QUASI-HEREDITARY STRUCTURES AND TILTING MODULES

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Abstract. In this article, we show that the set of IS-tilting modules is in one-to-one correspondence with the set of quasi-hereditary structures. Moreover, we give a characterization of a class of algebras such that all tilting modules are IS-tilting.

1. Introduction

In the seminal work ([1]) of Cline, Parshall and Scott, the notion of quasi-hereditary algebras was introduced to explore the representation theories of complex Lie algebras and algebraic groups. From the viewpoint of the representation theory of finite dimensional algebras, Dlab and Ringel intensively studied quasi-hereditary algebras (for example, [2, 4]). One of the important properties of quasi-hereditary algebras is the existence of tilting modules, called characteristic tilting modules ([4]). The aim of this article is to study quasi-hereditary algebras in terms of tilting theory.

Notation. Throughout this article, $\Bbbk$ is an algebraically closed filed, A is a basic finite dimensional $\Bbbk$-algebra and $\{e_x \mid x \in \Lambda\}$ is a complete set of primitive orthogonal idempotents of A. Let $\{S(x) := \operatorname{top} e_x A \mid x \in \Lambda\}$ be the set of representatives of isomorphism classes of simple A-modules. For $x \in \Lambda$, let $P(x)$ denote the projective cover of $S(x)$ and let $I(x)$ denote the injective envelop of $S(x)$. We write $\operatorname{mod} A$ for the category of finitely generated right A-modules. For a set $\mathcal{X}$ of indecomposable A-modules, let $\mathcal{F}(\mathcal{X})$ denote the full subcategory consisting of $M \in \operatorname{mod} A$ which admits a sequence of monomorphisms

$$0 = M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \to \cdots \xrightarrow{f_{l-1}} M_l = M$$

such that $\operatorname{Cok} f_i \in \mathcal{X}$ for all $0 \leq i \leq l-1$. For $M \in \operatorname{mod} A$, let $[M : S(x)]$ be the composition multiplicity of $S(x)$ in M.

2. Quasi-hereditary algebras

In this section, we recall the definition and a property of quasi-hereditary algebras.

Fix a partial order $\lhd$ on Λ. For $x \in \Lambda$, let $\Delta(x)$ be the standard A-module (that is, it is the maximal factor module of $P(x)$ such that each composition factor $S(y)$ satisfies $y \unlhd x$) and let $\nabla(x)$ be the costandard A-module (that is, it is the the maximal submodule of $I(x)$ such that each composition factor $S(y)$ satisfies $y \unlhd x$). Let $\Delta := \{\Delta(x) \mid x \in \Lambda\}$ be the set of standard modules and $\Delta(\rhd x) := \{\Delta(y) \mid y \rhd x\}$.

We recall the definition of quasi-hereditary algebras.

Definition 1 ([1])**.** Let $\lhd$ be a partial order on Λ. The pair $(A, \lhd)$ is called a *quasi-hereditary algebra* if it satisfies the following conditions.

The detailed version of this paper will be submitted for publication elsewhere.

(1) $[\Delta(x) : S(x)] = 1$ for each $x \in \Lambda$.
(2) There exists an exact sequence $0 \to K(x) \to P(x) \to \Delta(x) \to 0$ such that $K(x) \in \mathcal{F}(\Delta(\rhd x))$ for each $x \in \Lambda$.

We call a partial order $\lhd$ on Λ quasi-hereditary if $(A, \lhd)$ is a quasi-hereditary algebra.

Let $\lhd'$ be another partial order on Λ and $\{\Delta'(x) \mid x \in \Lambda\}$ the set of the standard modules with respect to $\lhd'$. We say that two partial orders $\lhd$ and $\lhd'$ on Λ are equivalent, and write $\lhd \sim \lhd'$, if $\Delta(x) = \Delta'(x)$ for all $x \in \Lambda$. Note that, for a refinement $\lhd'$ of quasi-hereditary partial order $\lhd$, the partial order $\lhd'$ is also quasi-hereditary and $\lhd' \sim \lhd$ by [2]. Denote by $[\lhd]$ an equivalence class of a quasi-hereditary partial order $\lhd$. Following [3], we call this class $[\lhd]$ a *quasi-hereditary structure* of A. Denote by $\mathsf{qhs}\, A$ the set of quasi-hereditary structures of A.

For $M \in \operatorname{mod} A$, let $\operatorname{add} M$ denote the subcategory of $\operatorname{mod} A$ whose objects are direct summands of finite direct sums of M. Recall that an A-module T is called a *tilting module* if the projective dimension of T is finite, $\operatorname{Ext}^i_A(T, T) = 0$ for all $i \geq 1$ and there exists an exact sequence

$$0 \to A \to T_0 \to T_1 \to \cdots \to T_n \to 0$$

such that all $T_i \in \operatorname{add} T$. A quasi-hereditary algebra admits a canonical tilting module as follows.

Proposition 2 ([4])**.** *Let $(A, \lhd)$ be a quasi-hereditary algebra. Then there exists a basic tilting A-module T such that* $\operatorname{add} T = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. *Moreover, there is an indecomposable decomposition $T = \bigoplus_{i \in \Lambda} T(x)$ such that*

$$[T(x) : S(x)] = 1, \text{ and } [T(x) : S(y)] \neq 0 \text{ implies } y \unlhd x.$$

We call the module T in Proposition 2 the *characteristic tilting module* associated to $(A, \lhd)$. When the partial order $\lhd$ needs to be clarified, we write $T_{\lhd} := T$.

3. Main results

In this section, we introduce the notion of IS-tilting modules and study their relationship with quasi-hereditary structures.

Let $(\Lambda, \lhd)$ be a partial ordered set. For $x \in \Lambda$, let

$$\Lambda_x := \{y \in \Lambda \mid y \ntriangleleft x\} \text{ and } \varepsilon_x := \sum_{y \in \Lambda_x} e_y.$$

To give a module theoretical interpretation of characteristic tilting modules, we introduce the notion of IS-tilting modules.

Definition 3. We call T an *IS-tilting module* if it satisfies the following conditions.

(1) T is a tilting A-module.
(2) There exist a partial order $\lhd$ and an indecomposable decomposition $T = \bigoplus_{x \in \Lambda} T(x)$ such that $T(x)\varepsilon_x \cong S(x)\varepsilon_x$ in $\operatorname{mod}(\varepsilon_x A \varepsilon_x)$ for all $x \in \Lambda$.

In this case, we write $(T, \lhd)$ to specify the associated partial order.

Assume that T admits another indecomposable decomposition $T = \bigoplus_{x\in\Lambda} T'(x)$ such that $T'(x)\varepsilon_x \cong S(x)\varepsilon_x$ in $\operatorname{mod}(\varepsilon_x A \varepsilon_x)$ for all $x \in \Lambda$. Then $T(x) \cong T'(x)$ holds for all $x \in \Lambda$. Hence we explicitly need not refer the exact labelling on its indecomposable direct summands.

Remark 4. For a partial order $\lhd$ and an indecomposable decomposition $T = \bigoplus_{x\in\Lambda} T(x)$, the pair $(T, \lhd)$ is an IS-tilting if and only if

$$[T(x) : S(x)] = 1 \text{ for all } x \in \Lambda, \text{ and } [T(x) : S(y)] \neq 0 \text{ implies } y \unlhd x.$$

Characteristic tilting modules are IS-tilting as follows.

Example 5. Let $(A, \lhd)$ be a quasi-hereditary algebra. Then the characteristic tilting module $T_\lhd$ is IS-tilting by Proposition 2 and Remark 4.

We give a concrete example of IS-tilting modules.

Example 6. Let $A = \Bbbk Q/I$ be the bound quiver algebra given by

$$Q: \quad 1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \xrightarrow{c} 3 \xrightarrow{d} 4 \quad \text{and} \quad I = \langle ac, ba \rangle.$$

Consider the module $T = S(1) \oplus P(1) \oplus (P(2)/\operatorname{rad}^2 P(2)) \oplus P(2)$. The Loewy structures of A and T are as follows.

$$A = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & & 4 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \end{smallmatrix}, \quad T = \begin{smallmatrix} 1 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & & 4 \end{smallmatrix}.$$

Let $(\Lambda, \lhd) = \{1 \lhd 2 \lhd 3 \lhd 4\}$. Then there is an indecomposable decomposition $T = \bigoplus_{x\in\Lambda} T(x)$, where

$$T(1) = \begin{smallmatrix} 1 \end{smallmatrix}, \quad T(2) = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}, \quad T(3) = \begin{smallmatrix} & 2 & \\ 1 & & 3 \end{smallmatrix}, \quad T(4) = \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & & 4 \end{smallmatrix},$$

so that $T(x)\varepsilon_x \cong S(x)\varepsilon_x$ for all $x \in \Lambda$. Thus $(T, \lhd)$ satisfies the condition (2) in Definition 3. Moreover, $T = \mu_X \mu_{P(3)\oplus P(4)}(A)$ is tilting, where $X = \operatorname{Cok}(P(3) \hookrightarrow P(2))$. Thus $(T, \lhd)$ is IS-tilting.

In Example 6, one can also check directly that $(A, \lhd)$ is quasi-hereditary. This turns out to be always true as we will show in Theorem 8.

Recall that a tilting A-module T is also a tilting (left) $\operatorname{End}_A(T)$-module. It turns out that the property of being an IS-tilting module is also inherited when transferring to the endomorphism ring side.

Proposition 7. *Let $(T, \lhd)$ be an IS-tilting A-module and $B := \operatorname{End}_A(T)$. Then $({}_BT, \lhd^{\mathrm{op}})$ is an IS-tilting B^{op}-module.*

An IS-tilting module yields a quasi-hereditary structure on the original algebra.

Theorem 8. *Let $(T, \lhd)$ be an IS-tilting module. Then $(A, \lhd)$ is a quasi-hereditary algebra.*

The following proposition tells us that IS-tilting modules are isomorphic to characteristic tilting modules.

Proposition 9. *Let $(T, \lhd)$ be an IS-tilting A-module. Then T is isomorphic to the characteristic tilting module $T_\lhd$ associated to a quasi-hereditary algebra $(A, \lhd)$. In particular, the following conditions are equivalent for IS-tilting modules $(T, \lhd), (T', \lhd')$.*

(1) $T \cong T'$.
(2) $T_{\lhd} \cong T_{\lhd'}$.
(3) $[\lhd] = [\lhd']$.

Let $\mathsf{IStilt}\, A$ denote the set of isomorphism classes of (basic) IS-tilting modules. By Proposition 9, an equivalence class $[\lhd]$ is uniquely determined by an IS-tilting module T. Thus we write such an equivalence class $[\lhd]_T$.

We provide a bijection between the set of IS-tilting modules and the set of quasi-hereditary structures.

Theorem 10. *Let A be a basic finite dimensional algebra over an algebraically closed filed. Then there exist mutually inverse bijections*

$$\mathsf{IStilt}\, A \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathsf{qhs}\, A\ ,$$

where $\varphi(T) := [\lhd]_T$ and $\psi([\lhd]) := (T_{\lhd}, \lhd)$. In particular, IS-tilting modules coincide with characteristic tilting modules.

Proof. By Theorem 8 and Proposition 9, the map φ is well-defined. By Example 5, the map ψ is well-defined. Since $\varphi\psi = 1$ is clear and $\psi\varphi = 1$ follows from Proposition 9, we obtain the assertion. □

We explore the class of algebras such that all tilting modules are IS-tilting. An algebra is called a *quadratic linear Nakayama algebra* if the quiver is a linearly oriented quiver of type $\mathbb{A}$ and the relations are generated by quadratic monomials.

Theorem 11. *Let A be a basic finite dimensional algebra. Then all tilting A-modules are IS-tilting if and only if A is a quadratic linear Nakayama algebra.*

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