LIFTING OF RELATIVE STABLE EQUIVALENCES OF MORITA TYPE FOR BLOCKS OF FINITE GROUPS

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Abstract. Wang-Zhang introduced the notion of relative stable equivalence of Morita type. Using this notion, we give a method of constructing Morita equivalences for the principal blocks of finite groups with a common central p-subgroup. We also give Morita equivalences for the principal 2-blocks of infinite series of 2-dimensional general linear groups $\mathrm{GL}_2(q)$ with wreathed Sylow 2-subgroups using this method.

1. Introduction

Let G be a finite group, and k a filed of characteristic $p > 0$. We can decompose kG as a direct product of indecomposable k-algebras:

$$kG = B_1 \times \cdots \times B_n.$$

Each B_i is called a *block* of G. We write k_G for the *trivial kG-module*, that is, a one-dimensional k-vector space on which every element of G acts trivially. There is a unique block B of G such that $k_G B = k_G$, called the *principal block* of G and denoted by $B_0(G)$.

For a block B of G, a defect group of B is a minimal p-subgroup D of G such that

$$B \otimes_{kD} B \to B : x \otimes y \to xy$$

is split as a homomorphism of B-B-bimodules, and D is determined up to G-conjugacy. A defect group of the principal block of G is a Sylow p-subgroup of G. Donovan conjecture states that, for a given p-subgroup P, there are only finitely many Morita equivalence classes of blocks of finite groups with defect groups isomorphic to P. In particular, we consider Morita equivalences for principal blocks.

Broué [1] introduced the notion of stable equivalence of Morita type. Morita equivalences for principal blocks have been constructed by lifting stable equivalences of Morita type. However, we cannot use this method when finite groups have a common nontrivial central p-subgroup. On the other hand, Wang and Zhang [12] introduced the notion of relative stable equivalence of Morita type for blocks of finite groups. In this report, we give another method of constructing Morita equivalences using relative stable equivalences of Morita type, which is applicable to finite groups having a common nontrivial central p-subgroup (see Section 3). We also give Morita equivalences for the principal 2-blocks of infinite series of 2-dimensional general linear groups $\mathrm{GL}_2(q)$ with wreathed Sylow 2-subgroups using this method (see Section 4).

Throughout this report, we assume that k is an algebraically closed field of characteristic $p > 0$, G is a finite group, and unless otherwise stated, modules are finitely generated right modules. We write $Z(G)$ for the center of G, $\Delta G = \{(g, g) | g \in G\}$, and U^* for the

See [5], [6] for the detailed version of this paper.

k-dual of a module U. For a p-subgroup P of G, the fusion system of G over P is the category $\mathcal{F}_P(G)$ whose objects are all subgroups of P, and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{F}_P(G)}(Q,R) = \{\varphi \in \mathrm{Hom}(Q,R) \mid \varphi = c_g \text{ for some } g \in G\},$$

where c_g is the conjugation map.

2. Stable equivalences of Morita type

In this section, we explain a method of constructing Morita equivalences for principal blocks.

Broué [1] introduced the notion of stable equivalence of Morita type.

Definition 1 (see [1])**.** Let Γ and Λ be finite dimensional algebras. Let M be a Γ-Λ-bimodule, and N a Λ-Γ-bimodule. We say that (M,N) induces a stable equivalence of Morita type between Γ and Λ if M and N are finitely generated and projective as left modules and right modules with the property that there are isomorphisms of bimodules

$$M \otimes_\Lambda N \cong \Gamma \oplus X \quad\text{and}\quad N \otimes_\Gamma M \cong \Lambda \oplus Y,$$

where X and Y are projective as bimodules.

Let Q be a p-subgroup of G. In any decomposition into the direct sum of indecomposable modules, $k_Q \otimes_{kQ} kG$ has a unique indecomposable summand having k_G as a submodule. This indecomposable summand is called the *Scott kG-module* with respect to Q, and denoted by $S(G,Q)$. For a kG-module M, the *Brauer construction* $M(Q)$ of M with respect to Q is the $kN_G(Q)$-module defined as follows:

$$M(Q) = M^Q / \sum_R \mathrm{tr}_R^Q(M^R),$$

where R runs over the set of proper subgroups of Q, M^Q is the set of fixed points of Q in M, and $\mathrm{tr}_R^Q : M^R \to M^Q$ is a linear map given by $\mathrm{tr}_R^Q(m) = \sum_{t\in[R\backslash Q]} mt$.

Broué [1] gave a method of constructing stable equivalences of Morita type for principal blocks.

Theorem 2. (see [1, Theorem 6.3]) *Let G and G' be finite groups with a common Sylow p-subgroup P such that $\mathcal{F}_P(G) = \mathcal{F}_P(G')$, and $M = S(G \times G', \Delta P)$. Then the following are equivalent.*

1. *For any nontrivial subgroup Q of P, the pair $(M(\Delta Q), M(\Delta Q)^*)$ induces a Morita equivalence between $B_0(C_G(Q))$ and $B_0(C_{G'}(Q))$.*
2. *(M, M^*) induces a stable equivalence of Morita type between $B_0(G)$ and $B_0(G')$.*

Linckelmann gave an equivalent condition for stable equivalences of Morita type between indecomposable selfinjective algebras to be in fact Morita equivalences.

Theorem 3. (see [7, Theorem 2.1]) *Let Γ and Λ be indecomposable nonsimple selfinjective k-algebras. Let M be a Γ-Λ-bimodule, and N a Λ-Γ-bimodule such that (M,N) induces a stable equivalence of Morita type between Γ and Λ. Then the following hold:*

1. *If M is indecomposable then for any simple Γ-module S, the Λ-module $S \otimes_B M$ is indecomposable.*

(2) *The pair (M, N) induces a Morita equivalence between Γ and Λ if and only if for any simple Γ-module S, the Λ-module $S \otimes_\Gamma M$ is simple.*

We may construct a stable equivalence of Morita type by Theorem 2 and lift it to a Morita equivalence by Theorem 3. In this way, Morita equivalences have been constructed in some cases (see for example [9], [2], and [4]). However we cannot use Theorem 2 when G and G' have a common nontrivial central p-subgroup Z as $C_G(Z) = G$ and $C_{G'}(Z) = G'$.

3. Relative stable equivalences of Morita type

In this section, we give another method of constructing Morita equivalences that is applicable to finite groups with a common nontrivial central p-subgroup.

Let H be a subgroup of G. We say that a kG-module U is relatively H-projective if there exists a kH-module V such that U is a direct summand of $V \otimes_{kH} kG$.

Wang-Zhang [12] introduced the notion of relative stable equivalence of Morita type.

Definition 4 (see [12, Definition 5.1])**.** Let G and G' be finite groups with a common p-subgroup P. Let B and B' be blocks of finite groups G and G', respectively. Let M be a B-B'-bimodule M and N a B'-B-bimodule. For a subgroup Q of P, we say that (M, N) induces a *relative Q-stable equivalence of Morita type* between B and B' if M and N are finitely generated and projective as left modules and right modules with the property that there are isomorphisms of bimodules

$$M \otimes_{B'} N \cong B \oplus X \quad \text{and} \quad N \otimes_B M \cong B' \oplus Y,$$

where X is $Q \times Q$-projective as a $k[G \times G]$-module and Y is $Q \times Q$-projective as a $k[G' \times G']$-module.

Remark 5. In [12, Definition 5.1], relative stable equivalence of Morita type is in fact defined using the notion of projectivity relative to modules introduced by Okuyama [8].

For finite groups G and G' having a common central p-subgroup Z, we give a method of constructing relative Z-stable equivalence between $B_0(G)$ and $B_0(G')$.

Main Result 6 (see [5])**.** *Let G and G' be finite groups with a common Sylow p-subgroup P such that $\mathcal{F}_P(G) = \mathcal{F}_P(G')$, and $M = S(G \times G', \Delta P)$. Assume that Z is a subgroup of P and contained in $Z(G)$ and $Z(G')$. Then the following are equivalent.*

(1) *For any subgroup Q of P properly containing Z, the pair $(M(\Delta Q), M(\Delta Q)^*)$ induces a Morita equivalence between $B_0(C_G(Q))$ and $B_0(C_{G'}(Q))$.*
(2) *(M, M^*) induces a relative Z-stable equivalence of Morita type between $B_0(G)$ and $B_0(G')$.*

We give an equivalent condition for relative Z-stable equivalences of Morita type to be Morita equivalences.

Main Result 7 (see [5])**.** *Let G and G' be finite groups with a common nontrivial Sylow p-subgroup P such that $\mathcal{F}_P(G) = \mathcal{F}_P(G')$, and $M = S(G \times G', \Delta P)$. Assume that Z is a proper subgroup of P and contained in $Z(G)$ and $Z(G')$. If (M, M^*) induces a relative Z-stable equivalence of Morita type between $B_0(G)$ and $B_0(G')$, then the following hold.*

(1) *For any simple $B_0(G)$-module S, the $B_0(G')$-module $S \otimes_{B_0(G)} M$ is indecomposable.*

(2) *The pair* (M, M^*) *induces a Morita equivalence between* $B_0(G)$ *and* $B_0(G')$ *if and only if for any simple* $B_0(G)$*-module* S*, the* $B_0(G')$*-module* $S \otimes_{B_0(G)} M$ *is simple.*

We may construct a relative Z-stable equivalence of Morita type by Main Result 6 and lift it to a Morita equivalence by Main Result 7. This method is applicable to finite groups with a common nontrivial central p-subgroup.

4. Application

In this section, we give splendid Morita equivalences for principal 2-blocks of 2-dimensional general linear groups $\mathrm{GL}_2(q)$.

For blocks B and B' of finite groups G and G', a Morita equivalence between B and B' is said to be *splendid* if it is induced by a B-B'-bimodule M that is a p-permutation module as an $k[G \times G']$-module. The methods stated in Section 2, 3 give in fact splendid Morita equivalences since Scott modules are p-permutation modules.

Puig's conjecture states that, for a given finite p-group P, there are only finitely many isomorphism classes of interior P-algebras arising as source algebras of p-blocks of finite groups with defect groups isomorphic to P (see [11, Conjecture 38.5]). This is equivalent to saying that there are only finitely many splendid Morita equivalence classes of p-blocks of finite groups with defect groups isomorphic to P.

Splendid Morita equivalence classes of principal 2-blocks of tame representation type have been classified (see [2], [3], and [4]). In the classifications, it was shown that there are only finitely many splendid Morita equivalence classes of the principal blocks arising from infinite series of finite groups. Thus the classifications imply that Puig's conjecture holds for the principal 2-blocks of tame representation type.

We consider splendid Morita equivalences of the principal blocks of 2-dimensional general linear groups. Let $G_i = \mathrm{GL}_2(q_i)$, $i = 1, 2$. If $(q_1 \pm 1)_p = (q_2 \pm 1)_p = p^s$, then G_1 and G_2 have a common Sylow p-subgroup P, where $(q_i \pm 1)_p$ is the p-part of $(q_i \pm 1)$, $s \geq 1$ if s is odd, and $s \geq 2$ if $p = 2$ (see Table 1).

Table 1

	$s \geq 2$	P	
p: odd	$(q_i + 1)_p = p^{s-1}$	$C_{p^{s-1}}$	(1)
	$(q_i - 1)_p = p^{s-1}$	$C_{p^{s-1}} \times C_{p^{s-1}}$	(2)
$p = 2$	$(q_i + 1)_2 = 2^s$	$SD_{2^{s+2}}$	(3)
	$(q_i - 1)_2 = 2^s$	$C_{2^s} \wr C_2$	(4)

In Table 1, $C_{p^{s-1}}$ is a cyclic group of order p^{s-1}, and $SD_{2^{s+2}}$ is a semidihedral group of order 2^{s+2}.

We get a splendid Morita equivalence between $B_0(G_1)$ and $B_0(G_2)$ by the general theory of cyclic defect blocks in case (1), and by the result of Puig [10] in case (2). Case (3) was constructed in [4]. However, for case (4), we cannot use Puig's result and also the method stated in Section 2 since G_1 and G_2 have a common nontrivial central 2-subgroup.

We get a splendid Morita equivalence in case (4) by the method stated in Section 3.

Main Result 8 (see [6]). *Let k be an algebraically closed field of characteristic 2. If q_1 and q_2 are odd prime powers with $q_1 \equiv q_2 \equiv 1 \pmod 4$ and $(q_1 - 1)_2 = (q_2 - 1)_2$, then $B_0(\mathrm{GL}_2(q_1))$ and $B_0(\mathrm{GL}_2(q_2))$ are splendidly Morita equivalent.*

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