CLIFFORD'S THEOREM IN WIDE SUBCATEGORIES

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Abstract. Clifford's theorem states that the restriction functor sends any simple modules to a semisimple modules. We formulate Clifford's theorem in wide subcategories satisfying a certain condition by using a result of Ringel which classifies wide subcategories via semibricks. As a corollary, it follows that the restriction functor sends any brick to a semibrick under some assumptions.

1. Introduction

In this note, we always assume that all modules are finitely generated right modules and all subcategories are full and closed under isomorphisms.

In representation theory of algebras, *wide subcategories* have been investigated in connection with torsion classes, ring epimorphisms [3], semibricks [4] and so on. A wide subcategory is a subcategory of an abelian category which is closed under taking extensions, kernels and cokernels, in other words, an exact abelian subcategory closed under taking extensions.

In representation theory of finite groups, *Clifford's theorem* given in [1] is one of the most important and fundamental results. It states that for any simple module S over a group algebra kG of a finite group G, the restriction $\mathsf{Res}S$ of S to any normal subgroup N of G is a semisimple kN-module.

In this note, we give a generalization of Clifford's theorem to a wide subcategory satisfying a certain condition using a result by Ringel [4].

2. Wide subcategories and semibricks

In this setion, we give definitions of wide subcategories and semibicks. Then we give the result by Ringel which classifies wide subcateogories by semibicks. Throughout this section, we denote by Λ a finite dimensional algebra over a field k and by $\mathsf{mod}\Lambda$ the category of finitely generated right Λ-modules.

Definition 1. A subcategory $\mathcal{W}$ of $\mathsf{mod}\Lambda$ is a *wide subcategory* if it satisfies the following conditions:

(1) $\mathcal{W}$ is *closed under extensions*, that is, for any short exact sequence

$$0 \to M_1 \to M \to M_2 \to 0$$

in $\mathsf{mod}\Lambda$, we have that $M_1, M_2 \in \mathcal{W}$ implies $M \in \mathcal{W}$.

(2) $\mathcal{W}$ is *closed under kernels and cokernels*, that is, for every morphism $\varphi\colon M_1 \to M_2$ with $M_1, M_2 \in \mathcal{W}$, we have $\mathsf{Ker}\varphi \in \mathcal{W}$ and $\mathsf{Coker}\varphi \in \mathcal{W}$.

The detailed version of this paper has been submitted for publication elsewhere.

A typical example of a wide subcategory of $\mathsf{mod}\Lambda$ is a *Serre subcategory*, a subcategory closed under taking extensions, quotients and submodules. Every wide subcategory $\mathcal{W}$ of $\mathsf{mod}\Lambda$ becomes an abelian category, and we always regard $\mathcal{W}$ as such.

Definition 2. (1) A Λ-module S is a *brick* if $\mathsf{End}_\Lambda(S)$ is a division k-algebra, that is, any nonzero endomorphism of S is an isomorphism.
(2) A set $\mathcal{S}$ of isoclasses of bricks in $\mathsf{mod}\Lambda$ is a *semibrick* in $\mathsf{mod}\Lambda$ if it satisfies the following: for any $S \neq S' \in \mathcal{S}$, we have $\mathsf{Hom}_\Lambda(S, S') = 0$.
(3) A Λ-module M is a *semibrick* if there is a semibrick $\{S_1, \dots, S_n\}$ in $\mathsf{mod}\Lambda$ such that $M \cong S_1^{a_1} \oplus \cdots \oplus S_n^{a_n}$ for some $a_i \in \mathbb{Z}_{>0}$.

Remark 3. (1) We do not regard a zero module as a brick.
(2) We regard the empty set as a semibrick.

It is easy to check the following implications:

$$\text{simple modules} \Rightarrow \text{bricks} \Rightarrow \text{indecomposable modules}.$$

An arbitrary set of isoclasses of simple Λ-modules is a semibrick since any morphism between pairwise non-isomrphic simple modules is zero by Schur's lemma. Therefore any semisimple module, that is a direct sum of simple modules, is a semibrick.

To end this section, we give a result by Ringel which states there are bijective correspondences between wide subcategories and semibricks.

For a semibrick $\mathcal{S}$ in $\mathsf{mod}\Lambda$, we define the subcategory $\mathsf{Filt}\mathcal{S}$ of $\mathsf{mod}\Lambda$ as follows: a module $X \in \mathsf{mod}\Lambda$ belongs to $\mathsf{Filt}\mathcal{S}$ if and only if there is a chain of submodules of X

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n-1} \subseteq X_n = X$$

such that for any $i \in \{1, \dots, n\}$, we have $X_i/X_{i-1} \cong S$ for some $S \in \mathcal{S}$. Note that $\mathsf{Filt}\mathcal{S}$ is the smallest subcategory of $\mathsf{mod}\Lambda$ which is closed under extensions and contains $\mathcal{S}$.

Since a wide subcategory $\mathcal{W}$ of $\mathsf{mod}\Lambda$ is an abelian category, we can consider simple objects in $\mathcal{W}$. We denote by $\mathsf{sim}\mathcal{W}$ the set of isoclasses of simple objects in $\mathcal{W}$.

Theorem 4. [4, 1.2] *There is a bijective correspondence between the following two sets:*

(1) *the set of semibricks in* $\mathsf{mod}\Lambda$*,*
(2) *the set of wide subcategories of* $\mathsf{mod}\Lambda$*.*

The map from (1) *to* (2) *is given by* $\mathcal{S} \mapsto \mathsf{Filt}\mathcal{S}$*. The converse is given by* $\mathcal{W} \mapsto \mathsf{sim}\mathcal{W}$*. Moreover the above restricts to the following two sets;*

(1) *the set of isoclasses of simple modules in* $\mathsf{mod}\Lambda$*,*
(2) *the set of Serre subcategories of* $\mathsf{mod}\Lambda$*.*

Thus Serre subcategories and simple modules are generalized to wide subcategories and bricks under the above correspondence as follows:

$$\begin{array}{rcl} \text{Serre subcategories} & \Rightarrow & \text{wide subcategories} \\ \text{simple modules} & \Rightarrow & \text{bricks} \\ \text{semisimple modules} & \Rightarrow & \text{semibricks} \end{array}$$

3. Clifford's theorem

In this section, we denote by G a finite group and by N a normal subgroup of G. We have the *induction functor* $\mathsf{Ind} = - \otimes_{kN} kG \colon \mathsf{mod}kN \to \mathsf{mod}kG$ and the *restriction functor* $\mathsf{Res} \colon \mathsf{mod}kG \to \mathsf{mod}kN$. These functors are exact functors, functors preserving any exact sequence.

Theorem 5 (Clifford's theorem)**.** [1] *Let S be a simple kG-module. Then its restriction $\mathsf{Res}S$ to N is a semisimple kN-module.*

In the previous section, we see that (semi)simple modules are generalized to (semi)bricks. Therefore we expect the following: for any brick S in $\mathsf{mod}kG$, we have that $\mathsf{Res}S$ is a semibrick in $\mathsf{mod}kN$. However this does not hold in general, see [2, Remark 3.16]. Then we consider a wide subcategory *stable under* $k[G/N] \otimes_k -$ and prove Clifford's theorem in the wide subcategory.

A natural surjection $G \twoheadrightarrow G/N$ induces a surjective k-algebra homorphism $kG \twoheadrightarrow k[G/N]$. Then we regard $k[G/N]$ as an kG-module via the algebra homomorphism.

Definition 6. A wide subcategory $\mathcal{W}$ of $\mathsf{mod}kG$ is *stable under* $k[G/N] \otimes_k -$ if for any $W \in \mathcal{W}$, we have $k[G/N] \otimes_k W \in \mathcal{W}$.

The kG-module $k[G/N]$ naturally appears in the isomophism of kG-modules:

$$\mathsf{Ind}(\mathsf{Res}X) \cong k[G/N] \otimes_k X. \tag{3.1}$$

Thanks to the above condition, we obtain the following:

Proposition 7. [2, Proposition 3.6] *Let $\mathcal{W}$ be a wide subcategory of $\mathsf{mod}kG$ stable under $k[G/N] \otimes_k -$. Then the following hold.*

1. *The subcategory*
$$\mathsf{Ind}^{-1}(\mathcal{W}) = \{X \in \mathsf{mod}kN \mid \mathsf{Ind}X \in \mathcal{W}\}$$
is a wide subcategory of $\mathsf{mod}kN$.
2. *The restriction functor Res induces an exact functor $\mathcal{W} \to \mathsf{Ind}^{-1}(\mathcal{W})$ between wide subcategories.*

Proof. (1) Since the induction functor Ind is an exact functor, it is easy to check that $\mathsf{Ind}^{-1}(\mathcal{W})$ is a wide subcategory of $\mathsf{mod}kN$.

(2) This follows easily from the isomorphism (3.1). □

The following is the main result of this note.

Theorem 8. [2, Theorem 3.8 (1)] *Let $\mathcal{W}$ be a wide subcategory of $\mathsf{mod}kG$ stable under $k[G/N] \otimes_k -$. Then for any simple object S in $\mathcal{W}$, the restriction $\mathsf{Res}S$ is a semisimple object in $\mathsf{Ind}^{-1}(\mathcal{W})$, in particular, a semibrick.*

We refer [2] to detailed statements of the above. Note that we recover Clifford's Theorem taking $\mathcal{W} = \mathsf{mod}kG$ in the above. The theorem says that the restriction functor Res sends any brick S obtained as a simple object of a wide subcategory stable under $k[G/N] \otimes_k -$ to a semibrick. Then it is natural to ask when bricks are obtained as simple objects of such a wide subcategory. As a suffitient condition, every brick is obtained by such a way when G/N is a p-group.

Lemma 9. [2, Lemma 3.12] *Suppose that k has a characteristic $p > 0$ and G/N is a p-group. Then every wide subcategory of* $\mathsf{mod}kG$ *is stable under* $k[G/N] \otimes_k -$.

Proof. Let $\mathcal{W}$ be a wide subcategory of $\mathsf{mod}kG$ and W an object of $\mathcal{W}$. Since G/N is a p-group, the only simple $k[G/N]$-module is the trivial module $k_{G/N}$. Then $k[G/N]$ has a composition series which has a trivial module as a only composition factor. Then $k[G/N] \otimes_k W$ is contained in $\mathsf{Filt}W$. Thus $k[G/N] \otimes_k W$ is contained in $\mathcal{W}$ since $\mathcal{W}$ is closed under extensions. □

Corollary 10. [2, Corollary 3.13] *Suppose that k has a characteristic $p > 0$ and G/N is a p-group. Then* $\mathsf{Res}S$ *is a semibrick for any brick S in* $\mathsf{mod}kG$.

Proof. Let S be a brick in $\mathsf{mod}kG$. Then S is a simple object in a wide subcategory $\mathsf{Filt}S$ of $\mathsf{mod}kG$ by Theorem 4. Since G/N is a p-group, $\mathsf{Filt}S$ is stable under $k[G/N] \otimes_k -$ by the previous lemma. Thus we can apply Theorem 8 to S. □

References

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