CLASSIFICATION OF LOCALLY FREE SHEAF BIMODULES OF RANK 2 OVER A PROJECTIVE LINE

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Abstract. Classification of noncommutative integral surfaces is one of the fundamental projects since the foundation of noncommutative algebraic geometry. Artin [1] conjectured that every noncommutative integral surface is birationally equivalent to (i) a noncommutative projective plane, (ii) a noncommutative ruled surface, or (iii) a noncommutative surface finite over its center. Since noncommutative projective planes were classified by Artin, Tate and Van den Bergh [3], the next step is to classify noncommutative ruled surfaces. In this paper, we will classify noncommutative Hirzebruch surfaces, which are defined to be noncommutative ruled surfaces over a projective line. To do this, it is essential to classify locally free sheaf bimodules of rank 2 over a projetive line.

1. Introduction

Throughout this paper, let k be an algebraically closed field of characteristic 0. For a right noetherian graded algebra A, we denote by $\mathbf{grmod}A$ the category of finitely generated graded right A-modules. The Gelfand-Kirillov dimension of $M \in \mathbf{grmod}A$ is defined by

$$\mathrm{GKdim} M := \min\left\{ d \in \mathbb{N} \;\middle|\; \lim_{n\to\infty} \frac{1}{n^d} \sum_{i=0}^{n} \dim_k M_i < \infty \right\}.$$

For $i \in \mathbb{N}$, we set $(\mathbf{grmod}A)_i := \{M \in \mathbf{grmod}A \mid \mathrm{GKdim} M \leq i\}$, which is a Serre subcategory of $\mathbf{grmod}A$. Artin and Zhang [4] defined the *noncommutative projective scheme* associated to A by the quotient category $\mathbf{Proj}_{\mathrm{nc}}A := \mathbf{grmod}A/(\mathbf{grmod}A)_0$. If $\mathrm{GKdim}A = 3$, then we call $\mathbf{Proj}_{\mathrm{nc}}A$ a *noncommutative surface*. Furthermore, if $\mathrm{GKdim}A = 3$ and A is a domain, then we call $\mathbf{Proj}_{\mathrm{nc}}A$ a *noncommutative integral surface*. For right noetherian graded algebras A and B, we say that $\mathbf{Proj}_{\mathrm{nc}}A$ and $\mathbf{Proj}_{\mathrm{nc}}B$ are *birationally equivalent* if

(1) $\mathrm{GKdim}A = \mathrm{GKdim}B = d$, and
(2) $\mathbf{grmod}A/(\mathbf{grmod}A)_{d-1} \cong \mathbf{grmod}B/(\mathbf{grmod}B)_{d-1}$.

Classification of noncommutative integral curves was completed by Artin and Stafford [2], however, classification of noncommutative integral surfaces is still open. In 1997, Artin [1] conjectured that every noncommutative integral surface is birationally equivalent to one of the following:

(1) a noncommutative projective plane.
(2) a noncommutative ruled surface.
(3) a noncommutative surface finite over its center.

The detailed version of this paper will be submitted for publication elsewhere.

Although this conjecture is still open, it is importatnt to classify each of the above three classes of noncommutative surfaces. Since noncommutative projective planes were classified by Artin, Tate and Van den Bergh [3], the next step is to classify noncommutative ruled surfaces.

In this paper, we classify noncommutative Hirzebruch surfaces, which are defined to be noncommutative ruled surfaces over $\mathbb{P}^1$. In section 2, we review the definition of a commutative $\mathbb{P}^1$-bundle. In section 3, we define a noncommutative $\mathbb{P}^1$-bundle. In section 4, we classify locally free sheaf bimodules $\mathcal{E}$ of rank 2 over $\mathbb{P}^1$, which is essential to classify noncommutative Hirzebruch surfaces.

2. COMMUTATIVE $\mathbb{P}^1$-BUNDLES (Review)

Let X be a scheme. We denote by $\mathbf{Mod}X$ the category of quasi-coherent sheaves on X and by $\mathbf{mod}X$ the category of coherent sheaves on X.

Definition 1. Let X be a scheme.

(1) Z is a $\mathbb{P}^1$*-bundle over* X if there exists a locally free sheaf $\mathcal{E} \in \mathbf{mod}X$ of rank 2 such that $Z \cong \mathbb{P}_X(\mathcal{E}) := \mathbf{Proj}\mathcal{S}_X(\mathcal{E})$ where $\mathcal{S}_X(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$ over X.
(2) Z is a *ruled surface* if Z is a $\mathbb{P}^1$-bundle over a curve X.
(3) Z is a *Hirzebruch surface* if Z is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

Example 2. If $\mathcal{E} \cong \mathcal{O}_X \oplus \mathcal{O}_X$, then

$$\mathbb{P}_X(\mathcal{E}) := \mathbf{Proj}\mathcal{S}_X(\mathcal{O}_X \oplus \mathcal{O}_X) = \mathbf{Proj}\mathcal{O}_X[x, y] \cong X \times \mathbb{P}^1.$$

Example 3. The only locally free sheaf of rank 2 on $X = \mathbf{Spec}k$ is k^2, so the only $\mathbb{P}^1$-bundle over $\mathbf{Spec}k$ is

$$\mathbb{P}_k(k^2) = \mathbf{Proj}S_k(k^2) \cong \mathbf{Proj}k[x, y] \cong \mathbb{P}^1 \cong \mathbf{Proj}_{\mathrm{nc}}\Pi(\cdot \rightrightarrows \cdot)$$

where $\Pi(\cdot \rightrightarrows \cdot)$ is the preprojective algebra of the 2-Kronecker quiver.

Recall that $\mathcal{L} \in \mathbf{mod}X$ is *invertible* if $- \otimes_X \mathcal{L} : \mathbf{Mod}X \to \mathbf{Mod}X$ is an autoequivalence. We denote by $\mathrm{Pic}X$ the group of invertible sheaves on X.

Theorem 4. ([5]) *Let* $\mathcal{E}, \mathcal{E}' \in \mathbf{mod}X$ *be locally free of rank 2. Then there exists* $\mathcal{L} \in \mathrm{Pic}X$ *such that* $\mathcal{E}' \cong \mathcal{E} \otimes_X \mathcal{L}$ *if and only if* $\mathbb{P}_X(\mathcal{E}) \cong \mathbb{P}_X(\mathcal{E}')$.

Since $\mathrm{Pic}\,\mathbb{P}^1 = \{\mathcal{O}_{\mathbb{P}^1}(a) \mid a \in \mathbb{Z}\}$, and $\mathcal{E} \in \mathbf{mod}\,\mathbb{P}^1$ is locally free of rank 2 if and only if $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ for some $a, b \in \mathbb{Z}$, every Hirzebruch surface Z is isomorphic to $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$ for some non-negative integer $d \in \mathbb{N}$ by the above theorem. We call $\mathbb{F}_d := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$ the Hirzebruch surface of degree d.

3. NONCOMMUTATIVE $\mathbb{P}^1$-BUNDLES

3.1. Sheaf bimodules and sheaf $\mathbb{Z}$-algebras.

Definition 5. Let X and Y be schemes.

(1) A coherent sheaf $\mathcal{E}$ on $X \times Y$ is called a *sheaf X-Y bimodule* if the scheme-theoretic support

$$W := \operatorname{Spec}_{X\times Y} \operatorname{Im}(\mathcal{O}_{X\times Y} \to \mathcal{E}nd_{X\times Y}\,\mathcal{E})$$

is finite over both X and Y.

(2) An adjoint pair of functors between $\mathbf{Mod}X$ and $\mathbf{Mod}Y$ is called an *X-Y bimodule.*

We denote by $\mathbf{bimod}(X\text{-}Y)$ the category of sheaf X-Y bimodules and by $\mathbf{BiMod}(X\text{-}Y)$ the category of X-Y bimodules.

Theorem 6. ([9]) *There exists a fully faithful functor*

$$\begin{array}{ccc} \mathbf{bimod}(X\text{-}Y) & \longrightarrow & \mathbf{BiMod}(X\text{-}Y) \\ \mathcal{E} & \mapsto & (-\otimes_X \mathcal{E}, \mathcal{H}om_Y(\mathcal{E}, -)) \end{array}$$

where $\mathrm{pr}_1 : X \times Y \to X$ *and* $\mathrm{pr}_2 : X \times Y \to Y$ *are projections, and*

$$-\otimes_X \mathcal{E} := \mathrm{pr}_{2*}(\mathrm{pr}_1^*(-) \otimes_{X\times Y} \mathcal{E})$$

(Fourier-Mukai transform).

Definition 7. A *sheaf $\mathbb{Z}$-algebra*

$$\mathcal{A} = ((\mathcal{A}_{ij})_{i,j\in\mathbb{Z}}, (\eta_i)_{i\in\mathbb{Z}}, (\mu_{ijk})_{i,j,k\in\mathbb{Z}})$$

over a scheme X consists of

(1) sheaf bimodules $\mathcal{A}_{ij} \in \mathbf{bimod}(X\text{-}X)$,

(2) morphisms $\eta_i : \mathcal{O}_{\Delta_X} \to \mathcal{A}_{ii}$ of sheaf bimodules called the *units* where

$$\Delta_X := \{(p,p) \in X \times X \mid p \in X\},$$

and

(3) morphisms $\mu_{ijk} : \mathcal{A}_{ij} \otimes_X \mathcal{A}_{jk} \to \mathcal{A}_{ik}$ called *multiplication maps*

satisfying the following two axioms:

(i) the compositions

$$\mathcal{A}_{ij} \xrightarrow{\sim} \mathcal{A}_{ij} \otimes_X \mathcal{O}_{\Delta_X} \xrightarrow{\mathrm{id}\otimes\eta_j} \mathcal{A}_{ij} \otimes_X \mathcal{A}_{jj} \xrightarrow{\mu_{ijj}} \mathcal{A}_{ij}$$

and

$$\mathcal{A}_{ij} \xrightarrow{\sim} \mathcal{O}_{\Delta_X} \otimes_X \mathcal{A}_{ij} \xrightarrow{\eta_i\otimes\mathrm{id}} \mathcal{A}_{ii} \otimes_X \mathcal{A}_{ij} \xrightarrow{\mu_{iij}} \mathcal{A}_{ij}$$

are the identity morphisms, and

(ii) the following diagrams

$$\begin{array}{ccc} \mathcal{A}_{ij} \otimes_X \mathcal{A}_{jk} \otimes_X \mathcal{A}_{kl} & \xrightarrow{\mu_{ijk}\otimes\mathrm{id}} & \mathcal{A}_{ik} \otimes_X \mathcal{A}_{kl} \\ {\scriptstyle \mathrm{id}\otimes\mu_{jkl}}\downarrow & & \downarrow{\scriptstyle \mu_{ikl}} \\ \mathcal{A}_{ij} \otimes_X \mathcal{A}_{jl} & \xrightarrow{\mu_{ijl}} & \mathcal{A}_{il} \end{array}$$

are commutative.

A *graded right module* $\mathcal{M} = ((\mathcal{M}_i)_{i\in\mathbb{Z}}, (h_{ij})_{i,j\in\mathbb{Z}})$ over a sheaf $\mathbb{Z}$-algebra $\mathcal{A}$ on a scheme X consists of

(1) sheaves $\mathcal{M}_i \in \mathbf{mod}X$, and

(2) morphisms $h_{ij} : \mathcal{M}_i \otimes_X \mathcal{A}_{ij} \to \mathcal{M}_j$ of sheaves called the *action*

satisfying the following two axioms:

(i) the composition

$$\mathcal{M}_i \xrightarrow{\sim} \mathcal{M}_i \otimes_X \mathcal{O}_{\Delta_X} \xrightarrow{\mathrm{id}\otimes\eta_i} \mathcal{M}_i \otimes_X \mathcal{A}_{ii} \xrightarrow{h_{ii}} \mathcal{M}_i$$

is the identity morphism, and

(ii) the following diagrams

$$\begin{array}{ccc} \mathcal{M}_i \otimes_X \mathcal{A}_{ij} \otimes_X \mathcal{A}_{jk} & \xrightarrow{h_{ij}\otimes\mathrm{id}} & \mathcal{M}_j \otimes_X \mathcal{A}_{jk} \\ {\scriptstyle \mathrm{id}\otimes\mu_{ijk}}\downarrow & & \downarrow{\scriptstyle h_{jk}} \\ \mathcal{M}_i \otimes_X \mathcal{A}_{ik} & \xrightarrow{h_{ik}} & \mathcal{M}_k \end{array}$$

are commutative.

A *morphism* $f = (f_i)_{i\in\mathbb{Z}} : \mathcal{M} \to \mathcal{N}$ of graded right $\mathcal{A}$-modules consists of morphisms $f_i : \mathcal{M}_i \to \mathcal{N}_i$ of sheaves such that the diagrams

$$\begin{array}{ccc} \mathcal{M}_i \otimes_X \mathcal{A}_{ij} & \xrightarrow{f_i\otimes\mathrm{id}} & \mathcal{N}_i \otimes_X \mathcal{A}_{ij} \\ {\scriptstyle h^{\mathcal{M}}_{ij}}\downarrow & & \downarrow{\scriptstyle h^{\mathcal{N}}_{ij}} \\ \mathcal{M}_j & \xrightarrow{f_j} & \mathcal{N}_j \end{array}$$

are commutative.

For a right noetherian sheaf $\mathbb{Z}$-algebra $\mathcal{A}$, we denote by $\mathbf{grmod}\mathcal{A}$ the category of finitely generated graded right $\mathcal{A}$-modules. A graded right $\mathcal{A}$-module $M \in \mathbf{grmod}\mathcal{A}$ is called *right-bounded* if $\mathcal{M}_i = 0$ for $i \gg 0$. We denote by $\mathbf{tors}\mathcal{A}$ the full subcategory of $\mathbf{grmod}\mathcal{A}$ consisting of right bounded modules. We define the *noncommutative projective scheme* associated to $\mathcal{A}$ by the quotient category $\mathbf{Proj}_{\mathrm{nc}}\mathcal{A} := \mathbf{grmod}\mathcal{A}/\mathbf{tors}\mathcal{A}$.

3.2. **Noncommutative symmetric algebras and noncommutative $\mathbb{P}^1$-bundles.** Let X, Y be schemes, and $\mathrm{pr}_1 : X \times Y \to X, \mathrm{pr}_2 : X \times Y \to Y$ projections. A sheaf X-Y bimodule $\mathcal{E}$ is called *locally free of rank* r if both $\mathrm{pr}_{1*}\mathcal{E} \in \mathbf{mod}X$ and $\mathrm{pr}_{2*}\mathcal{E} \in \mathbf{mod}Y$ are locally free of rank r.

If $\mathcal{E} \in \mathbf{bimod}(X\text{-}X)$ is a locally free sheaf bimodule of rank 2 on a smooth projective scheme X of dimension n, then

$$\mathcal{E}^* := \mathrm{pr}_2^*\omega_X^{-1} \otimes_{X\times X} \mathcal{E}^D$$
$${}^*\mathcal{E} := \mathcal{E}^D \otimes_{X\times X} \mathrm{pr}_1^*\omega_X^{-1}$$

are also locally free sheaf bimodules of rank 2 on X where

$$\mathcal{E}^D := \mathcal{E}xt^n_{X\times X}(\mathcal{E}, \omega_{X\times X}),$$

so we define a sequence $(\mathcal{E}^{*i})_{i\in\mathbb{Z}}$ of sheaf bimodules of rank 2 on X inductively by

$$\mathcal{E}^{*i} := \begin{cases} (\mathcal{E}^{*(i-1)})^* & i \geq 1, \\ \mathcal{E} & i = 0, \\ {}^*({}^{*(i+1)}\mathcal{E}) & i \leq -1. \end{cases}$$

Since $(- \otimes_X \mathcal{E}^{*i}, - \otimes_X \mathcal{E}^{*(i+1)})$ is an adjoint pair of fucntors for each i, we can define a morphism

$$\phi_i : \mathcal{O}_{\Delta_X} \to \mathcal{E}^{*i} \otimes_X \mathcal{E}^{*(i+1)}$$

to be the canonical morphism coming from the adjunction

$$\mathrm{Hom}(\mathcal{E}^{*i}, \mathcal{E}^{*i}) \cong \mathrm{Hom}(\mathcal{O}_{\Delta_X}, \mathcal{E}^{*i} \otimes_X \mathcal{E}^{*(i+1)})$$

in $\mathbf{bimod}(X\text{-}X)$. Let $\mathcal{Q}_i \subset \mathcal{E}^{*i} \otimes_{\mathcal{O}_X} \mathcal{E}^{*(i+1)}$ be the image of ϕ_i.

Definition 8. The *noncommutative symmetric algebra* of $\mathcal{E}$ over X is the sheaf $\mathbb{Z}$-algebra $\mathcal{S}_X(\mathcal{E})$ on X generated by $\mathcal{E}^{*i}$ subject to the relations $\mathcal{Q}_i$. To be more explicit, it is a sheaf $\mathbb{Z}$-algebra with

$$\mathcal{S}(\mathcal{E})_{ij} = \begin{cases} 0 & i > j, \\ \mathcal{O}_{\Delta_X} & i = j, \\ \mathcal{E}^{*i} & j = i+1, \\ (\mathcal{E}^{*i} \otimes_X \cdots \otimes_X \mathcal{E}^{*(j-1)})/\mathcal{R}_{ij} & j > i+1, \end{cases}$$

where

$$\mathcal{R}_{ij} := \sum_{k=i}^{j-2} \mathcal{E}^{*i} \otimes_X \cdots \otimes_X \mathcal{E}^{*(k-1)} \otimes_X \mathcal{Q}_k \otimes_X \mathcal{E}^{*(k+2)} \otimes_X \cdots \otimes_X \mathcal{E}^{*(j-1)}.$$

Now we are ready to define a noncommutative $\mathbb{P}^1$-bundle.

Definition 9. Let X be a smooth projective scheme.

(1) Z is a *noncommutative* $\mathbb{P}^1$*-bundle over* X if there exists a locally free sheaf bimodule $\mathcal{E} \in \mathbf{bimod}(X\text{-}X)$ of rank 2 such that $Z \cong \mathbb{P}_X(\mathcal{E}) := \mathbf{Proj}_{\mathrm{nc}}\mathcal{S}_X(\mathcal{E})$.
(2) Z is a *noncommutative ruled surface* if Z is a noncommutative $\mathbb{P}^1$-bundle over a curve X.
(3) Z is a *noncommutative Hirzebruch surface* if Z is a noncommutative $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

We say that $\mathcal{L} \in \mathbf{bimod}(X\text{-}X)$ is *invertible* if $- \otimes_X \mathcal{L} : \mathbf{Mod}X \to \mathbf{Mod}X$ is an autoequivalence.

Theorem 10. ([6]) *Let $\mathcal{E}, \mathcal{E}' \in \mathbf{bimod}(X\text{-}X)$ be locally free sheaf bimodules of rank 2. If there exist invertible sheaf bimodules $\mathcal{L}_1, \mathcal{L}_2 \in \mathbf{bimod}(X\text{-}X)$ such that $\mathcal{E}' \cong \mathcal{L}_1 \otimes_X \mathcal{E} \otimes_X \mathcal{L}_2$, then $\mathbf{P}_X(\mathcal{E}) \cong \mathbf{P}_X(\mathcal{E}')$.*

By the above theorem, to classify noncommutative $\mathbb{P}^1$-bundles over X, it is enough to classify locally free sheaf bimodules $\mathcal{E} \in \mathbf{bimod}(X\text{-}X)$ of rank 2 up to tensoring with invertible bimodules on both sides.

4. CLASSIFICATION OF NONCOMMUTATIVE HIRZEBRUCH SURFACES

In this section, we fix the following notations:

(1) X and Y are smooth projective schemes of the same dimension.
(2) $\mathcal{E} \in \mathbf{bimod}(X\text{-}Y)$ is a locally free sheaf bimodule of rank 2.

(3) $\iota : W := \mathrm{Supp}\,\mathcal{E} \to X \times Y$ is the embedding.

We denote by $\mathrm{CM}(W)$ the set of isomorphism classes of maximal Cohen-Macaulay sheaves on W.

Lemma 11. ([9]) *There exists unique $\mathcal{U} \in \mathrm{CM}(W)$ such that $\iota_*\mathcal{U} \cong \mathcal{E}$.*

By the above lemma, we will classify the pair $(W, \mathcal{U})$ where $\mathcal{U} \in \mathrm{CM}(W)$ such that $\iota_*\mathcal{U} \cong \mathcal{E}$ instead of $\mathcal{E} \in \mathbf{bimod}(X\text{-}Y)$.

Theorem 12. ([7]) *If $\mathcal{U} \in \mathrm{CM}(W)$ such that $\iota_*\mathcal{U} \cong \mathcal{E}$, then one of the following cases occurs:*

	W	u, v
(i)	*integral*	*u, v are isomorphisms*
(ii)	*integral*	$\deg u = \deg v = 2$
(iii)	*$W = W_1 \cup W_2$ reduced*	*$u\|_{W_i}, v\|_{W_i}$ are isomorphisms for $i = 1, 2$*
(iv)	*irreducible and non-reduced*	*$u\|_{W_{red}}, v\|_{W_{red}}$ are isomorphisms*

Moreover,

(1) *$\mathcal{U} \in \mathrm{CM}(W)$ is locally free of rank 2 in the case (i), and generically locally free of rank 1 in the other cases.*
(2) *Except for the case (ii), $X \cong Y$.*
(3) *If X, Y are curves, then W is a Cartier divisor of $X \times Y$.*

From now on we will assume that $X = Y = \mathbb{P}^1$.

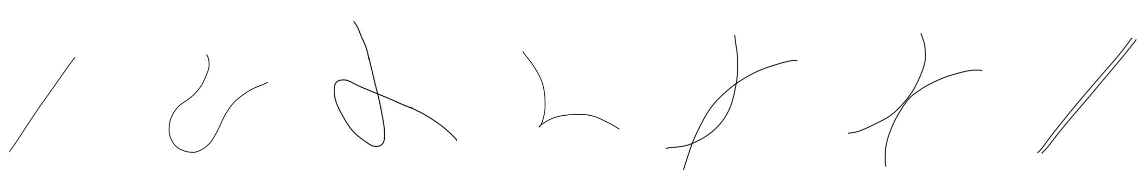
For $W, W' \subset \mathbb{P}^1 \times \mathbb{P}^1$, we define $W \sim W'$ if there exist $\tau_1, \tau_2 \in \mathrm{Aut}\mathbb{P}^1$ such that $(\tau_1 \times \tau_2)(W) = W'$.

Lemma 13. ([7]) *Let $\mathcal{E} \in \mathbf{bimod}(\mathbb{P}^1\text{-}\mathbb{P}^1)$ be locally free sheaf bimodule of* rank 2*, and $W = \mathrm{Supp}\,\mathcal{E} \subset \mathbb{P}^1 \times \mathbb{P}^1$. If $W \sim W'$, then there exists locally free sheaf bimodules $\mathcal{E}' \in \mathbf{bimod}(\mathbb{P}^1\text{-}\mathbb{P}^1)$ of* rank 2 *such that* $\mathrm{Supp}\,\mathcal{E}' = W'$ *and $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}')$.*

By the above lemma, it is enough to classify W up to equivalence $\sim$.

Theorem 14. ([7], [8]) *For a locally free sheaf bimodule $\mathcal{E} \in \mathbf{bimod}(\mathbb{P}^1\text{-}\mathbb{P}^1)$ of rank* 2*, $W = \mathrm{Supp}\,\mathcal{E} \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a Cartier divisor of bidegree $(1,1)$ or $(2,2)$. In fact, W is equivalent to one of the following types:*

Type P *Type* EC *Type* NC *Type* CC *Type* S *Type* T *Type* WL



We now classify $\mathcal{U}$ in each Type.

(I) If W is of Type P, then $W \sim \Delta_{\mathbb{P}^1}$. In this case, $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \cong \mathbb{F}_d$ for some $d \in \mathbb{N}$ (a commutative Hirzebruch surface).

If W is not of Type P, then we define the non-invertible locus of $\mathcal{U} \in \mathrm{CM}(W)$ by

$$\mathrm{Ninv}(\mathcal{U}) := \{p \in W \mid \mathcal{U}_p \not\cong \mathcal{O}_{W,p}\} \subset \mathrm{Sing}\, W$$

so that $\mathrm{Ninv}(\mathcal{U}) = \emptyset$ if and only if $\mathcal{U} \in \mathrm{Pic}\, W$. If $\mathcal{U} \in \mathrm{Pic}\, W \subset \mathrm{CM}(W)$, then $\iota_*\mathcal{U} \in$ **bimod**$(\mathbb{P}^1\text{-}\mathbb{P}^1)$ is locally free of rank 2. For each type, $\mathrm{Pic} W$ is known:

Type	EC	NC	CC	S	T	WL
$\mathrm{Pic} W$	$W \times \mathbb{Z}$	$k^\times \times \mathbb{Z}$	$k \times \mathbb{Z}$	$k^\times \times \mathbb{Z}$	$k \times \mathbb{Z}$	$k \times \mathbb{Z}$

(II) If W is smooth (Type EC), then $\mathrm{Sing}\, W = \emptyset$, so $\mathcal{U} \in \mathrm{CM}(W) = \mathrm{Pic}\, W \cong W \times \mathbb{Z}$.

(III) If W is singular and reduced (Type NC, CC, S, T), then we have the following classification.

Theorem 15. ([7]) *For $\mathcal{U} \in \mathrm{CM}(W)$ such that $\iota_*\mathcal{U} \cong \mathcal{E}$, there exists $\widetilde{\mathcal{U}} \in \mathrm{Pic}\, \widetilde{W}$ such that $\nu_*\widetilde{\mathcal{U}} \cong \mathcal{U}$ where $\nu : \widetilde{W} := \textbf{Spec}\, \mathcal{E}nd_W(\mathcal{U}) \to W$. For each pair $(W,\mathcal{U})$, possible $\widetilde{W}$ and $\mathrm{Pic}\, \widetilde{W}$ are listed as follows:*

Type	$\widetilde{W}$	$\mathrm{Pic}\, \widetilde{W}$
NC, CC	$\mathbb{P}^1$	$\mathbb{Z}$
S, T	$\mathbb{P}^1 \cup \mathbb{P}^1$	$\mathbb{Z} \times \mathbb{Z}$
S, T	$\mathbb{P}^1 \sqcup \mathbb{P}^1$	$\mathbb{Z} \times \mathbb{Z}$

(IV) If W is non-reduced (Type WL), then we have the following classification.

Theorem 16. ([7]) *For $\mathcal{U} \in \mathrm{CM}(W)$ such that $\iota_*\mathcal{U} \cong \mathcal{E}$, there exists $\mathcal{L} \in \mathrm{Pic}\, W \cong k \times \mathbb{Z}$ such that*

$$0 \to \mathcal{U} \to \mathcal{L} \to \mathcal{O}_{\mathrm{Ninv}(\mathcal{U})} \to 0$$

is exact.

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