object* $C \in \mathcal{C}$ *admits a finite* $\mathcal{X}$*-resolution, then we have an isomorphism*

$$\begin{array}{rcl} K_0(\mathcal{C}, \mathbb{E}, \mathfrak{s}) & \xrightarrow{\cong} & K_0(\mathcal{X}, \mathbb{E}|_{\mathcal{X}}, \mathfrak{s}|_{\mathcal{X}}) \\ [C] & \longmapsto & \displaystyle\sum_{i=0}^{n} (-1)^i [X_i] \end{array}$$

where we consider an $\mathcal{X}$-resolution (3.5) *of $C \in \mathcal{C}$.*

4. Applications

Lastly we demonstrate some usages of our localization and resolution theorem. As expected, an exact sequence in ET induces an exact sequence in wWald. In such a case, we may apply the localization theorem to get a right exact sequence of the Grothendieck groups in Ab, recovering Enomoto-Saito's extriangulated localization theorem [6, Cor. 4.32]. A benefit of weak Waldhausen structures sits in the fact that such a construction still holds for the abelian localization in the sense of Theorem 5. Exact sequences appearing in this article are related to each other as summarized below.



Thus, although the "right exact" sequence (2.2) does not exsist in ET, it induces a natural exact sequences $(\mathcal{N}^{\mathsf{W}}, \mathsf{Seq}', \mathsf{V}') \xrightarrow{\mathsf{inc}} (\mathcal{C}, \mathsf{Seq}, \mathsf{V}) \xrightarrow{\mathsf{id}} (\mathcal{C}, \mathsf{Seq}, \mathsf{W})$ in wWald to which Theorem 12 can apply. Thus, like the case of Enomoto-Saito's theorem, it also induces a right exact sequence in Ab as below.

$$K_0(\mathcal{N}, \mathbb{E}|_{\mathcal{N}}, \mathfrak{s}|_{\mathcal{N}}) \xrightarrow{K_0(\mathsf{inc})} K_0(\mathcal{C}, \mathbb{E}^R_{\mathcal{N}}, \mathfrak{s}^R_{\mathcal{N}}) \xrightarrow{K_0(\mathsf{id})} K_0(\mathcal{C}, \mathsf{Seq}, \mathsf{W}) \longrightarrow 0$$

Furthermore, thanks to the assumption $\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$ in Theorem 5, (the dual of) the resolution theorem applies to the inclusion $\mathcal{N} \subseteq (\mathcal{C}, \mathbb{E}^R_{\mathcal{N}}, \mathfrak{s}^R_{\mathcal{N}})$. It shows the leftmost arrow is an isomorphism $K_0(\mathcal{N}, \mathbb{E}|_{\mathcal{N}}, \mathfrak{s}|_{\mathcal{N}}) \xrightarrow{\cong} K_0(\mathcal{C}, \mathbb{E}^R_{\mathcal{N}}, \mathfrak{s}^R_{\mathcal{N}})$. This isomorphism has been already appeared in the literature, which we now describe.

Example 15. (cf. Example 6) Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a triangulated category and $\mathcal{X} \subseteq \mathcal{C}$ a 2-cluster tilting subcategory. Put $\mathcal{N} := \mathcal{X}[1] = \mathcal{X}^{\perp_0}$. Then the aforementioned isomorphism can be described as follows,

$$\begin{array}{rcl} K_0(\mathcal{C}, \mathbb{E}^R_{\mathcal{N}}, \mathfrak{s}^R_{\mathcal{N}}) & \xrightarrow{\cong} & K_0^{\mathsf{sp}}(\mathcal{X}) \\ [C] & \longmapsto & [X_0] - [X_1] \end{array}$$

where we consider a triangle $X_1 \to X_0 \to C \to X_1[1]$ comming from the defining cotorsion pair $(\mathcal{X}, \mathcal{X})$. This isomorphism is known as the *index isomorphism* [17]. In the case of $\mathcal{X} = \mathcal{X}[1]$, by a closer look at this isomorphism, Fedele interpreted the Grothendieck group $K_0(\mathcal{C})$ of the triangulated category as that of the 4-angulated category $\mathcal{X}$ [7, Thm. C].

Due to the very generality of our abstract theorems, we expand their results to wider setup containing the n-cluster tilting subcategory case.

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Center for Educational Research of Science and Mathematics
Nara University of Education
Takabatake-cho, Nara, 630-8528, JAPAN
Email address: `ogawa.yasuaki.gh@cc.nara-edu.ac.jp`

EMBEDDINGS INTO MODULES OF FINITE PROJECTIVE DIMENSIONS AND THE n-TORSIONFREENESS OF SYZYGIES

YUYA OTAKE

Abstract. Let R be a commutative noetherian ring. In this article, we find out close relationships between the module M being embedded in a module of projective dimension at most n and the $(n+1)$-torsionfreeness of the nth syzygy of M. As an application, we consider the n-torsionfreeness of syzygies of the residue field k over a local ring R.

Key Words: n-torsionfree module, n-syzygy module, projective dimension, Gorenstein ring.

2000 *Mathematics Subject Classification*: 13D02, 13D07.

1. Introduction

Throughout this article, let R be a commutative noetherian ring. We assume that all modules are finitely generated ones. It is a natural and classical question to ask when a given R-module can be embedded in an R-module of finite projective dimension. Auslander and Buchweitz [2] proved that over a Gorenstein local ring any module admits a *finite projective hull*, which is a dual notion of a *Cohen–Macaulay approximation*.

Theorem 1 (Auslander–Buchweitz). *Let R be a Gorenstein local ring and M an R-module. Then there exists an exact sequence $0 \to M \to Y^M \to X^M \to 0$ of R-modules such that Y^M has finite projective dimension and X^M is maximal Cohen-Macaulay.*

In particular, every module over a Gorenstein local ring can be embedded in a module of finite projective dimension. Conversely, Foxby [5] proved that if R is a Cohen–Macaulay local ring and every R-module can be embedded in an R-module of finite projective dimension, then R is Gorenstein. Takahashi, Yassemi and Yoshino [13] succeeded in removing from Foxby's theorem the assumption of Cohen–Macaulayness of the ring R.

Theorem 2 (Foxby, Takahashi–Yassemi–Yoshino). *Let R be a local ring of depth t. Let k be the residue field of R. Then the following are equivalent.*

(1) *The ring R is Gorenstein.*
(2) *Any R-module can be embedded in an R-module of finite projective dimension.*
(3) *The module $\operatorname{Tr}\Omega^t k$ can be embedded in an R-module of finite projective dimension.*

Here, we denote by $\operatorname{Tr}(-)$ and $\Omega^n(-)$ the (Auslander–Bridger) transpose and n-th syzygy, respectively. In the present article, for a fixed integer n, we consider embedding a given module in a module of projective dimension at most n. Our answer to this question is Theorem 3, which says that the question is closely related to the $(n+1)$-torsionfreeness of nth syzygies. The notion of n-torsionfree modules was introduced by Auslander and

The detailed version [11] of this article has been submitted for publication elsewhere.

Bridger [1] as a generalization of the notion of torsionfree modules over integral domains: An R-module M is called *n-torsionfree* if $\mathrm{Ext}_R^i(\mathrm{Tr}\, M, R) = 0$ for all $1 \le i \le n$. Various studies on the n-torsionfreeness have been done so far; see [1, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13]. As an application of Theorem 3, we can recover Theorems 1 and 2.

Next, let us consider the case where R is local with residue field k, and has depth t. Recently, Dey and Takahashi [3] studied the torsionfreeness of syzygies of k. They especially proved in [3, Theorems 4.1(2) and 4.5(1)] that $\Omega^t k$ is $(t+1)$-torsionfree, and it is a $(t+2)$nd syzygy if and only if the local ring R has type one. Motivated by their results, as another application of our main theorem, we consider the n-torsionfreeness of syzygies of the residue field k.

2. Modules embedded in modules of finite projective dimension

The following theorem is the first main result of this article. The following theorem gives an answer to the question of when a given R-module can be embedded in an R-module of projective dimension at most n, under the assumption that the given module is locally of finite Gorenstein dimension. Let M be an R-module. We denote by $\mathrm{Gdim}_R\, M$ the *Gorenstein dimension* of M; see [1] for details.

Theorem 3. *Let M be an R-module and n a nonnegative integer. Consider the following conditions.*

(1) *The module $\Omega^n M$ is $(n+1)$-torsionfree.*
(2) *There exists an exact sequence $0 \to M \to Y \to X \to 0$ of R-modules such that Y has projective dimension at most n and $\mathrm{Ext}_R^i(X, R) = 0$ for all $1 \le i \le n+1$.*
(3) *The module M can be embedded in an R-module of projective dimension at most n.*

Then the implications (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) *hold. If $\mathrm{Gdim}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$ for all prime ideals $\mathfrak{p}$ of R with $\mathrm{depth}\, R_\mathfrak{p} < n$, then all the three conditions are equivalent.*

Let us consider an application of the above theorem. We can deduce Theorem 2 due to Foxby [5] and Takahashi, Yassemi and Yoshino [13] directly from Theorem 3.

Proof of Theorem 2. Assume that R is Gorenstein. Then for any R-module M the tth syzygy $\Omega^t M$ is maximal Cohen–Macaulay, in particular, $(t+1)$-torsionfree. The implication (1) $\Rightarrow$ (2) follows from Theorem 3. The implication (2) $\Rightarrow$ (3) is clear. Suppose that $\mathrm{Tr}\,\Omega^t k$ is a submodule of an R-module of finite projective dimension. It follows from Theorem 3 that $\Omega^t\, \mathrm{Tr}\,\Omega^t k$ is $(t+1)$-torsionfree. In particular, $\mathrm{Ext}^1(\Omega^t\,\mathrm{Tr}\,\Omega^t\,\mathrm{Tr}\,\Omega^t k, R) = \mathrm{Ext}^{t+1}(\mathrm{Tr}\,\Omega^t\,\mathrm{Tr}\,\Omega^t k, R) = 0$. Since $\mathrm{Ext}^1(\Omega^t k, R)$ is a direct summand of $\mathrm{Ext}^1(\Omega^t\,\mathrm{Tr}\,\Omega^t\,\mathrm{Tr}\,\Omega^t k, R)$, we have $\mathrm{Ext}^{t+1}(k, R) = \mathrm{Ext}^1(\Omega^t k, R) = 0$ and the implication (3) $\Rightarrow$ (1) holds. $\square$

Grades of Ext modules are one of the main subjects of the theory of Auslander and Bridger; see [1, Chapters 2 and 4]. Recall that the *grade* of an R-module M is defined to be the infimum of integers i such that $\mathrm{Ext}_R^i(M, R) \ne 0$, and denoted by $\mathrm{grade}_R\, M$. We state the relationship between Theorem 3 and the grade condition given by Auslander and Bridger.

Corollary 4. *Let $n \geq 0$ be an integer and M an R-module. If $\Omega^n M$ is $(n+1)$-torsionfree, then* $\operatorname{grade}_R \operatorname{Ext}^i_R(M,R) \geq i$ *for all integers* $1 \leq i \leq n$.

3. The n-torsionfreeness of syzygies of the residue field of local rings

Let M and N be R-modules. By $M \approx N$ we mean that there are projective modules P and Q such that $M \oplus P \cong N \oplus Q$.

The following corollary is necessary to prove Theorem 7, which is one of the main theorems in this article. For a local ring $(R, \mathfrak{m}, k)$ we denote by $\mathrm{r}(R)$ the *type* of R, that is, $\mathrm{r}(R)$ is the dimension of the vector space $\operatorname{Ext}_R^{\operatorname{depth} R}(k,R)$ over the residue field k of R.

Corollary 5. *Suppose that R is local and with depth t. Let k be the residue field of R. Then the following hold.*

(1) [3, Theorem 4.1(2)] *The module $\Omega^t k$ is $(t+1)$-torsionfree.*
(2) *There exists an exact sequence $0 \to k \to Y^k \to X^k \to 0$ such that Y^k has projective dimension t and $X^k \approx \operatorname{Tr}\Omega^{t+1}\operatorname{Tr}\Omega^t k$. Moreover, if $t > 0$, then $Y^k \approx \operatorname{Tr}\Omega^{t-1}(k^{\oplus \mathrm{r}(R)})$.*

Proof. We note that the residue field k can be embedded in a module of finite projective dimension. Hence, by Theorem 3, the module $\Omega^t k$ is $(t+1)$-torsionfree, and there exists an exact sequence $0 \to k \to Y^k \to X^k \to 0$ such that Y^k has projective dimension at most t and $X^k \approx \operatorname{Tr}\Omega^{t+1}\operatorname{Tr}\Omega^t k$. We assume that t is positive. Then since $\operatorname{Ext}^i(k,R) = 0 = \operatorname{Ext}^i(X^k,R)$ for all $1 \leq i \leq t-1$, so does Y^k. Also, we have $\operatorname{Ext}^t(Y^k,R) \cong \operatorname{Ext}^t(k,R) \cong k^{\oplus \mathrm{r}(R)}$. By the following lemma, we obtain that $Y^k \approx \operatorname{Tr}\Omega^{t-1}\operatorname{Ext}^t(Y^k,R) \cong \operatorname{Tr}\Omega^{t-1}(k^{\oplus \mathrm{r}(R)})$. □

Lemma 6. [9, Theorem 2.7] *Let Y be an R-module and $s > 0$ an integer. If $\operatorname{Ext}^i_R(Y,R) = 0$ for all $1 \leq i < s$ and Y has projective dimension at most s, then $Y \approx \operatorname{Tr}\Omega^{s-1}\operatorname{Ext}^s_R(Y,R)$.*

Theorem 7. *Let $(R, \mathfrak{m}, k)$ be local and with depth t. The following hold.*

(1) *The local ring R has type one if and only if the module $\Omega^t k$ is $(t+2)$-torsionfree.*
(2) *The local ring R is Gorenstein if and only if the module $\Omega^t k$ is $(t+3)$-torsionfree, if and only if one has $\operatorname{Ext}^i_R(\operatorname{Tr}\Omega^t k, R) = 0$ for some integer $i \geq t+3$*

Proof. We only need to prove the case where $t > 0$. In this case, by Corollary 5, there exists an exact sequence $0 \to \operatorname{Tr} X^k \to \operatorname{Tr} Y^k \to \operatorname{Tr} k \to 0$, and we have $\operatorname{Tr} X^k \approx \Omega^{t+1}\operatorname{Tr}\Omega^t k$ and $\operatorname{Tr} Y^k \approx \Omega^{t-1}(k^{\oplus \mathrm{r}(R)})$. So we obtain the long exact sequence

$$0 \to \operatorname{Ext}^1(\operatorname{Tr} k, R) \to \operatorname{Ext}^1(\operatorname{Tr} Y^k, R) \to \operatorname{Ext}^1(\operatorname{Tr} X^k, R) \to \operatorname{Ext}^2(\operatorname{Tr} k, R) \to \cdots.$$

Since the module $\operatorname{Tr} k$ has projective dimension one, the assertions follow. □

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Graduate School of Mathematics, Nagoya University
Furocho, Chikusaku, Nagoya 464-8602, Japan
Email address: `m21012v@math.nagoya-u.ac.jp`

THE FIRST EULER CHARACTERISTIC AND THE DEPTH OF ASSOCIATED GRADED RINGS

KAZUHO OZEKI

Abstract. The homological property of the associated graded ring of an ideal is an important problem in commutative algebra. In this talk, we explore the structure of the associated graded ring of $\mathfrak{m}$-primary ideals in the case where the first Euler characteristic attains almost minimal value in a Cohen-Macaulay local ring.

Key Words: commutative ring, Cohen-Macaulay local ring, associated graded ring, first Euler characteristic, Hilbert function, Hilbert coefficient, stretched ideal.

2020 *Mathematics Subject Classification*: Primary 13H10; Secondary 13D40.

1. Introduction

Throughout this report, let A be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. For simplicity, we may assume the residue class field $A/\mathfrak{m}$ is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and let

$$R = \mathrm{R}(I) := A[It] \quad \subseteq A[t] \text{ and } R' = \mathrm{R}'(I) := A[It, t^{-1}] \quad \subseteq A[t, t^{-1}]$$

denote, respectively, the Rees algebra and the extended Rees algebra of I. Let

$$G = \mathrm{G}(I) := R'/t^{-1}R' \cong \bigoplus_{n \geq 0} I^n/I^{n+1}$$

denotes the associated graded ring of I. Let $M = \mathfrak{m}G + G_+$ be the graded maximal ideal in G. Let $\ell_A(N)$ denote, for an A-module N, the length of N.

Let $Q = (a_1, a_2, \cdots, a_d) \subseteq I$ be a parameter ideal in A which forms a reduction of I. Then, we set

$$\chi_1(a_1t, a_2t, \ldots, a_dt; G) := \ell(G/(a_1t, a_2t, \ldots, a_dt)G) - \mathrm{e}(a_1t, a_2t, \ldots, a_dt; G_M)$$

and call it the *first Euler characteristic* of G relative to $a_1t, a_2t, \ldots, a_dt$ (c.f. [1, 2, 11]), where $\mathrm{e}(a_1t, a_2t, \ldots, a_dt; G_M)$ denotes the multiplicity of G_M with respect to $a_1t, a_2t, \ldots, a_dt$.

It is well-known that $\chi_1(a_1t, a_2t, \ldots, a_dt; G) \geq 0$ holds true, and the equality

$$\chi_1(a_1t, a_2t, \ldots, a_dt : G) = 0$$

holds true if and only if the associated graded ring G is Cohen-Macaulay. The aim of this talk is to explore the structure of the associated graded ring G with $\chi_1(a_1t, a_2t, \ldots, a_dt; G) = 1$ and, in particular, we prove that $\operatorname{depth} G = d - 1$.

The detailed version of this paper will be submitted for publication elsewhere.

In this report we will also study the Hilbert series and coefficients of $\mathfrak{m}$-primary ideals. We set the power series

$$HS_I(z) = \sum_{n=0}^{\infty} \ell_A(I^n/I^{n+1})z^n$$

and call it the Hilbert series of I. It is also well known that this series is rational and that there exists a polynomial $h_I(z)$ with integer coefficients such that $h_I(1) \neq 0$ and

$$HS_I(z) = \frac{h_I(z)}{(1-z)^d}.$$

As is well known, for a given $\mathfrak{m}$-primary ideal I, there exist integers $\{\mathrm{e}_k(I)\}_{0 \le k \le d}$ such that the equality

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_d(I)$$

holds true for all integers $n \gg 0$. For each $0 \le k \le d$, $\mathrm{e}_k(I)$ is called the k-th *Hilbert coefficient* of I.

The main result of this report is the following.

Theorem 1. *The following conditions are equivalent to each other.*

(1) $\chi_1(a_1t, a_2t, \cdots, a_dt; G) = 1,$
(2) $\mathrm{e}_0(I) = \ell_A(A/I) + \sum_{n \ge 1} \ell_A(I^n/QI^{n-1} + I^{n+1}) - 1,$
(3) *the Hilbert series* $HS_I(z)$ *of* I *is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \sum_{n=1}^{r_I} \ell_A(I^n/QI^{n-1} + I^{n+1})z^n - z^s}{(1-z)^d}$$

for some $s > 0$.

When this is the case we have the following.

(*i*) $s = \min\{n \ge 1 \mid QI^{n-1} \cap I^{n+1} \neq QI^n\},$
(*ii*) $\mathrm{e}_k(I) = \sum_{n=k}^{r_I} \binom{n}{k} \ell_A(I^n/QI^{n-1} + I^{n+1}) - \binom{s}{k}$ *for* $1 \le k \le d,$
(*iii*) $\mathrm{a}_{d-1}(G) := \sup\{n \in \mathbb{Z} \mid [\mathrm{H}_M^{d-1}(G)]_n \neq (0)\} = s - d,$ *and* $\ell_A([\mathrm{H}_M^{d-1}(G)]_{s-d}) = 1,$
(*iv*) $\operatorname{depth} G = d - 1.$

We can get the following result as a corollary of Theorem 1.

Corollary 2. *Suppose that* $\chi_1(a_1t, a_2t, \ldots, a_dt; G) \le 1,$ *then* $\operatorname{depth} G \ge d - 1.$

2. The structure of Sally modules

In this report we need the notion of Sally modules which was introduced by W. V. Vasconcelos [12]. The purpose of this section is to summarize some results and techniques on the Sally modules which we need throughout this report. Remark that in this section $\mathfrak{m}$-primary ideals I are not necessarily stretched.

Let $T = \mathrm{R}(Q) = A[Qt] \subseteq A[t]$ denotes the Rees algebra of Q. Following Vasconcelos [12], we consider

$$S = \mathrm{S}_Q(I) = IR/IT \cong \bigoplus_{n \geq 1} I^{n+1}/Q^n I$$

the Sally module of I with respect to Q.

We give one remark about Sally modules. See [5, 12] for further information.

Remark 3 ([5, 12]). We notice that S is a finitely generated graded T-module and $\mathfrak{m}^n S = (0)$ for all $n \gg 0$. We have $\mathrm{Ass}_T S \subseteq \{\mathfrak{m}T\}$ so that $\dim_T S = d$ if $S \neq (0)$.

From now on, let us introduce some techniques, being inspired by [3, 4], which plays a crucial role throughout this report. See [7, Section 3] (also [6, Section 2] for the case where $I = \mathfrak{m}$) for the detailed proofs.

We denote by $E(m)$, for a graded module E and each $m \in \mathbb{Z}$, the graded module whose grading is given by $[E(m)]_n = E_{m+n}$ for all $n \in \mathbb{Z}$.

We have an exact sequence

$$0 \to K^{(-1)} \to F \overset{\varphi_{-1}}{\to} G \to R/IR + T \to 0 \qquad (\dagger_{-1})$$

of graded T-modules induced by tensoring the canonical exact sequence

$$0 \to T \overset{i}{\hookrightarrow} R \to R/T \to 0$$

of graded T-modules with A/I where $\varphi_{-1} = A/I \otimes i$, $K^{(-1)} = \mathrm{Ker}\, \varphi_{-1}$, and $F = T/IT \cong (A/I)[X_1, X_2, \cdots, X_d]$ is a polynomial ring with d indeterminates over the residue class ring A/I.

Lemma 4. ([7]) *There exists an exact sequence*

$$0 \to K^{(0)}(-1) \to ([R/IR + T]_1 \otimes F)(-1) \overset{\varphi_0}{\to} R/IR + T \to S/IS(-1) \to 0 \quad (\dagger_0)$$

of graded T-modules where $K^{(0)} = \mathrm{Ker}\, \varphi_0$.

Notice that $\mathrm{Ass}_T K^{(m)} \subseteq \{\mathfrak{m}T\}$ for all $m = -1, 0$, because $F \cong (A/I)[X_1, X_2, \cdots, X_d]$ is a polynomial ring over the residue ring A/I and $[R/IR+T]_1 \otimes F$ is a maximal Cohen-Macaulay module over F.

We then have the following proposition by the exact sequences $(\dagger_{-1})$ and $(\dagger_0)$.

Proposition 5. ([7, Lemma 3.3]) *We have*

$$\begin{aligned} \ell_A(I^n/I^{n+1}) &= \ell_A(A/[I^2+Q])\binom{n+d-1}{d-1} - \ell_A(I/[I^2+Q])\binom{n+d-2}{d-2} \\ &+ \ell_A([S/IS]_{n-1}) - \ell_A(K^{(-1)}_n) - \ell_A(K^{(0)}_{n-1}) \end{aligned}$$

for all $n \geq 0$.

We also need the notion of *filtration of the Sally module* which was introduced by M. Vaz Pinto [13] as follows.

Definition 6. ([13]) We set, for each $m \geq 1$,

$$S^{(m)} = I^m t^{m-1} R / I^m t^{m-1} T (\cong I^m R / I^m T(-m+1)).$$

We notice that $S^{(1)} = S$, and $S^{(m)}$ are finitely generated graded T-modules for all $m \geq 1$, since R is a module-finite extension of the graded ring T.

The following lemma follows by the definition of the graded module $S^{(m)}$.

Lemma 7. *Let $m \geq 1$ be an integer. Then the following assertions hold true.*

(1) $\mathfrak{m}^n S^{(m)} = (0)$ *for integers $n \gg 0$; hence* $\dim_T S^{(m)} \leq d$.

(2) *The homogeneous components $\{S_n^{(m)}\}_{n\in\mathbb{Z}}$ of the graded T-module $S^{(m)}$ are given by*

$$S_n^{(m)} \cong \begin{cases} (0) & \text{if } n \leq m-1, \\ I^{n+1}/Q^{n-m+1}I^m & \text{if } n \geq m. \end{cases}$$

Let $L^{(m)} = TS_m^{(m)}$ be a graded T-submodule of $S^{(m)}$ generated by $S_m^{(m)}$ and

$$\begin{aligned} D^{(m)} &= (I^{m+1}/QI^m) \otimes (A/\operatorname{Ann}_A(I^{m+1}/QI^m))[X_1, X_2, \cdots, X_d] \\ &\cong (I^{m+1}/QI^m)[X_1, X_2, \cdots, X_d] \end{aligned}$$

for $m \geq 1$ (c.f. [13, Section 2]).

We then have the following lemma.

Lemma 8. ([13, Section 2]) *The following assertions hold true for $m \geq 1$.*

(1) $S^{(m)}/L^{(m)} \cong S^{(m+1)}$ *so that the sequence*

$$0 \to L^{(m)} \to S^{(m)} \to S^{(m+1)} \to 0$$

is exact as graded T-modules.

(2) *There is a surjective homomorphism $\theta_m : D^{(m)}(-m) \to L^{(m)}$ graded T-modules.*

For each $m \geq 1$, tensoring the exact sequence

$$0 \to L^{(m)} \to S^{(m)} \to S^{(m+1)} \to 0$$

and the surjective homomorphism $\theta_m : D^{(m)}(-m) \to L^{(m)}$ of graded T-modules with A/I, we get the exact sequence

$$0 \to K^{(m)}(-m) \to D^{(m)}/ID^{(m)}(-m) \stackrel{\varphi_m}{\to} S^{(m)}/IS^{(m)} \to S^{(m+1)}/IS^{(m+1)} \to 0 \qquad (\dagger_m)$$

of graded F-modules where $K^{(m)} = \operatorname{Ker}\varphi_m$.

Notice here that, for all $m \geq 1$, we have $\operatorname{Ass}_T K^{(m)} \subseteq \{\mathfrak{m}T\}$ because $D^{(m)}/ID^{(m)} \cong (I^{m+1}/QI^m + I^{m+2})[X_1, X_2, \cdots, X_d]$ is a maximal Cohen-Macaulay module over F.

We then have the following result by Proposition 5 and exact sequences $(\dagger_m)$ for $m \geq 1$.

Proposition 9. *The following assertions hold true:*

(1) *We have*

$$\begin{aligned} \ell_A(I^n/I^{n+1}) &= \{\ell_A(A/I^2+Q) + \sum_{m=1}^{r_I-1} \ell_A(I^{m+1}/QI^m + I^{m+2})\}\binom{n+d-1}{d-1} \\ &+ \sum_{k=1}^{r_I}(-1)^k \left\{\sum_{m=k-1}^{r_I-1} \binom{m+1}{k} \ell_A(I^{m+1}/QI^m + I^{m+2})\right\}\binom{n+d-k-1}{d-k-1} \\ &- \sum_{m=-1}^{r_I-1} \ell_A(K_{n-m-1}^{(m)}) \end{aligned}$$

for all $n \geq \max\{0, r_I - d + 1\}$.

(2) $\mathrm{e}_0(I) = \ell_A(A/I^2+Q)+\sum_{m=1}^{r_I-1} \ell_A(I^{m+1}/QI^m+I^{m+2})-\sum_{m=-1}^{r_I-1} \ell_{T_\mathcal{P}}(K_\mathcal{P}^{(m)})$ *where* $\mathcal{P} = \mathfrak{m}T$.

3. Proof of Main Theorem

In this section, let us introduce a proof of Theorem 1.

Let us begin with the following remark, where $\mathrm{e}(a_1t, a_2t, \cdots, a_dt; G)$ denotes the multiplicity of G with respect to $a_1t, a_2t, \cdots, a_dt$, and

$$\chi_1(a_1t, a_2t, \cdots, a_dt; G) = \ell_A(G/(a_1t, a_2t, \cdots, a_dt)G) - \mathrm{e}(a_1t, a_2t, \cdots, a_dt; G) \geq 0$$

is called the first Euler characteristic of G with respect to $a_1t, a_2t, \cdots, a_dt$.

Remark 10. We have, by Proposition 9,

$$\chi_1(a_1t, a_2t, \cdots, a_dt; G) = \sum_{m\geq -1} \ell_{T_\mathcal{P}}(K_\mathcal{P}^{(m)})$$

because $\mathrm{e}(a_1t, a_2t, \cdots, a_dt; G) = \mathrm{e}_0(I)$ and $[G/(a_1t, a_2t, \cdots, a_dt)G]_n \cong I^n/QI^{n-1} + I^{n+1}$ for all $n \geq 1$.

The following corollary seems well known by the basic properties of Cohen-Macaulay rings.

Corollary 11. *The following conditions are equivalent to each other;*

(1) $\chi_1(a_1t, a_2t, \cdots, a_dt; G) = 0$,

(2) $\mathrm{e}_0(I) = \ell_A(A/I) + \sum_{n\geq 1} \ell_A(I^n/QI^{n-1} + I^{n+1})$,

(3) *the Hilbert series* $HS_I(z)$ *of* I *is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \sum_{n=1}^{r_I} \ell_A(I^n/QI^{n-1} + I^{n+1})z^n}{(1-z)^d},$$

(4) *G is Cohen-Macaulay.*

Let $B = T/\mathfrak{m}T \cong (A/\mathfrak{m})[X_1, X_2, \cdots, X_d]$ which is a polynomial ring with d indeterminates over the field $A/\mathfrak{m}$.

The following proposition plays an important role for our proof of Theorem 1.

Proposition 12. *The following conditions are equivalent to each other, where* $s = \min\{n \geq 1 \mid QI^{n-1} \cap I^{n+1} \neq QI^n\}$.

(1) $\chi_1(a_1t, a_2t, \cdots, a_dt; G) = 1$,

(2) $K^{(m)} \cong B(-u)$ *as graded T-modules for some* $-1 \leq m \leq s-2$ *and* $1 \leq u \leq s$, *and* $K^{(n)} = (0)$ *for all* $n \neq m$.

When this is the case we have the following.

(*i*) $\mathrm{e}_k(I) = \sum_{n=k}^{r_I} \binom{n}{k} \ell_A(I^n/QI^{n-1} + I^{n+1}) - \binom{s}{k}$ *for* $1 \leq k \leq d$,

(*ii*) *the Hilbert series* $HS_I(z)$ *of* I *is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \sum_{n=1}^{r_I} \ell_A(I^n/QI^{n-1} + I^{n+1})z^n - z^s}{(1-z)^d},$$

(*iii*) $\mathrm{a}_{d-1}(G) = s - d$, *and* $\ell_A([\mathrm{H}^{d-1}_M(G)]_{s-d}) = 1$,
(*iv*) $\operatorname{depth} G = d - 1$.

4. Applications for stretched ideals

In this section let us introduce some applications of Theorem 1 for stretched ideals.

The notion of *stretched* Cohen-Macaulay local rings was introduced by J. Sally to extend the rings of minimal or almost minimal multiplicity.

We say that the ring A is *stretched* if $\ell_A(\mathfrak{m}^2 + Q/\mathfrak{m}^3 + Q) = 1$ holds true, i.e. the ideal $(\mathfrak{m}/Q)^2$ is principal, for some parameter ideal Q in A which forms a reduction of $\mathfrak{m}$ ([10]). We note here that this condition depends on the choice of a reduction Q (see [9, Example 2.3]).

In 2001, Rossi and Valla [9] gave the notion of stretched $\mathfrak{m}$-primary ideals. We say that the $\mathfrak{m}$-primary ideal I is stretched if the following two conditions

(1) $Q \cap I^2 = QI$ and
(2) $\ell_A(I^2 + Q/I^3 + Q) = 1$

hold true for some parameter ideal Q in A which forms a reduction of I. We notice that the first condition is naturally satisfied if $I = \mathfrak{m}$ so that this extends the classical definition of stretched local rings given in [10].

The following lemma which was essentially given by Rossi and Valla.

Lemma 13. ([9, Lemma 2.4]) *Suppose that I is stretched. Then we have the following.*

(1) *There exists $x, y \in I \backslash Q$ such that $I^{n+1} = QI^n + (x^n y)$ holds true for all $n \geq 1$.*
(2) *The map*

$$I^{n+1}/QI^n \overset{\widehat{x}}{\to} I^{n+2}/QI^{n+1}$$

is surjective for all $n \geq 1$. Therefore $\alpha_n \geq \alpha_{n+1}$ for all $n \geq 1$.
(3) $\mathfrak{m}x^n y \subseteq QI^n + I^{n+2}$ *and hence* $\ell_A(I^n/QI^{n-1} + I^{n+1}) \leq 1$ *for all* $n \geq 1$.

We set

$$\Lambda := \Lambda_I = \Lambda_Q(I) = \{n \geq 1 \mid QI^{n-1} \cap I^{n+1}/QI^n \neq (0)\}$$

and $|\Lambda|$ denotes the cardinality of the set Λ. Let

$$n_I = n_Q(I) = \min\{n \geq 0 \mid I^{n+1} \subseteq Q\}.$$

It is easy to see that the inequality $r_I \geq n_I$ holds true.

Then the following proposition is satisfied.

Proposition 14. *Suppose that I is stretched. Then $\chi_1(a_1t, a_2t, \cdots, a_dt; G) = |\Lambda| = r_I - n_I$.*

The following result was essentially given by Sally and Rossi-Valla.

Corollary 15. ([9, 10]) *Suppose that I is stretched, then the following conditions are equivalent to each other.*

(1) $r_I = n_I$,
(2) $\Lambda = \emptyset$,

(3) *the Hilbert series $HS_I(z)$ of I is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \{\mathrm{e}_0(I) - \ell_A(A/I) - n_I + 1\}z + \sum_{2 \le n \le r_I} z^n}{(1-z)^d}$$

(4) *G is Cohen-Macaulay.*

We can get the following corollary for the case where the reduction number r_I attains almost minimal value $n_I + 1$.

Corollary 16. ([8, Theorem 1.1]) *Suppose that I is stretched, then the following conditions are equivalent to each other.*

(1) $r_I = n_I + 1$,
(2) $|\Lambda| = 1$,
(3) *the Hilbert series $HS_I(z)$ of I is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \{\mathrm{e}_0(I) - \ell_A(A/I) - n_I + 1\}z + \sum_{2 \le n \le r_I, n \neq s} z^n}{(1-z)^d}$$

for some $s > 0$.

When this is the case, the following conditions also hold true.

(*i*) $\Lambda = \{s\}$,
(*ii*) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \binom{n_I+1}{2} - s + 1$,
(*iii*) $\mathrm{e}_k(I) = \binom{n_I+2}{k+1} - \binom{s}{k}$ *for all* $2 \le k \le d$,
(*iv*) $\mathrm{a}_{d-1}(G) = s - d$ *and* $\ell_A([\mathrm{H}_M^{d-1}(G)]_{s-d}) = 1$, *and*
(*v*) $\operatorname{depth} G = d - 1$.

Corollary 17. ([8, Corollary 1.2]) *Suppose that I is stretched and assume that $r_I \le n_I + 1$. Then* $\operatorname{depth} G \ge d - 1$.

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Department of Mathematics
College of Humanities and Sciences
Nihon University
3-25-40 Sakurajyousui Setagaya-ku Tokyo 156-8550 JAPAN
Email address: ozeki.kazuho@nihon-u.ac.jp

CLASSIFYING SEVERAL SUBCATEGORIES OF THE CATEGORY OF MAXIMAL COHEN-MACAULAY MODULES

SHUNYA SAITO

Abstract. In this summary, we introduce the classification of several subcategories of a torsion-free class of the module category over a commutative noetherian ring. More precisely, we classify Serre subcategories and torsion(-free) classes of a torsion-free class in the sense of exact categories. This result extends Gabriel's classification of Serre subcategories of the module category to torsionfree classes. As an immediate consequence, we classify the Serre subcategories and the torsion(-free) classes of the category of maximal Cohen-Macaulay modules over a one-dimensional Cohen-Macaulay ring.

Key Words: torsion-free classes; exact categories; Cohen-Macaulay modules.

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部分圏の分類問題は，傾理論やスキームの圏論的復元問題など様々な分野との関連から長い間研究されてきた．特に可換ネーター環の場合には，Gabriel による Serre 部分圏の分類 [2] や，高橋によるトーション・フリー類の分類 [7] を筆頭に，これまで様々な部分圏が分類されてきた．本稿では [4] と [5] に基づいて，これらの部分圏の分類の完全圏への拡張を紹介する．

第 1 節では，アーベル圏の様々な部分圏とその分類に関する先行研究を紹介する．第 2 節では，前節で述べたアーベル圏における部分圏の分類の，トーション・フリー類や極大 Cohen-Macaulay 加群の圏 $\mathsf{cm}\, R$ といった完全圏への拡張を紹介する．

本稿において，任意の部分圏は充満部分圏であり同型で閉じているとする．またネーター環 Λ に対して $\mathsf{mod}\,\Lambda$ で有限生成 (右)Λ 加群の圏を表す．可換環 R に対して，$\operatorname{Spec} R$ で R の素イデアルの集合を表す．また R 上の加群 M に対して，$\operatorname{Supp} M$ で M の台 (support) を表し，$\operatorname{Ass} M$ で M の素因子の集合を表す．

1. アーベル圏の様々な部分圏

この節では，本稿の主な考察対象であるアーベル圏の様々な部分圏とその分類を紹介する．アーベル圏では，短完全列による拡大や，射の核，余核，像を取るなどの操作がある．まずはこれらの操作で閉じるような部分圏を導入する．

Definition 1. アーベル圏 $\mathcal{A}$ の加法部分圏 $\mathcal{X}$ を考える．

(1) $\mathcal{X}$ が**拡大で閉じる**とは，任意の $\mathcal{A}$ の短完全列 $0 \to A \to B \to C \to 0$ に対して $A, C \in \mathcal{X}$ ならば $B \in \mathcal{X}$ となるときに言う．
(2) $\mathcal{X}$ が**部分対象で閉じる**とは，任意の $\mathcal{A}$ の短完全列 $0 \to A \to B \to C \to 0$ に対して $B \in \mathcal{X}$ ならば $A \in \mathcal{X}$ となるときに言う．
(3) $\mathcal{X}$ が**商で閉じる**とは，任意の $\mathcal{A}$ の短完全列 $0 \to A \to B \to C \to 0$ に対して $B \in \mathcal{X}$ ならば $C \in \mathcal{X}$ となるときに言う．

The detailed version of this paper will be submitted for publication elsewhere.

(4) $\mathcal{X}$ が**核で閉じる**とは，任意の $\mathcal{A}$ の射 $f\colon X \to Y$ に対して，$X, Y \in \mathcal{X}$ ならば $\operatorname{Ker} f \in \mathcal{X}$ となるときに言う．
(5) $\mathcal{X}$ が**余核で閉じる**とは，任意の $\mathcal{A}$ の射 $f\colon X \to Y$ に対して，$X, Y \in \mathcal{X}$ ならば $\operatorname{Cok} f \in \mathcal{X}$ となるときに言う．
(6) $\mathcal{X}$ が**像で閉じる**とは，任意の $\mathcal{A}$ の射 $f\colon X \to Y$ に対して，$X, Y \in \mathcal{X}$ ならば $\operatorname{Im} f \in \mathcal{X}$ となるときに言う．

これらの性質を組み合わせることで様々なアーベル圏の部分圏を定義することができる．

Definition 2. アーベル圏 $\mathcal{A}$ の加法部分圏 $\mathcal{X}$ を考える．

(1) $\mathcal{X}$ が **Serre 部分圏**であるとは，拡大と部分対象，商で閉じるときに言う．
(2) $\mathcal{X}$ が**トーション・フリー類**であるとは，拡大と部分対象で閉じるときに言う．
(3) $\mathcal{X}$ が**トーション類**であるとは，拡大と商で閉じるときに言う．
(4) $\mathcal{X}$ が**ワイド部分圏**（あるいは **CKE 閉部分圏**）であるとは，余核と核，拡大で閉じるときに言う．
(5) $\mathcal{X}$ が **IKE 閉部分圏**であるとは，像と核，拡大で閉じるときに言う．
(6) $\mathcal{X}$ が **ICE 閉部分圏**であるとは，像と余核，拡大で閉じるときに言う．
(7) $\mathcal{X}$ が **IE 閉部分圏**であるとは，像と拡大で閉じるときに言う．
(8) $\mathcal{X}$ が **KE 閉部分圏**であるとは，核と拡大で閉じるときに言う．
(9) $\mathcal{X}$ が **CE 閉部分圏**であるとは，余核と拡大で閉じるときに言う．

これらの部分圏の関係は次のように図示できる：

$$
\begin{tikzcd}
 & & \text{Serre} \arrow[dll, Rightarrow] \arrow[d, Rightarrow] \arrow[drr, Rightarrow] & & \\
\text{トーション・フリー類} \arrow[d, Rightarrow] & & \text{ワイド} \arrow[dll, Rightarrow] \arrow[drr, Rightarrow] & & \text{トーション類} \arrow[d, Rightarrow] \\
\text{IKE 閉} \arrow[d, Rightarrow] \arrow[drr, Rightarrow] & & & & \text{ICE 閉} \arrow[dll, Rightarrow] \arrow[d, Rightarrow] \\
\text{KE 閉} \arrow[drr, Rightarrow] & & \text{IE 閉} \arrow[d, Rightarrow] & & \text{CE 閉} \arrow[dll, Rightarrow] \\
 & & \text{拡大で閉じる} & &
\end{tikzcd}.
$$

可換ネーター環 R 上の有限生成加群の圏 $\mathsf{mod}\, R$ に関しては，これらの部分圏の多くが分類されてきた．

Theorem 3 ([2])**.** R を可換ネーター環とする．このとき対応

$$\mathcal{X} \mapsto \operatorname{Supp} \mathcal{X} := \bigcup_{X \in \mathcal{X}} \operatorname{Supp} X, \quad Z \mapsto \mathsf{mod}_Z\, R := \{M \in \mathsf{mod}\, R \mid \operatorname{Supp} M \subseteq Z\}$$

は次の集合の間に互いに逆な全単射対応を与える：

- $\mathsf{mod}\, R$ の *Serre* 部分圏の集合．
- $\operatorname{Spec} R$ の特殊化閉部分集合の集合．ここで部分集合 $Z \subseteq \operatorname{Spec} R$ が**特殊化閉 *(specialization-closed)*** であるとは，任意の $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$ に対して $\mathfrak{p} \in Z$ かつ $\mathfrak{p} \subseteq \mathfrak{q}$ ならば $\mathfrak{q} \in Z$ となるときに言う．

つまり $\mathsf{mod}\,R$ の *Serre* 部分圏は $\operatorname{Spec} R$ の特殊化閉部分集合で分類される．

Theorem 4 ([7])**.** R を可換ネーター環とする．このとき対応

$$\mathcal{X} \mapsto \operatorname{Ass}\mathcal{X} := \bigcup_{X\in\mathcal{X}} \operatorname{Ass} X, \quad \Phi \mapsto \mathsf{mod}^{\mathsf{ass}}_{\Phi} R := \{M \in \mathsf{mod}\,R \mid \operatorname{Ass} M \subseteq \Phi\}$$

は次の集合の間に互いに逆な全単射対応を与える：

- アーベル圏 $\mathsf{mod}\,R$ のトーション・フリー類の集合.
- $\operatorname{Spec} R$ のべき集合.

つまり $\mathsf{mod}\,R$ のトーション・フリー類は $\operatorname{Spec} R$ の部分集合で分類される．

Theorem 5 ([1, 6, 7])**.** R を可換ネーター環とする．

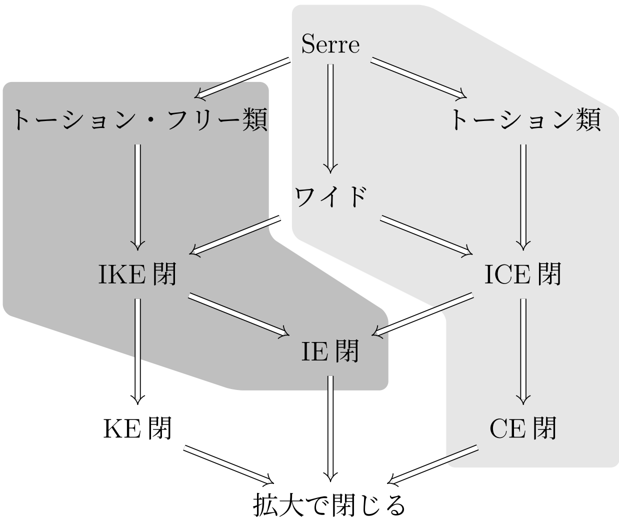
(1) $\mathsf{mod}\,R$ の加法部分圏 $\mathcal{X}$ に対して次は同値である.
- $\mathcal{X}$ は *Serre* 部分圏である.
- $\mathcal{X}$ はトーション類である.
- $\mathcal{X}$ はワイド部分圏である.
- $\mathcal{X}$ は *ICE* 閉部分圏である.
- $\mathcal{X}$ は *CE* 閉部分圏である.

つまり，上記の部分圏は $\operatorname{Spec} R$ の特殊化閉部分集合で分類される．

(2) $\mathsf{mod}\,R$ の加法部分圏 $\mathcal{X}$ に対して次は同値である.
- $\mathcal{X}$ はトーション・フリー類である.
- $\mathcal{X}$ は *IKE* 閉部分圏である.
- $\mathcal{X}$ は *IE* 閉部分圏である.

つまり，上記の部分圏は $\operatorname{Spec} R$ の部分集合で分類される．

つまり $\mathsf{mod}\,R$ の部分圏のクラスは次のように分けられる：



これが可換ネーター環 R 上の加群圏 $\mathsf{mod}\,R$ の部分圏の分類に関する先行研究である．次節では，これらの分類のトーション・フリー類への拡張を紹介する．これらの部分圏の分類のネーター代数やスキームへの拡張に関してはそれぞれ [3] と [4] を見よ．

2. 完全圏の部分圏の分類

前節までは，加群圏 $\mathsf{mod}\, R$ などのアーベル圏の部分圏について論じていたのに対して，この節では，極大 Cohen-Macaulay 加群の圏 $\mathsf{cm}\, R$ などの完全圏（＝アーベル圏の拡大で閉じた部分圏）の部分圏について論じたい．そのためにまず次の概念を導入する．

Definition 6. アーベル圏 $\mathcal{A}$ の拡大で閉じた部分圏 $\mathcal{X}$ を考える．このとき $\mathcal{X}$ の**許容短完全列**とは，$\mathcal{A}$ の短完全列 $0 \to A \to B \to C \to 0$ であって $A, B, C \in \mathcal{X}$ となるものである．

許容短完全列を考えることで，$\mathcal{X}$ の中でアーベル圏のようなホモロジー代数的議論を行うことができるようになる．とくにアーベル圏の短完全列を $\mathcal{X}$ の許容短完全列に置き換えることで，$\mathcal{X}$ の Serre 部分圏やトーション（・フリー）類を考えることが出来る．

Definition 7. アーベル圏 $\mathcal{A}$ の拡大で閉じた部分圏 $\mathcal{X}$ と $\mathcal{X}$ の加法部分圏 $\mathcal{S}$ を考える．

(1) $\mathcal{S}$ が**拡大で閉じる**とは，任意の $\mathcal{X}$ の許容短完全列 $0 \to X \to Y \to Z \to 0$ に対して $X, Z \in \mathcal{S}$ ならば $Y \in \mathcal{S}$ となるときに言う．
(2) $\mathcal{S}$ が**許容部分対象で閉じる**とは，任意の $\mathcal{X}$ の許容短完全列 $0 \to X \to Y \to Z \to 0$ に対して $Y \in \mathcal{S}$ ならば $X \in \mathcal{S}$ となるときに言う．
(3) $\mathcal{X}$ が**許容商で閉じる**とは，任意の $\mathcal{X}$ の許容短完全列 $0 \to X \to Y \to Z \to 0$ に対して $Y \in \mathcal{S}$ ならば $Z \in \mathcal{S}$ となるときに言う．
(4) $\mathcal{S}$ が $\mathcal{X}$ の **Serre 部分圏**であるとは，拡大と許容部分対象，許容商で閉じるときに言う．
(5) $\mathcal{S}$ が $\mathcal{X}$ の**トーション・フリー類**であるとは，拡大と許容部分対象で閉じるときに言う．
(6) $\mathcal{S}$ が $\mathcal{X}$ の**トーション類**であるとは，拡大と許容商で閉じるときに言う．

この節で紹介する主結果は，可換ネーター環 R 上の加群圏 $\mathsf{mod}\, R$ のトーション・フリー類の部分圏の分類である．まず定理 4 より $\mathsf{mod}\, R$ のトーション・フリー類は，$\operatorname{Spec} R$ の部分集合 Φ を用いて次のような形で記述されたことを思い出そう：

$$\mathsf{mod}^{\mathsf{ass}}_{\Phi} R := \{M \in \mathsf{mod}\, R \mid \operatorname{Ass} M \subseteq \Phi\}.$$

実は完全圏 $\mathsf{mod}^{\mathsf{ass}}_{\Phi} R$ の Serre 部分圏やトーション・フリー類は Φ のある部分集合を用いて記述することができる．

Theorem 8 ([4])**.** R を可換ネーター環とし，Φ を $\operatorname{Spec} R$ の部分集合とする．このとき対応

$$\mathcal{X} \mapsto \operatorname{Ass} \mathcal{X} := \bigcup_{X \in \mathcal{X}} \operatorname{Ass} X, \quad \Psi \mapsto \mathsf{mod}^{\mathsf{ass}}_{\Psi} R$$

は次の集合の間に互いに逆な全単射対応を与える：

- $\mathsf{mod}^{\mathsf{ass}}_{\Phi} R$ の *Serre* 部分圏の集合．
- Φ の特殊化閉部分集合の集合．ここで部分集合 $\Psi \subseteq \Phi$ が**特殊化閉** ***(specialization-closed)*** であるとは，任意の $\mathfrak{p}, \mathfrak{q} \in \Phi$ に対して $\mathfrak{p} \in \Psi$ かつ $\mathfrak{p} \subseteq \mathfrak{q}$ ならば $\mathfrak{q} \in \Psi$ となるときに言う．

つまり完全圏 $\mathsf{mod}^{\mathsf{ass}}_{\Phi} R$ の *Serre* 部分圏は Φ の特殊化閉部分集合で分類される．

この定理において $\Phi = \operatorname{Spec} R$ とすれば，定理 3 が復元される．この意味でこの定理は，定理 3 の完全圏への拡張だと思える．

Theorem 9 ([5]). R を可換ネーター環とし，Φ を $\operatorname{Spec} R$ の部分集合とする．このとき $\mathsf{mod}_{\Phi}^{\mathsf{ass}} R$ の中で *Serre* 部分圏とトーション類は一致する．とくに完全圏 $\mathsf{mod}_{\Phi}^{\mathsf{ass}} R$ のトーション類は Φ の特殊化閉部分集合で分類される．

Theorem 10 ([5]). R を 1 次元可換ネーター環とし，Φ を $\operatorname{Spec} R$ の部分集合とする．このとき対応

$$\mathcal{X} \mapsto \operatorname{Ass} \mathcal{X} := \bigcup_{X \in \mathcal{X}} \operatorname{Ass} X, \quad \Psi \mapsto \mathsf{mod}_{\Psi}^{\mathsf{ass}} R$$

は次の集合の間に互いに逆な全単射対応を与える：

- $\mathsf{mod}_{\Phi}^{\mathsf{ass}} R$ のトーション・フリー類の集合．
- Φ のべき集合．

つまり完全圏 $\mathsf{mod}_{\Phi}^{\mathsf{ass}} R$ のトーション・フリー類は Φ の部分集合で分類される．

この定理において $\Phi = \operatorname{Spec} R$ とすれば，1 次元の場合の定理 4 が復元される．この意味でこの定理は，定理 4 の完全圏への拡張だと思える．次元が 2 以上の可換ネーター環に関しては，この定理の反例が存在する．

1 次元 Cohen-Macaulay 環に対しては，極大 Cohen-Macaulay 加群の圏 $\mathsf{cm}\, R$ は $\mathsf{mod}\, R$ のトーション・フリー類となり，$\mathsf{cm}\, R = \mathsf{mod}_{\operatorname{Min} R}^{\mathsf{ass}} R$ となる．ここで $\operatorname{Min} R$ は R の極小素イデアルの集合である．よってここまでの定理を合わせることで次を得る．

Corollary 11. R を 1 次元 *Cohen-Macaulay* 環とする．このとき完全圏 $\mathsf{cm}\, R$ の中で *Serre* 部分圏，トーション類およびトーション・フリー類は一致する．またこれらの部分圏は $\operatorname{Min} R$ の部分集合で分類される．

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Graduate School of Mathematics
Nagoya University
Chikusa-ku, Nagoya. 464-8602, Japan
Email address: m19018i@math.nagoya-u.ac.jp

A CLASSIFICATION OF T-STRUCTURES BY A LATTICE OF TORSION CLASSES

ARASHI SAKAI (酒井 嵐士)

Abstract. We introduce the notion of ICE sequences to investigate t-structures on the bounded derived category of the module categories $\mathsf{mod}\Lambda$ over a finite dimensional algebra Λ. We give a correspondence between bounded t-structures and ICE sequences. Moreover we give a description of ICE sequences in $\mathsf{mod}\Lambda$ in terms of the lattice consisting of torsion classes in $\mathsf{mod}\Lambda$.

1. Introduction

Let Λ be a finite dimensional algebra over a field k. We denote by $\mathsf{mod}\Lambda$ the category of finitely generated right Λ-modules and $D^b(\mathsf{mod}\Lambda)$ the bounded derived category of $\mathsf{mod}\Lambda$. It is one of the main subjects of representation theory of finite dimensional algebras to study subcategories of $\mathsf{mod}\Lambda$ and $D^b(\mathsf{mod}\Lambda)$. For example, torsion classes are studied actively, and correspond to intermediate t-structures on $D^b(\mathsf{mod}\Lambda)$ bijectively [6]. In this note, we always assume that all subcategories are full and closed under isomorphisms.

We focus on t-structures on $D^b(\mathsf{mod}\Lambda)$. For subcategories $\mathcal{U}$ and $\mathcal{V}$ of $D^b(\mathsf{mod}\Lambda)$, we denote by $\mathcal{U} * \mathcal{V}$ the subcategory of $D^b(\mathsf{mod}\Lambda)$ consisting of objects X such that there exists an exact triangle $U \to X \to V \to \Sigma U$ in $D^b(\mathsf{mod}\Lambda)$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

Definition 1. [2, Définition 1.3.1] A pair of subcategories $(\mathcal{U}, \mathcal{V})$ of $D^b(\mathsf{mod}\Lambda)$ is a *t-structure* on $D^b(\mathsf{mod}\Lambda)$ if it satisfies the following conditions:

(1) $\mathsf{Hom}(\mathcal{U}, \mathcal{V}) = 0$.
(2) $D^b(\mathsf{mod}\Lambda) = \mathcal{U} * \mathcal{V}$.
(3) $\Sigma\mathcal{U} \subseteq \mathcal{U}$.

We call $\mathcal{U}$ an *aisle*. A t-structure $(\mathcal{U}, \mathcal{V})$ is *bounded* if it satisfies

$$\bigcup_{n\in\mathbb{Z}} \Sigma^{-n}\mathcal{U} = D^b(\mathsf{mod}\Lambda) = \bigcup_{n\in\mathbb{Z}} \Sigma^{n}\mathcal{V}.$$

For a t-structure $(\mathcal{U}, \mathcal{V})$ on $D^b(\mathsf{mod}\Lambda)$, we have $\mathcal{U} = {}^{\perp}\mathcal{V}$, therefore a t-structure is determined by its aisle. Hence we focus on aisles, and we call a subcategory of $D^b(\mathsf{mod}\Lambda)$ an aisle if it is an aisle of a certain t-structure.

A subcategory $\mathcal{X}$ of $D^b(\mathsf{mod}\Lambda)$ is *closed under extensions* if it satisfies $\mathcal{X} * \mathcal{X} \subseteq \mathcal{X}$.

Definition 2. A subcategory $\mathcal{U}$ of $D^b(\mathsf{mod}\Lambda)$ is a *preaisle* if $\mathcal{U}$ is closed under extensions and positive shifts.

It is easy to check that an aisle of a t-structure is a preaisle. Actually, aisles are exactly contravariantly finite preaisles:

The detailed version of this paper will be submitted for publication elsewhere.

Proposition 3. [8, Proposition 1.3] *The following are equivalent for a subcategory* $\mathcal{U}$ *of* $D^b(\mathsf{mod}\Lambda)$.

(1) $\mathcal{U}$ *is an aisle.*
(2) $\mathcal{U}$ *is a coreflective preaisle, that is,* $\mathcal{U}$ *is a preaisle and the inclusion* $\mathcal{U} \to D^b(\mathsf{mod}\Lambda)$ *has a right adjoint functor.*
(3) $\mathcal{U}$ *is a contravariantly finite preaisle closed under direct summands.*

Proof. (1) $\Leftrightarrow$ (2): This is well-known.
(2) $\Leftrightarrow$ (2): This follows from [3, Corollary 4.5]. $\square$

At first, we deel with preaisles. In [10], homology-determined preaisles are classified by *narrow sequences.* We denote by H^k the k-th cohomology functor.

Definition 4. A preaisle $\mathcal{U}$ of $D^b(\mathsf{mod}\Lambda)$ is *homology-determined* if for any $X \in D^b(\mathsf{mod}\Lambda)$, we have $X \in \mathcal{U}$ if and only if $\Sigma^{-k}(H^k X) \in \mathcal{U}$ for any $k \in \mathbb{Z}$.

Note that if Λ is hereditary, then every aisle is homology-determined since every complex X in $D^b(\mathsf{mod}\Lambda)$ is isomorphic to a direct sum $\oplus\Sigma^{-k}(H^k X)$. For homology-determined preaisle $\mathcal{U}$ of $D^b(\mathsf{mod}\Lambda)$, we can consider a sequence $\{H^k\mathcal{U}\}_{k\in\mathbb{Z}}$ of subcategories of $\mathsf{mod}\Lambda$. In the next section, we give a characterization of the sequence.

2. Aisles and ICE sequences

In this section, we introduce ICE sequences to study preaisles. We recall basic definitions of subcategories of an abelian category.

Definition 5. Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a subcategory of $\mathcal{A}$.

(1) $\mathcal{C}$ is *closed under extensions* if for every short exact sequence
$$0 \to L \to M \to N \to 0$$
in $\mathcal{A}$ with $L, N \in \mathcal{C}$, we have $M \in \mathcal{C}$.
(2) $\mathcal{C}$ is *closed under quotients (resp. subobjects) in* $\mathcal{A}$ if, for every object $C \in \mathcal{C}$, every quotient (resp. subobject) of C in $\mathcal{A}$ belongs to $\mathcal{C}$.
(3) $\mathcal{C}$ is a *torsion class (resp. torsion-free class) in* $\mathcal{A}$ if $\mathcal{C}$ is closed under extensions and quotients in $\mathcal{A}$ (resp. extensions and subobjects).
(4) $\mathcal{C}$ is closed under *images (resp. kernels, cokernels)* if, for every map $\varphi\colon C_1 \to C_2$ with $C_1, C_2 \in \mathcal{C}$, we have $\mathsf{Im}\varphi \in \mathcal{C}$ (resp. $\mathsf{Ker}\varphi \in \mathcal{C}$, $\mathsf{Coker}\varphi \in \mathcal{C}$).
(5) $\mathcal{C}$ is a *wide subcategory of* $\mathcal{A}$ if $\mathcal{C}$ is closed under kernels, cokernels, and extensions.
(6) $\mathcal{C}$ is an *ICE-closed subcategory of* $\mathcal{A}$ if $\mathcal{C}$ is closed under images, cokernels and extensions.

It is easy to check that torsion classes and wide subcategories are ICE-closed subcategories. Moreover, every torsion class in a wide subcategory (viewed as an abelian category) is ICE-closed, see [5, Lemma 2.2]. In [7], Ingalls and Thomas introduced an operation α which associates to a torsion class a wide subcategory. In [4, Proposition 4.2], the operation was generalized to ICE-closed subcategories. The following is shown by the same argument of [7, Proposition 2.12].

Proposition 6. *Let $\mathcal{C}$ be an ICE-closed subcategory of $\mathcal{A}$. Define a subcategory of $\mathcal{C}$ by*

$$\alpha\mathcal{C} = \{A \in \mathcal{C} \mid {}^{\forall}(f\colon C \to A) \in \mathcal{C},\ \ker f \in \mathcal{C}\}.$$

Then $\alpha\mathcal{C}$ is a wide subcategory of $\mathcal{A}$.

Next we give a definition of ICE sequences. This is the key notion in this note.

Definition 7. A sequence $\{\mathcal{C}(k)\}_{k\in\mathbb{Z}}$ of subcategories of $\mathsf{mod}\Lambda$ is an *ICE sequence* if for any k, the subcategory $\mathcal{C}(k)$ is an ICE-closed subcategory of $\mathsf{mod}\Lambda$ and the subcategory $\mathcal{C}(k+1)$ is a torsion class in $\alpha(\mathcal{C}(k))$.

Clealy, we have $\mathcal{C}(k+1) \subseteq \mathcal{C}(k)$ for any $k \in \mathbb{Z}$. Actually, ICE sequnces are the same notion of narrow sequences introduced in [10, Definition 4.1], see [9, Proposition 4.2]. Combining this fact and the result [10, Theorem 4.11], we obtain the following result.

Theorem 8. [9, Theorem 4.5] *There exist mutually bijective correspondences between*

(1) *the set of homology-determined preaisles in $D^b(\mathsf{mod}\Lambda)$.*
(2) *the set of ICE sequences in $\mathsf{mod}\Lambda$,*

The map from (1) to (2) is given by $\mathcal{U} \mapsto \{H^k\mathcal{U}\}_{k\in\mathbb{Z}}$. The converse is given by $\{\mathcal{C}(k)\}_{k\in\mathbb{Z}} \mapsto \{X \in D^b(\mathsf{mod}\Lambda) \mid H^kX \in \mathcal{C}(k)$ for any $k\}$.

Finally, we restrict the above result to aisles of bounded t-structures.

Definition 9. Let $\{\mathcal{C}(k)\}_{k\in\mathbb{Z}}$ be an ICE sequence in $\mathsf{mod}\Lambda$.

(1) $\{\mathcal{C}(k)\}_{k\in\mathbb{Z}}$ is *contravariantly finite* if $\mathcal{C}(k)$ is contravariantly finite in $\mathsf{mod}\Lambda$ for any $k \in \mathbb{Z}$.
(2) $\{\mathcal{C}(k)\}_{k\in\mathbb{Z}}$ is *full* if there exist integers $m \leq n$ such that $\mathcal{C}(m) = 0$ and $\mathcal{C}(n) = \mathsf{mod}\Lambda$.
(3) For a positive integer n, we say that $\{\mathcal{C}(k)\}_{k\in\mathbb{Z}}$ is *of length* $n+1$ if we have $\mathcal{C}(1) = 0$ and $\mathcal{C}(-n) = \mathsf{mod}\Lambda$.

Note that an ICE-closed subcategory of $\mathsf{mod}\Lambda$ is contravariantly finite if and only if it is coreflective by [3, Corollary 7.2]. If Λ is τ-tilting finite, then every ICE-closed subcategory of $\mathsf{mod}\Lambda$ is contravariantly finite, see [5, Proposition 4.20].

The following is the main result in this section.

Theorem 10. [9, Theorem 5.5, Corollary 5.6] *There exist bijective correspondences between*

(1) *the set of contravariantly finite full ICE sequences in $\mathsf{mod}\Lambda$,*
(2) *the set of bounded t-structures on $D^b(\mathsf{mod}\Lambda)$ whose aisles are homology-determined.*

Let n be a positive integer. Then the above restrict to the following.

(1) *the set of contravariantly finite ICE sequences in $\mathsf{mod}\Lambda$ of length $n+1$,*
(2) *the set of $(n+1)$-intermediate t-structures on $D^b(\mathsf{mod}\Lambda)$ whose aisles are homology-determined.*

Thus we can construct t-structures on $D^b(\mathsf{mod}\Lambda)$ from ICE sequences in $\mathsf{mod}\Lambda$. In the next section, we give a description of ICE sequences by a lattice-theoretical notion.

3. A Lattice of torsion classes

In this section, we fix a positive integer n, and focus on $(n+1)$-intermediate t-structures. We give a description of ICE sequences of length $n+1$ in $\mathsf{mod}\Lambda$ from the viewpoint of a lattice consisting of torsion classes in $\mathsf{mod}\Lambda$. We denote by $\mathsf{tors}\Lambda$ the set of torsion classes in $\mathsf{mod}\Lambda$, which forms a partially ordered set by inclusion. Moreover $\mathsf{tors}\Lambda$ is a complete lattice since there are arbitrary intersections. We collect some definitions and results.

Definition 11. To $\mathcal{T}, \mathcal{U} \in \mathsf{tors}\Lambda$, we associate the set

$$[\mathcal{U}, \mathcal{T}] := \{\mathcal{C} \in \mathsf{tors}\Lambda \mid \mathcal{U} \subseteq \mathcal{C} \subseteq \mathcal{T}\}$$

called an *interval* in $\mathsf{tors}\Lambda$. To an interval $[\mathcal{U}, \mathcal{T}]$, we associate a subcategory $\mathcal{H}_{[\mathcal{U},\mathcal{T}]} = \mathcal{T} \cap \mathcal{U}^{\perp}$ called the *heart* of $[\mathcal{U}, \mathcal{T}]$. We call an interval $[\mathcal{U}, \mathcal{T}]$ a *wide interval* if the heart is a wide subcategory of $\mathsf{mod}\Lambda$. We denote by $\mathsf{Hasse}(\mathsf{tors}\Lambda)$ the *Hasse quiver* of $\mathsf{tors}\Lambda$, the quiver whose vertex set is $\mathsf{tors}\Lambda$, and there is an arrow $\mathcal{T} \to \mathcal{U}$ in $\mathsf{tors}\Lambda$ if and only if $\mathcal{U} \subsetneq \mathcal{T}$ holds and there is no $\mathcal{C} \in \mathsf{tors}\Lambda$ satisfying $\mathcal{U} \subsetneq \mathcal{C} \subsetneq \mathcal{T}$.

Wide intervals are characterized as a lattice-theoretical property in $\mathsf{tors}\Lambda$ as follows:

Proposition 12. [1, Theorem 5.2] *Let $[\mathcal{U}, \mathcal{T}]$ be an interval in $\mathsf{tors}\Lambda$. Then the following conditions are equivalent:*
(1) *$[\mathcal{U}, \mathcal{T}]$ is a wide interval.*
(2) *$[\mathcal{U}, \mathcal{T}]$ is a meet interval, that is, it holds*

$$\mathcal{U} = \mathcal{T} \bigcap \{\mathcal{C} \in [\mathcal{U}, \mathcal{T}] \mid \textit{there is an arrow } \mathcal{T} \to \mathcal{C} \textit{ in } \mathsf{Hasse}(\mathsf{tors}\Lambda)\}.$$

The operation α is understood from the viewpoint of wide intervals:

Proposition 13. *Let $\mathcal{T}$ be a torsion class in $\mathsf{mod}\Lambda$. Then the following statements hold.*
(1) [1, Proposition 6.3] *$\alpha\mathcal{T}$ equals to the heart of the interval $[\mathcal{T} \cap {}^{\perp}\alpha\mathcal{T}, \mathcal{T}]$.*
(2) [5, Proposition 3.3] *We set*

$$\mathcal{T}^{-} = \mathcal{T} \bigcap \{\mathcal{C} \in \mathsf{tors}\Lambda \mid \textit{there is an arrow } \mathcal{T} \to \mathcal{C} \textit{ in } \mathsf{Hasse}(\mathsf{tors}\Lambda)\}.$$

Then we have $\mathcal{T}^{-} = \mathcal{T} \cap {}^{\perp}\alpha\mathcal{T}$ and $\mathcal{H}_{[\mathcal{T}^{-},\mathcal{T}]} = \alpha\mathcal{T}$.

Thus we can understand α in terms of $\mathsf{tors}\Lambda$. We introduce the following notion.

Definition 14. (1) We call an interval of the form $[\mathcal{T}^{-}, \mathcal{T}]$ a *maximal meet interval* in $\mathsf{tors}\Lambda$. More generally, we call an interval $[\mathcal{U}', \mathcal{T}']$ contained in a wide interval $[\mathcal{U}, \mathcal{T}]$ in $\mathsf{tors}\Lambda$ a *maximal meet interval* in $[\mathcal{U}, \mathcal{T}]$ if we have

$$\mathcal{U}' = \mathcal{T}' \bigcap \{\mathcal{C} \in [\mathcal{U}, \mathcal{T}] \mid \text{there is an arrow } \mathcal{T}' \to \mathcal{C} \text{ in } \mathsf{Hasse}(\mathsf{tors}\Lambda)\}.$$

(2) We call a sequence $\{[\mathcal{U}_k, \mathcal{T}_k]\}_{k=1}^{n}$ of intervals in $\mathsf{tors}\Lambda$ a *decreasing sequence of maximal meet intervals* in $\mathsf{tors}\Lambda$ provided that $[\mathcal{U}_{k+1}, \mathcal{T}_{k+1}]$ is a maximal meet interval in $[\mathcal{U}_k, \mathcal{T}_k]$ for any $k = 0, \ldots, n-1$ where we set $\mathcal{U}_0 = 0$ and $\mathcal{T}_0 = \mathsf{mod}\Lambda$. We call n the *length* of the sequence.

Now we obtain a classification of $(n+1)$-intermediate t-structures whose aisles are homology-determined via ICE sequences and the lattice of torsion classes:

Theorem 15. *Let Λ be a τ-tilting finite algebra and* $\mathsf{tors}\Lambda$ *the lattice consisting of torsion classes in* $\mathsf{mod}\Lambda$*. Then there are one-to-one correspondences between*

(1) *the set of $(n+1)$-intermediate t-structures on $D^b(\mathsf{mod}\Lambda)$ whose aisles are homology-determined,*
(2) *the set of ICE sequences in* $\mathsf{mod}\Lambda$ *of length $n+1$,*
(3) *the set of decreasing sequences of maximal meet intervals in* $\mathsf{tors}\Lambda$ *of length n,*

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Graduate School of Mathematics
Nagoya University
Nagoya, Aichi 464-8602 JAPAN
Email address: m20019b@math.nagoya-u.ac.jp

RESOLVING SUBCATEGORIES OF DERIVED CATEGORIES

RYO TAKAHASHI

Abstract. Let R be a commutative noetherian ring. Denote by $\mathsf{D}^{\mathsf{b}}(R)$ the bounded derived category of finitely generated R-modules. In this article we classify the preaisles of $\mathsf{D}^{\mathsf{b}}(R)$ containing R and closed under direct summands, when R is a complete intersection. This classification includes as restrictions the classification of thick subcategories of the singularity category due to Stevenson, and the classification of resolving subcategories of the module category due to Dao and Takahashi.

1. Main result

Throughout this article, we assume that all subcategories are strictly full. First of all, we introduce a setup to explain our main result.

Setup 1. Let (R, V) be a pair, where R and V satisfy either of the following two conditions.

(1) R is a commutative noetherian ring which is locally a hypersurface, and V is the singular locus of R.
(2) R is a quotient ring of the form $S/(\boldsymbol{a})$ where S is a regular ring of finite Krull dimension and $\boldsymbol{a} = a_1, \dots, a_c$ is a regular sequence, and V is the singular locus of the zero subscheme of $a_1x_1 + \cdots + a_cx_c \in \Gamma(X, \mathcal{O}_X(1))$ where $X = \mathbb{P}_S^{c-1} = \mathrm{Proj}(S[x_1, \dots, x_c])$.

Here, a commutative noetherian ring R is said to be *locally* a *hypersurface* if the local ring $R_\mathfrak{p}$ is a hypersurface for every prime ideal $\mathfrak{p}$ of R. When R is a local ring with maximal ideal $\mathfrak{m}$, we say that R is a *hypersurface* if the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is a quotient of a regular local ring by a principal ideal. A *regular sequence* on R is a sequence $\boldsymbol{x} = x_1, \dots, x_n$ of elements of R such that the residue class of x_i in $R/(x_1, \dots, x_{i-1})$ is a non-zerodivisor for each $i = 1, \dots, n$ and that $(x_1, \dots, x_n)$ is not a unit ideal of R.

For a commutative noetherian ring R, we denote by $\operatorname{\mathsf{mod}} R$ the category of finitely generated R-modules, by $\mathsf{D}^{\mathsf{b}}(R)$ the bounded derived category of $\operatorname{\mathsf{mod}} R$, by $\mathsf{D}^{\mathsf{perf}}(R)$ the subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ consisting of perfect complexes, and by $\mathsf{D}_{\mathsf{sg}}(R)$ the singularity category of R, i.e.,

$$\mathsf{D}_{\mathsf{sg}}(R) = \mathsf{D}^{\mathsf{b}}(R)/\mathsf{D}^{\mathsf{perf}}(R).$$

Recall that a *thick subcategory* of a triangulated category is by definition a triangulated subcategory closed under direct summands. Under the above setup, Stevenson [2] proved the following classification theorem of thick subcategories.

The detailed version [3] of this article has been submitted for publication elsewhere.

Theorem 2 (Stevenson)**.** *Let (R, V) be as in Setup 1. Then there are one-to-one correspondences*

$$\left\{\begin{matrix}\textit{thick}\\ \textit{subcategories}\\ \textit{of } \mathsf{D_{sg}}(R)\end{matrix}\right\} \cong \left\{\begin{matrix}\textit{thick}\\ \textit{subcategories}\\ \textit{of } \mathsf{D^b}(R)\\ \textit{containing } R\end{matrix}\right\} \overset{\text{(a)}}{\cong} \left\{\begin{matrix}\textit{specialization-}\\ \textit{closed}\\ \textit{subsets of } V\end{matrix}\right\}.$$

Recall that a *resolving subcategory* of $\mathsf{mod}\, R$ is defined to be a subcategory of $\mathsf{mod}\, R$ containing R and closed under direct summands, extensions and kernels of epimorphisms. Dao and Takahashi [1] gave a complete classification of the resolving subcategories of $\mathsf{mod}\, R$ under the setup introduced above.

Theorem 3 (Dao–Takahashi)**.** *Let (R, V) be as in Setup 1. Then there is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{resolving}\\ \textit{subcategories}\\ \textit{of } \mathsf{mod}\, R\end{matrix}\right\} \overset{\text{(b)}}{\cong} \left\{\begin{matrix}\textit{grade-}\\ \textit{consistent}\\ \textit{functions}\\ \textit{on } \operatorname{Spec} R\end{matrix}\right\} \times \left\{\begin{matrix}\textit{specialization-}\\ \textit{closed}\\ \textit{subsets of } V\end{matrix}\right\}.$$

Here, a *grade-consistent function* on $\operatorname{Spec} R$ is defined as an order-preserving map $f : \operatorname{Spec} R \to \mathbb{N}$ which satisfies the inequality $f(\mathfrak{p}) \leqslant \operatorname{grade} \mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec} R$, where

$$\operatorname{grade} \mathfrak{p} = \inf\{i \in \mathbb{N} \mid \operatorname{Ext}^i_R(R/\mathfrak{p}, R) \neq 0\}.$$

Recall that a *preaisle* (resp. *precoaisle*) of a triangulated category is defined as a subcategory closed under extensions and positive (resp. negative) shifts. Mimicking the definition of a resolving subcategory of $\mathsf{mod}\, R$, we define a *resolving subcategory* of $\mathsf{D^b}(R)$ as a subcategory of $\mathsf{D^b}(R)$ containing R and closed under direct summands, extensions and cocones.

The main result of this article is the following theorem. This theorem provides a classification of preaisles of $\mathsf{D^b}(R)$ that satisfy some mild and natural conditions. Also, the theorem includes both the classification of thick subcategories by Stevenson and the classification of resolving subcategories by Dao and Takahashi.

Theorem 4. *Let (R, V) be a pair as in Setup 1. Then there are one-to-one correspondences*

$$\left\{\begin{matrix}\textit{preaisles}\\ \textit{of } \mathsf{D^b}(R)\\ \textit{containing}\\ R \textit{ and closed}\\ \textit{under direct}\\ \textit{summands}\end{matrix}\right\} \cong \left\{\begin{matrix}\textit{resolving}\\ \textit{subcategories}\\ \textit{of } \mathsf{D^b}(R)\end{matrix}\right\} \overset{(*)}{\cong} \left\{\begin{matrix}\textit{order-}\\ \textit{preserving}\\ \textit{maps}\\ \textit{from } \operatorname{Spec} R\\ \textit{to } \mathbb{N}\cup\{\infty\}\end{matrix}\right\} \times \left\{\begin{matrix}\textit{specialization-}\\ \textit{closed}\\ \textit{subsets of } V\end{matrix}\right\}.$$

The restriction of the bijection $(*)$ to the thick subcategories of $\mathsf{D^b}(R)$ containing R is identified with the bijection (a) *in Theorem 2. The composition of the bijection $(*)$ with the map*

$$\mathcal{X} \mapsto \mathsf{res}_{\mathsf{D^b}(R)}\, \mathcal{X}$$

coincides with the bijection (b) *in Theorem 3.*

Here, $\mathsf{res}_{\mathsf{D^b}(R)}\mathcal{X}$ stands for the *resolving closure* of $\mathcal{X}$ in $\mathsf{D^b}(R)$, that is, the smallest resolving subcategory of $\mathsf{D^b}(R)$ containing $\mathcal{X}$.

2. Outline

This section is devoted to (roughly) explaining how to deduce Theorem 4.

We say that R is *locally* a *Gorenstein ring* if the local ring $R_\mathfrak{p}$ is Gorenstein for every prime ideal $\mathfrak{p}$ of R. When R is a local ring, we say that R is *Gorenstein* if R has finite injective dimension as an R-module.

The following proposition is immediately obtained by using the fact that the functor $\mathbf{R}\mathrm{Hom}_R(-,R)$ gives a duality of $\mathsf{D^b}(R)$ if R is locally a Gorenstein ring, and comparing the definitions of a resolving subcategory and a precoaisle.

Proposition 5. *Let R be a commutative noetherian ring. Suppose that R is locally a Gorenstein ring. Assigning to each subcategory $\mathcal{X}$ of $\mathsf{D^b}(R)$ the subcategory*

$$\mathbf{R}\mathrm{Hom}_R(\mathcal{X},R)=\{\mathbf{R}\mathrm{Hom}_R(X,R)\mid X\in\mathcal{X}\}$$

of $\mathsf{D^b}(R)$, one obtains a one-to-one correspondence

$$\left\{\begin{array}{c}\textit{preaisles}\\ \textit{of } \mathsf{D^b}(R)\\ \textit{containing}\\ R \textit{ and closed}\\ \textit{under direct}\\ \textit{summands}\end{array}\right\}\cong\left\{\begin{array}{c}\textit{precoaisles}\\ \textit{of } \mathsf{D^b}(R)\\ \textit{containing}\\ R \textit{ and closed}\\ \textit{under direct}\\ \textit{summands}\end{array}\right\}=\left\{\begin{array}{c}\textit{resolving}\\ \textit{subcategories}\\ \textit{of } \mathsf{D^b}(R)\end{array}\right\}.$$

We say that R is *locally* a *complete intersection* if the local ring $R_\mathfrak{p}$ is a complete intersection for every prime ideal $\mathfrak{p}$ of R. When R is a local ring with maximal ideal $\mathfrak{m}$, we say that R is a *complete intersection* if the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is a quotient of a regular local ring by an ideal generated by a regular sequence. Denote by $\mathsf{D^{CM}}(R)$ the subcategory of $\mathsf{D^b}(R)$ consisting of *maximal Cohen–Macaulay complexes*, that is, complexes $C\in\mathsf{D^b}(R)$ such that

$$\operatorname{depth}_{R_\mathfrak{p}} C_\mathfrak{p}\geqslant\dim R_\mathfrak{p}$$

for all prime ideals $\mathfrak{p}$ of R, where dim denotes Krull dimension. When R is a local ring with residue field k, for each $X\in\mathsf{D^b}(R)$ we set

$$\operatorname{depth}_R X=\inf\{i\in\mathbb{Z}\mid \operatorname{Ext}_R^i(k,X)\neq 0\}.$$

The following two theorems are the most essential parts of our work. In the first theorem, $\mathsf{thick}_{\mathsf{D^b}(R)}\mathcal{X}$ stands for the *thick closure* of $\mathcal{X}$ in $\mathsf{D^b}(R)$, that is, the smallest thick subcategory of $\mathsf{D^b}(R)$ containing $\mathcal{X}$. The assumption of locally a complete intersection in the first theorem is necessary to deduce that each resolving subcategory of $\mathsf{D^b}(R)$ contained in $\mathsf{D^{CM}}(R)$ is closed under exact triangles of maximal Cohen–Macaulay complexes. The proof of the second theorem uses subtle arguments on Koszul complexes, and the notion of an *NE-locus*, which is a certain Zariski-closed subset of $\operatorname{Spec} R$.

Theorem 6. *Let R be a commutative noetherian ring. Suppose that R is locally a complete intersection.*

(1) *There are mutually inverse bijections*

$$\left\{\begin{matrix}\textit{resolving}\\ \textit{subcategories}\\ \textit{of } \mathsf{D^b}(R)\end{matrix}\right\} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \left\{\begin{matrix}\textit{resolving}\\ \textit{subcategories}\\ \textit{of } \mathsf{D^b}(R)\\ \textit{contained}\\ \textit{in } \mathsf{D^{perf}}(R)\end{matrix}\right\} \times \left\{\begin{matrix}\textit{resolving}\\ \textit{subcategories}\\ \textit{of } \mathsf{D^b}(R)\\ \textit{contained}\\ \textit{in } \mathsf{D^{CM}}(R)\end{matrix}\right\},$$

where the maps ϕ, ψ *are given by*

$$\phi(\mathcal{X}) = (\mathcal{X} \cap \mathsf{D^{perf}}(R), \mathcal{X} \cap \mathsf{D^{CM}}(R))$$

for each element $\mathcal{X}$ *of the left-hand side, and*

$$\psi(\mathcal{Y}, \mathcal{Z}) = \mathsf{res}_{\mathsf{D^b}(R)}(\mathcal{Y} \cup \mathcal{Z})$$

for each element $(\mathcal{Y}, \mathcal{Z})$ *of the right-hand side.*

(2) *There are mutually inverse bijections*

$$\left\{\begin{matrix}\textit{resolving}\\ \textit{subcategories}\\ \textit{of } \mathsf{D^b}(R)\\ \textit{contained}\\ \textit{in } \mathsf{D^{CM}}(R)\end{matrix}\right\} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \left\{\begin{matrix}\textit{thick}\\ \textit{subcategories}\\ \textit{of } \mathsf{D^b}(R)\\ \textit{containing } R\end{matrix}\right\} \cong \left\{\begin{matrix}\textit{thick}\\ \textit{subcategories}\\ \textit{of } \mathsf{D_{sg}}(R)\end{matrix}\right\}.$$

where the maps ϕ, ψ *are given by*

$$\phi(\mathcal{X}) = \mathsf{thick}_{\mathsf{D^b}(R)}\, \mathcal{X}$$

for each element $\mathcal{X}$ *of the left-hand side, and*

$$\psi(\mathcal{Y}) = \mathcal{Y} \cap \mathsf{D^{CM}}(R)$$

for each element $\mathcal{Y}$ *of the right-hand side.*

Theorem 7. *Let* R *be any commutative noetherian ring. Then there are mutually inverse bijections*

$$\left\{\begin{matrix}\textit{resolving}\\ \textit{subcategories}\\ \textit{of } \mathsf{D^b}(R)\\ \textit{contained}\\ \textit{in } \mathsf{D^{perf}}(R)\end{matrix}\right\} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \left\{\begin{matrix}\textit{order-}\\ \textit{preserving}\\ \textit{maps from}\\ \operatorname{Spec} R \textit{ to}\\ \mathbb{N} \cup \{\infty\}\end{matrix}\right\}.$$

where the maps ϕ, ψ *are given by*

$$\phi(\mathcal{X})(\mathfrak{p}) = \sup_{X \in \mathcal{X}} \{\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p}\}$$

for each element $\mathcal{X}$ *of the left-hand side and each prime ideal* $\mathfrak{p}$ *of* R*, and*

$$\psi(f) = \{X \in \mathsf{D^b}(R) \mid \operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p} \leqslant f(\mathfrak{p}) \textit{ for all } \mathfrak{p} \in \operatorname{Spec} R\}$$

for each element f *of the right-hand side.*

Here, pd stands for projective dimension.

Taking the combination of Proposition 5 with Theorems 6 and 7, we obtain the following theorem.

Theorem 8. *Let R be a commutative noetherian ring which is locally a complete intersection. Then there are one-to-one correspondences*

$$\left\{\begin{matrix}\textit{preaisles}\\ \textit{of } \mathsf{D}^{\mathsf{b}}(R)\\ \textit{containing}\\ R \textit{ and closed}\\ \textit{under direct}\\ \textit{summands}\end{matrix}\right\} \cong \left\{\begin{matrix}\textit{resolving}\\ \textit{subcategories}\\ \textit{of } \mathsf{D}^{\mathsf{b}}(R)\end{matrix}\right\} \cong \left\{\begin{matrix}\textit{order-}\\ \textit{preserving}\\ \textit{maps from}\\ \operatorname{Spec} R \textit{ to}\\ \mathbb{N}\cup\{\infty\}\end{matrix}\right\} \times \left\{\begin{matrix}\textit{thick}\\ \textit{subcategories}\\ \textit{of } \mathsf{D}_{\mathsf{sg}}(R)\end{matrix}\right\}.$$

Finally, combining Theorem 8 with Theorems 2 and 3 completes the proof of Theorem 4.

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Graduate School of Mathematics
Nagoya University
Furocho, Chikusaku, Nagoya 464-8602, Japan
Email address: takahashi@math.nagoya-u.ac.jp

PERIODIC DIMENSIONS OF MODULES AND ALGEBRAS

SATOSHI USUI

Abstract. For an eventually periodic module, we obtain the degree n and the period p of its first periodic syzygy. In this note, in order to study the degree n, we introduce the notion of the periodic dimension of a module and report results on periodic dimensions obtained so far.

1. Introduction

Throughout this note, let k be a field, and we assume that all rings are left Noetherian semiperfect rings (that are associative and unital). By a module, we mean a finitely generated left module.

Homological algebra [7] has been playing an important role in the representation theory of rings, and one of the fundamental tools is a projective resolution of a module. So it is natural to study the behavior of projective resolutions. In this note, we are concerned with eventually periodic modules (i.e., modules whose minimal projective resolutions become periodic in sufficiently large degrees) and study when their minimal projective resolutions become periodic. For this, we will introduce the notion of the *periodic dimension* of a module. From the definition, a module M is of finite periodic dimension if and only if M is eventually periodic. In this case, the value of the periodic dimension equals the degree of the first periodic syzygy of M. We first provide some of the basic properties of periodic dimensions and then investigate the relationship between Gorenstein and periodic dimensions. Moreover, motivated by a recent result of Dotsenko-Gélinas-Tamaroff [9], we determine the bimodule periodic dimension of a finite dimensional eventually periodic Gorenstein algebra.

2. Eventually periodic modules

This section recalls the definition of eventually periodic modules and some related results. Let R be a ring. For an R-module M and an integer $i \geq 0$, we denote by $\Omega^i_R(M)$ the i-th syzygy of the R-module M. It is understood that $\Omega^0_R(M) = M$.

Definition 1. An R-module M is called *periodic* if there exists an integer $p > 0$ such that $\Omega^p_R(M) \cong M$ as R-modules. The smallest $p > 0$ with this property is called the *period* of M. We call M *eventually periodic* if there exists an integer $n \geq 0$ such that $\Omega^n_R(M)$ is periodic.

We say that an R-module M is (n, p)-*eventually periodic* if M is eventually periodic over R and satisfies that its n-th syzygy is the first periodic syzygy of period p. We call a $(0, p)$-eventually periodic module a p-*periodic* module.

The detailed version of this paper will be submitted for publication elsewhere.

Modules of finite projective dimension n are $(n+1, 1)$-eventually periodic. The following example exhibits (n, p)-eventually periodic modules (with infinite projective dimension).

Example 2. Fix two integers $n \geq 0$ and $p > 0$, and consider the finite dimensional radical square zero algebra $\Lambda = kQ/R_Q^2$, where Q is the following quiver:

$$\begin{tikzcd} n \arrow[r] & n-1 \arrow[r] & \cdots \arrow[r] & 1 \arrow[r] & 0 \arrow[r] & -1 \arrow[r] & \cdots \arrow[r] & -p+1 \arrow[lll, bend right] \end{tikzcd}$$

and R_Q is the arrow ideal of the path algebra kQ. We denote by S_i the simple Λ-module associated with the vertex i. A direct calculation shows that S_i is (i, p)-eventually periodic if $1 \leq i \leq n$ and is p-periodic if $-p+1 \leq i \leq 0$. In particular, S_n is (n, p)-eventually periodic.

The integers n and p associated with an (n, p)-eventually periodic module are studied in the literature, for example [3, 6, 8, 10, 11]. We recall the following result of Avramov [3].

Theorem 3 ([3, Theorem 7.3.1])**.** *Let R be a commutative local ring, and let M be an R-module of finite complete intersection dimension. Then the following conditions are equivalent.*

(1) *M is (n, p)-eventually periodic with $n \leq \mathrm{depth} R - \mathrm{depth}_R M + 1$ and $p = 1$ or 2.*

(2) *M has bounded Betti numbers.*

Using [2, Lemma 1.2.6], one can check that any (n, p)-eventually periodic module M over a commutative local ring R satisfies that $\mathrm{depth} R - \mathrm{depth}_R M \leq n$. Thus, for any (n, p)-eventually periodic R-modules satisfying the assumption of Theorem 3, we obtain the following formula

$$\mathrm{depth} R - \mathrm{depth}_R M \leq n \leq \mathrm{depth} R - \mathrm{depth}_R M + 1. \tag{2.1}$$

3. Periodic dimensions

In this section, we will introduce the notion of the periodic dimension of a module and provide our main results. Throughout this section, let R denote a ring.

Observe that if M is a periodic module, then all its syzygies are periodic and have the same period as M. Thus it is natural to introduce the following notion.

Definition 4. Let M be an R-module. Then we define the *periodic dimension* of M by

$$\mathrm{per.dim}_R M := \inf \{ \, n \geq 0 \mid \Omega_R^n(M) \text{ is periodic} \, \}.$$

By definition, M is eventually periodic if and only if $\mathrm{per.dim}_R M < \infty$. In this case, $\mathrm{per.dim}_R M$ equals the degree n of the first periodic syzygy $\Omega_R^n(M)$ of M. For instance, if M has finite projective dimension, then $\mathrm{per.dim}_R M = \mathrm{proj.dim}_R M + 1$. Also, if M is of finite periodic dimension n, then we have

$$\mathrm{per.dim}_R \Omega_R^i(M) = \begin{cases} n - i & \text{if } 0 \leq i \leq n, \\ 0 & \text{if } i > n \, . \end{cases}$$

Recall from [1, 4] that an R-module X, where R is an arbitrary ring, is called *totally reflexive* if $X \cong X^{**}$ and $\mathrm{Ext}^i_R(X,R) = 0 = \mathrm{Ext}^i_{R^{\mathrm{op}}}(X^*,R)$ for all $i > 0$, where we set $(-)^* := \mathrm{Hom}_R(-,R)$. The *Gorenstein dimension* $\text{G-dim}_R M$ of an R-module M is defined to be the infimum of the length n of an exact sequence of R-modules

$$0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$

with each X_i totally reflexive. The following proposition states the property of periodic dimensions with respect to direct sums.

Proposition 5. *For any finite family $\{{}_R M_i\}_{i\in I}$ of R-modules, we have*

$$\mathrm{per.dim}_R \bigoplus_{i\in I} M_i \leq \sup\{\, \mathrm{per.dim}_R M_i \mid i \in I\,\}$$

The equality holds if R is left artin, and $\text{G-dim}_R M_i < \infty$ *for all $i \in I$.*

The following is our first main result.

Theorem 6. *Let M be an (n,p)-eventually periodic R-module of finite Gorenstein dimension r. Then we have $r \leq n \leq r+1$. If, furthermore, R is left artin, then the following assertions hold.*

(1) *$n = r$ if and only if $\Omega^r_R(M)$ has no non-zero projective direct summand.*
(2) *If $\Omega^{n-1}_R(M) = X \oplus Q$ for some R-module X without non-zero projective direct summand and some projective R-module Q, then $r = n-1$ if and only if $X \cong \Omega^{n+p-1}_R(M)$ as R-modules.*

Remark 7. (1) Let M be an (n,p)-eventually periodic R-module of finite complete intersection dimension, where R is a commutative local ring. Then, since we know from [3, Theorems 8.7 and 8.8] that $\mathrm{depth} R - \mathrm{depth}_R M = \text{G-dim}_R M$, the obtained bounds $r \leq n \leq r+1$ in this case are noting but (2.1).
(2) If R is a CM-finite Gorenstein artin algebra, then any R-modules satisfy the assumption of the theorem. Here, CM-finite [5] means that there are only finitely many pairwise non-isomorphic indecomposable totally reflexive R-modules, and Gorenstein [4] means that the injective dimension of R is finite as a left and as a right R-module.

In what follows, let Λ be a finite dimensional algebra over the filed k. We say that Λ is *eventually periodic* if $\Omega^n_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}(\Lambda)$ is eventually periodic as a $\Lambda\otimes_k\Lambda^{\mathrm{op}}$-module for some $n \geq 0$. In case $n = 0$, we call Λ a *periodic* algebra. The following is a result of Dotsenko-Gélinas-Tamaroff [9].

Theorem 8 ([9, the proof of Theorem 6.3])**.** *Let Λ be a monomial Gorenstein algebra. Then* $\mathrm{per.dim}_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}\Lambda$ *is finite and at most* $\mathrm{inj.dim}_\Lambda \Lambda + 1$*, where* $\mathrm{inj.dim}_\Lambda\Lambda$ *stands for the injective dimension of the regular Λ-module Λ.*

Motivated by the theorem, we first obtain the following observation.

Proposition 9. *The following statements hold for a finite dimensional algebra Λ.*

(1) *If Λ is eventually periodic, then* $\text{G-dim}_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}\Lambda < \infty$ *if and only if Λ is Gorenstein.*
(2) *If Λ is Gorenstein, then* $\text{G-dim}_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}\Lambda = \mathrm{inj.dim}_\Lambda\Lambda$.

As a consequence of Theorem 6, we then have the following second main result of this note.

Theorem 10. *Let Λ be a finite dimensional eventually periodic Gorenstein algebra. Then we have*

$$\mathrm{inj.dim}_{\Lambda}\Lambda \leq \mathrm{per.dim}_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}\Lambda \leq \mathrm{inj.dim}_{\Lambda}\Lambda + 1.$$

Moreover, $\mathrm{per.dim}_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}\Lambda = \mathrm{inj.dim}_{\Lambda}\Lambda$ *if and only if* $\Omega^{\mathrm{inj.dim}_{\Lambda}\Lambda}_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}(\Lambda)$ *has no non-zero projective direct summand.*

We end this section by explaining that the bounds given in the theorem are the best possible.

Proposition 11 ([12, Proposition 4.3])**.** *Let Λ and Γ be finite dimensional algebras. Assume that Λ is periodic and Γ has finite global dimension d. Then the tensor product $A = \Lambda\otimes_k\Gamma$ is a Gorenstein algebra with* $\mathrm{per.dim}_{A\otimes_k A^{\mathrm{op}}}A = \mathrm{inj.dim}_{A}A$.

Example 12. Let Λ be the finite dimensional monomial algebra given by the following quiver with relations:

$$\beta \circlearrowleft d \xrightarrow{\alpha_d} d-1 \xrightarrow{\alpha_{d-1}} \cdots \longrightarrow 1 \xrightarrow{\alpha_1} 0 \qquad \beta^2,\ \alpha_{i-1}\alpha_i \text{ for } 2\leq i\leq d.$$

A direct calculation shows that Λ is a Gorenstein algebra with $\mathrm{per.dim}_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}\Lambda = \mathrm{inj.dim}_{\Lambda}\Lambda + 1$.

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Department of general education
Tokyo Metropolitan College of Industrial Technology
8-17-1, Minamisenju, Arakawa-ku, Tokyo 116-8523 JAPAN
Email address: usui@metro-cit.ac.jp