THE MODULI OF 5-DIMENSIONAL SUBALGEBRAS OF THE FULL MATRIX RING OF DEGREE 3

KAZUNORI NAKAMOTO AND TAKESHI TORII

Abstract. We describe the moduli $\mathrm{Mold}_{3,5}$ of 5-dimensional subalgebras of the full matrix ring of degree 3. We show that $\mathrm{Mold}_{3,5}$ has three irreducible components, whose relative dimensions over $\mathbb{Z}$ are all 4.

Key Words: moduli of subalgebras, full matrix ring.

2020 *Mathematics Subject Classification*: Primary 14D22; Secondary 16S80, 16S50.

1. Introduction

Let k be a field. We say that k-subalgebras A and B of $\mathrm{M}_n(k)$ are equivalent (or $A \sim B$) if $P^{-1}AP = B$ for some $P \in \mathrm{GL}_n(k)$. If k is an algebraically closed field, then there are 26 equivalence classes of k-subalgebras of $\mathrm{M}_3(k)$ over k ([5]).

Definition 1 ([2, Definition 1.1], [3, Definition 3.1])**.** We say that a subsheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras of $\mathrm{M}_n(\mathcal{O}_X)$ is a *mold* of degree n on a scheme X if $\mathrm{M}_n(\mathcal{O}_X)/\mathcal{A}$ is a locally free sheaf. We denote by $\mathrm{rank}\mathcal{A}$ the rank of $\mathcal{A}$ as a locally free sheaf.

Proposition 2 ([2, Definition and Proposition 1.1], [3, Definition and Proposition 3.5])**.** *The following contravariant functor is representable by a closed subscheme of the Grassmann scheme* $\mathrm{Grass}(d, n^2)$:

$$\begin{array}{rcccl} \mathrm{Mold}_{n,d} & : & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \mathcal{A} \mid \ \mathcal{A} \textit{ is a rank } d \textit{ mold of degree } n \textit{ on } X \ \right\}. \end{array}$$

We consider the moduli $\mathrm{Mold}_{3,d}$ of rank d molds of degree 3 over $\mathbb{Z}$. In the case $d \neq 5$, we have the following theorem:

Theorem 3 ([5])**.** *Let $n = 3$. If $d \leq 4$ or $d \geq 6$, then*

$$\begin{array}{rcl} \mathrm{Mold}_{3,1} & = & \mathrm{Spec}\mathbb{Z}, \\ \mathrm{Mold}_{3,2} & \cong & \mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}, \\ \mathrm{Mold}_{3,3} & = & \overline{\mathrm{Mold}^{\mathrm{reg}}_{3,3}} \cup \overline{\mathrm{Mold}^{\mathrm{S}_2}_{3,3}} \cup \overline{\mathrm{Mold}^{\mathrm{S}_3}_{3,3}}, \textit{ where the relative dimensions of} \\ & & \overline{\mathrm{Mold}^{\mathrm{reg}}_{3,3}}, \overline{\mathrm{Mold}^{\mathrm{S}_2}_{3,3}}, \textit{ and } \overline{\mathrm{Mold}^{\mathrm{S}_3}_{3,3}} \textit{ over } \mathbb{Z} \textit{ are } 6, 4, \textit{ and } 4, \textit{ respectively,} \\ \mathrm{Mold}_{3,4} & \cong & Q(V) \coprod \mathbb{P}^2_{\mathbb{Z}} \coprod \mathbb{P}^2_{\mathbb{Z}}, \textit{ where the relative dimension of } Q(V) \textit{ over } \mathbb{Z} \textit{ is } 5, \\ \mathrm{Mold}_{3,6} & \cong & \mathrm{Flag}_3 := \mathrm{GL}_3/\{(a_{ij}) \in \mathrm{GL}_3 \mid a_{ij} = 0 \textit{ for } i > j\}, \\ \mathrm{Mold}_{3,7} & \cong & \mathbb{P}^2_{\mathbb{Z}} \coprod \mathbb{P}^2_{\mathbb{Z}}, \end{array}$$

The detailed version of this paper will be submitted for publication elsewhere.

$$
\begin{aligned}
\mathrm{Mold}_{3,8} &= \emptyset, \\
\mathrm{Mold}_{3,9} &= \mathrm{Spec}\mathbb{Z}.
\end{aligned}
$$

Remark 4 ([4, Definition 14]). The scheme $Q(V)$ is a smooth scheme over $\mathbb{Z}$ defined by

$$
\begin{aligned}
Q(V) &= \mathrm{Flag}(V) \times_{\mathbb{P}_*(V)} \mathrm{Flag}(V) \times_{\mathbb{P}^*(V)} \mathrm{Flag}(V) \\
&= \{(L_1, W_2; L_1, W_1; L_2, W_1) \mid \dim_k L_i = 1, \dim_k W_i = 2\} \\
&= \{(L_1, L_2, W_1, W_2) \mid L_1 \subset W_1, L_1 \subset W_2, L_2 \subset W_1\}.
\end{aligned}
$$

The case $d = 5$ remains. In this paper, we describe the moduli $\mathrm{Mold}_{3,5}$ of rank 5 molds of degree 3. We introduce several rank 5 molds of degree 3 on a commutative ring R.

Definition 5 ([5])**.** For a commutative ring R, we define

(1) $(\mathrm{M}_2 \times \mathrm{D}_1)(R) = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(R) \right\}$,

(2) $\mathrm{S}_{10}(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & e \end{pmatrix} \middle| \, a, b, c, d, e \in R \right\}$,

(3) $\mathrm{S}_{11}(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & e & d \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c, d, e \in R \right\}$,

(4) $\mathrm{S}_{12}(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & e & d \\ 0 & 0 & e \end{pmatrix} \middle| \, a, b, c, d, e \in R \right\}$,

(5) $\mathrm{S}_{13}(R) = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(R) \right\}$,

(6) $\mathrm{S}_{14}(R) = \left\{ \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(R) \right\}$.

There are 6 equivalence classes of 5-dimensional subalgebras of $\mathrm{M}_3(k)$ over an algebraically closed field k: $(\mathrm{M}_2 \times \mathrm{D}_1)(k)$, $\mathrm{S}_{10}(k)$, $\mathrm{S}_{11}(k)$, $\mathrm{S}_{12}(k)$, $\mathrm{S}_{13}(k)$, and $\mathrm{S}_{14}(k)$.

The following theorem is our main result in this paper.

Theorem 6 (Theorem 19)**.** *When $d = 5$, we have an irreducible decomposition*

$$\mathrm{Mold}_{3,5} = \overline{\mathrm{Mold}_{3,5}^{\mathrm{M}_2 \times \mathrm{D}_1}} \coprod \overline{\mathrm{Mold}_{3,5}^{\mathrm{S}_{13}}} \coprod \overline{\mathrm{Mold}_{3,5}^{\mathrm{S}_{14}}},$$

whose irreducible components are all connected components. The relative dimensions of $\overline{\mathrm{Mold}_{3,5}^{\mathrm{M}_2 \times \mathrm{D}_1}}$, $\overline{\mathrm{Mold}_{3,5}^{\mathrm{S}_{13}}}$, *and* $\overline{\mathrm{Mold}_{3,5}^{\mathrm{S}_{14}}}$ *over* $\mathbb{Z}$ *are all* 4*. Moreover,*

$$
\begin{aligned}
\overline{\mathrm{Mold}_{3,5}^{\mathrm{M}_2 \times \mathrm{D}_1}} &= \mathrm{Mold}_{3,5}^{\mathrm{M}_2 \times \mathrm{D}_1} \cup \mathrm{Mold}_{3,5}^{\mathrm{S}_{11}}, \\
\overline{\mathrm{Mold}_{3,5}^{\mathrm{S}_{13}}} &= \mathrm{Mold}_{3,5}^{\mathrm{S}_{13}} \cup \mathrm{Mold}_{3,5}^{\mathrm{S}_{12}},
\end{aligned}
$$

$$\overline{\mathrm{Mold}_{3,5}^{\mathrm{S}_{14}}} = \mathrm{Mold}_{3,5}^{\mathrm{S}_{14}} \cup \mathrm{Mold}_{3,5}^{\mathrm{S}_{10}}.$$

Remark 7 ([1]). We need to say the relation between $\mathrm{Mold}_{d,d}$ and the variety Alg_d of algebras defined by Gabriel in [1]. Let $V = ke_1 \oplus ke_2 \oplus \cdots \oplus ke_d$ be a d-dimensional vector space over a field k. For $\varphi \in \mathrm{Hom}_k(V \otimes_k V, V)$, put $\varphi(e_i \otimes e_j) = \sum_{l=1}^{n} c_{ij}^l e_l$. We say that φ *determines an algebra structure* on V with 1 if the multiplication $e_i \cdot e_j = c_{ij}^l e_l$ defines an algebra V over k with 1. Then we define the variety Alg_d of d-dimensional algebras in the sense of Gabriel by

$$\mathrm{Alg}_d = \left\{ \varphi \in \mathrm{Hom}_k(V \otimes_k V, V) \,\middle|\, \begin{array}{c} \varphi \textit{ determines an} \\ \text{algebra structure} \\ \text{on } V \text{ with } 1 \end{array} \right\} \subset \mathbb{A}_k^{d^3}.$$

Then we can define a morphism $\Psi_d : \mathrm{Alg}_d \to \mathrm{Mold}_{d,d}$ by

$$\varphi \mapsto \{\varphi(v \otimes -) \in \mathrm{End}_k(V) \cong \mathrm{M}_d(k) \mid v \in V\}.$$

If we could prove that $U_d = \{A \subset \mathrm{M}_d(k) \mid A \text{ is a } d\text{-dimensional tame algebra } \}$ is open in $\mathrm{Mold}_{d,d}$ for any d, then $\Psi_d^{-1}(U_d) = \{A \mid\ d\text{-dimensional tame algebra } \}$ would also be open in Alg_d, which gives an affirmative answer to "Tame type is open conjecture". Hence, we believe that $\mathrm{Mold}_{n,d}$ is an important geometric object. This is one of our motivations to investigate $\mathrm{Mold}_{n,d}$.

2. Several Tools

In this section, we introduce several tools for describing $\mathrm{Mold}_{3,5}$. Let A be an associative algebra over a commutative ring R. Assume that A is projective over R. Let $A^e = A \otimes_R A^{op}$ be the enveloping algebra of A. For an A-bimodule M over R, we can regard it as an A^e-module. We define the i-th Hochschild cohomology group $\mathrm{HH}^i(A, M)$ of A with coefficients in M as $\mathrm{Ext}^i_{A^e}(A, M)$.

Let $\mathcal{A}$ be the universal mold on $\mathrm{Mold}_{n,d}$. For $x \in \mathrm{Mold}_{n,d}$, denote by $\mathcal{A}(x) = \mathcal{A} \otimes_{\mathcal{O}_{\mathrm{Mold}_{n,d}}} k(x) \subset \mathrm{M}_n(k(x))$ the mold corresponding to x, where $k(x)$ is the residue field of x. As applications of Hochschild cohomology to the moduli $\mathrm{Mold}_{n,d}$, we have the following tools.

Theorem 8 ([3, Theorem 1.1])**.** *For each point $x \in \mathrm{Mold}_{n,d}$, the dimension of the tangent space $T_{\mathrm{Mold}_{n,d}/\mathbb{Z},x}$ of $\mathrm{Mold}_{n,d}$ at x is given by*

$$\dim_{k(x)} T_{\mathrm{Mold}_{n,d}/\mathbb{Z},x} = \dim_{k(x)} \mathrm{HH}^1(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x)) + n^2 - \dim_{k(x)} N(\mathcal{A}(x)),$$

where $N(\mathcal{A}(x)) = \{b \in \mathrm{M}_n(k(x)) \mid [b, a] = ba - ab \in \mathcal{A}(x) \text{ for any } a \in \mathcal{A}(x)\}$.

Theorem 9 ([3, Theorem 1.2])**.** *Let $x \in \mathrm{Mold}_{n,d}$. If $\mathrm{HH}^2(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x)) = 0$, then the canonical morphism $\mathrm{Mold}_{n,d} \to \mathbb{Z}$ is smooth at x.*

For a rank d mold A of degree n on a locally noetherian scheme S, we can consider a $\mathrm{PGL}_{n,S}$-orbit $\{P^{-1}AP \mid P \in \mathrm{PGL}_{n,S}\}$ in $\mathrm{Mold}_{n,d} \otimes_{\mathbb{Z}} S$, where $\mathrm{PGL}_{n,S} = \mathrm{PGL}_n \otimes_{\mathbb{Z}} S$. For $x \in S$, put $A(x) = A \otimes_{\mathcal{O}_S} k(x)$, where $k(x)$ is the residue field of x. By using $\mathrm{HH}^1(A(x), \mathrm{M}_n(k(x))/A(x))$, we have:

Theorem 10 ([3, Theorem 1.3])**.** *Assume that* $\mathrm{HH}^1(A(x), \mathrm{M}_n(k(x))/A(x)) = 0$ *for each* $x \in S$. *Then the* $\mathrm{PGL}_{n,S}$*-orbit* $\{P^{-1}AP \mid P \in \mathrm{PGL}_{n,S}\}$ *is open in* $\mathrm{Mold}_{n,d} \otimes_{\mathbb{Z}} S$.

These tools are useful for investigating $\mathrm{Mold}_{3,5}$. For each rank 5 molds of $\mathrm{M}_3(R)$ over a commutative ring R, we obtained the following table:

TABLE 1. Hochschild cohomology $H^*(A, \mathrm{M}_3(R)/A)$ for R-subalgebras A of $\mathrm{M}_3(R)$

A	$d = \mathrm{rank} A$	$H^* = H^*(A, \mathrm{M}_3(R)/A)$	${}^tA$	$N(A)$	$\dim T_{\mathrm{Mold}_{3,d}/\mathbb{Z},A}$
$(\mathrm{M}_2 \times \mathrm{D}_1)(R) = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\}$	5	$H^i = 0$ for $i \geq 0$	$(\mathrm{M}_2 \times \mathrm{D}_1)(R)$	$(\mathrm{M}_2 \times \mathrm{D}_1)(R)$	4
$\mathrm{S}_{10}(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & e \end{pmatrix} \right\}$	5	$H^i \cong \begin{cases} R \oplus \mathrm{Ann}(2) & (i : \text{even}) \\ R \oplus (R/2R) & (i : \text{odd}) \end{cases}$	$\mathrm{S}_{12}(R)$	$\left\{ \begin{pmatrix} * & * & * \\ a & * & * \\ 0 & 0 & * \end{pmatrix} \middle\vert\ 2a = 0 \right\}$	4
$\mathrm{S}_{11}(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & e & d \\ 0 & 0 & a \end{pmatrix} \right\}$	5	$H^i \cong \begin{cases} R & (i = 0, 1) \\ 0 & (i \geq 2) \end{cases}$	$\mathrm{S}_{11}(R)$	$\mathrm{B}_3(R)$	4
$\mathrm{S}_{12}(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & e & d \\ 0 & 0 & e \end{pmatrix} \right\}$	5	$H^i \cong \begin{cases} R \oplus \mathrm{Ann}(2) & (i : \text{even}) \\ R \oplus (R/2R) & (i : \text{odd}) \end{cases}$	$\mathrm{S}_{10}(R)$	$\left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & a & * \end{pmatrix} \middle\vert\ 2a = 0 \right\}$	4
$\mathrm{S}_{13}(R) = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\}$	5	$H^i = 0$ for $i \geq 0$	$\mathrm{S}_{14}(R)$	$\mathrm{S}_{13}(R)$	4
$\mathrm{S}_{14}(R) = \left\{ \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$	5	$H^i = 0$ for $i \geq 0$	$\mathrm{S}_{13}(R)$	$\mathrm{S}_{14}(R)$	4

3. DESCRIPTION OF $\mathrm{Mold}_{3,5}$

In this section, we describe $\mathrm{Mold}_{3,5}$. Let V be a free module of rank 3 over $\mathbb{Z}$. Fix a canonical basis $\{e_1, e_2, e_3\}$ of V over $\mathbb{Z}$. We define schemes $\mathbb{P}^*(V)$, $\mathbb{P}_*(V)$, and $\mathrm{Flag}(V)$ over $\mathbb{Z}$ as contravariant functors from the category of schemes to the category of sets in the following way:

$$\begin{aligned} \mathbb{P}^*(V)(X) &= \left\{ W \mid W \text{ is a rank 2 subbundle of } \mathcal{O}_X \otimes_{\mathbb{Z}} V \text{ on } X \right\}, \\ \mathbb{P}_*(V)(X) &= \left\{ L \mid L \text{ is a rank 1 subbundle of } \mathcal{O}_X \otimes_{\mathbb{Z}} V \text{ on } X \right\}, \\ \mathrm{Flag}(V)(X) &= \left\{ (L, W) \in (\mathbb{P}_*(V) \times \mathbb{P}^*(V))(X) \mid L \subset W \right\} \end{aligned}$$

for a scheme X.

Remark 11. If we consider the case over a field k, then $\mathbb{P}^*(V)$, $\mathbb{P}_*(V)$, and $\mathrm{Flag}(V)$ over k are regarded as

$$\begin{aligned} \mathbb{P}^*(V) &= \{W \subset V \mid W \text{ is a 2-dimensional subspace of } V\}, \\ \mathbb{P}_*(V) &= \{L \subset V \mid L \text{ is a 1-dimensional subspace of } V\}, \\ \mathrm{Flag}(V) &= \{(L, W) \in \mathbb{P}_*(V) \times \mathbb{P}^*(V) \mid 0 \subset L \subset W \subset V\}, \end{aligned}$$

respectively.

Definition 12. Let $A = \mathrm{M}_2 \times \mathrm{D}_1, \mathrm{S}_{10}, \mathrm{S}_{11}, \mathrm{S}_{12}, \mathrm{S}_{13}$, or S_{14}. We define

$$\mathrm{Mold}_{3,5}^A = \{x \in \mathrm{Mold}_{3,5} \mid \mathcal{A}(x) \otimes_{k(x)} \overline{k(x)} \sim A(\overline{k(x)})\},$$

where $\overline{k(x)}$ is an algebraic closure of $k(x)$.

Definition 13. Let us define morphisms

$$\begin{array}{rcl} \Phi_{2,2} & : & \mathrm{Mold}_{2,2} \rightarrow \mathrm{Mold}_{3,5} \\ & & A \mapsto \left(\begin{array}{ccc} * & * & * \\ 0 & & \\ 0 & \multicolumn{2}{c}{A} \end{array}\right) \end{array}$$

and

$$\begin{array}{rcl} \Phi'_{2,2} & : & \mathrm{Mold}_{2,2} \rightarrow \mathrm{Mold}_{3,5} \\ & & A \mapsto \left(\begin{array}{ccc} \multicolumn{2}{c}{A} & * \\ & & * \\ 0 & 0 & * \end{array}\right). \end{array}$$

Example 14. Recall that

$$\begin{array}{rcl} \mathbb{P}(\mathrm{M}_2/\langle I_2\rangle) \cong \mathbb{P}^2_{\mathbb{Z}} & \rightarrow & \mathrm{Mold}_{2,2} \\ [A] & \mapsto & \langle A\rangle \end{array}$$

is an isomorphism.

There are two types of rank 2 molds of degree 2:

$$\mathrm{D}_2 = \left\{\left(\begin{array}{cc} * & 0 \\ 0 & * \end{array}\right)\right\}, \ \mathrm{N}_2 = \left\{\left(\begin{array}{cc} a & b \\ 0 & a \end{array}\right)\right\}.$$

By the isomorphism above, we have:

$$\begin{array}{rcl} \mathrm{Mold}^{\mathrm{D}_2}_{2,2} & \cong & \{[A] \in \mathbb{P}^2_{\mathbb{Z}} \mid \mathrm{tr}(A)^2 - 4\det(A) \neq 0\}, \\ \mathrm{Mold}^{\mathrm{N}_2}_{2,2} & \cong & \{[A] \in \mathbb{P}^2_{\mathbb{Z}} \mid \mathrm{tr}(A)^2 - 4\det(A) = 0\}. \end{array}$$

Note that GL_2 acts on $\mathrm{Mold}_{2,2}$ by $A \mapsto PAP^{-1}$. Set

$$\mathrm{P}_{1,2} = \left\{\left(\begin{array}{ccc} * & * & * \\ 0 & * & * \\ 0 & * & * \end{array}\right) \in \mathrm{GL}_3\right\}, \ \mathrm{P}_{2,1} = \left\{\left(\begin{array}{ccc} * & * & * \\ * & * & * \\ 0 & 0 & * \end{array}\right) \in \mathrm{GL}_3\right\}.$$

We define the action of $\mathrm{P}_{1,2}$ (or $\mathrm{P}_{2,1}$) on $\mathrm{Mold}_{2,2}$ by

$$\left(\begin{array}{ccc} * & * & * \\ 0 & & \\ 0 & \multicolumn{2}{c}{P'} \end{array}\right) \cdot A = P'AP'^{-1}$$

$$\left(\text{or } \left(\begin{array}{ccc} \multicolumn{2}{c}{P'} & * \\ & & * \\ 0 & 0 & * \end{array}\right) \cdot A = P'AP'^{-1}, \text{respectively}\right), \text{ where } P' \in \mathrm{GL}_2.$$

Let us consider $\mathrm{GL}_3 \times_{\mathrm{P}_{1,2}} \mathrm{Mold}_{2,2}$ and $\mathrm{GL}_3 \times_{\mathrm{P}_{2,1}} \mathrm{Mold}_{2,2}$. For example, $\mathrm{GL}_3 \times_{\mathrm{P}_{1,2}} \mathrm{Mold}_{2,2} = \{(g, A) \mid g \in \mathrm{GL}_3, A \in \mathrm{Mold}_{2,2}\}/\sim$, where $(g, A) \sim (gb^{-1}, bAb^{-1})$ for $b \in \mathrm{P}_{1,2}$.

Definition 15. The morphisms $\Phi_{2,2}$ and $\Phi'_{2,2}$ induce $\Psi_{2,2}$ and $\Psi'_{2,2}$, respectively:

$$\begin{array}{rcccl} \Psi_{2,2} & : & \mathrm{GL}_3 \times_{\mathrm{P}_{1,2}} \mathrm{Mold}_{2,2} & \to & \mathrm{Mold}_{3,5} \\ & & (g, A) & \mapsto & g \begin{pmatrix} * & * & * \\ 0 & & \\ 0 & \multicolumn{2}{c}{A} \end{pmatrix} g^{-1} \end{array}$$

and

$$\begin{array}{rcccl} \Psi'_{2,2} & : & \mathrm{GL}_3 \times_{\mathrm{P}_{2,1}} \mathrm{Mold}_{2,2} & \to & \mathrm{Mold}_{3,5} \\ & & (g, A) & \mapsto & g \begin{pmatrix} \multicolumn{2}{c}{A} & * \\ & & * \\ 0 & 0 & * \end{pmatrix} g^{-1}. \end{array}$$

Theorem 16. *The morphism* $\Psi_{2,2} : \mathrm{GL}_3 \times_{\mathrm{P}_{1,2}} \mathrm{Mold}_{2,2} \to \mathrm{Mold}_{3,5}$ *induces an isomorphism between* $\mathrm{GL}_3 \times_{\mathrm{P}_{1,2}} \mathrm{Mold}_{2,2}$ *and* $\overline{\mathrm{Mold}_{3,5}^{\mathrm{S}_{13}}}$*. Moreover, we have the following correspondences as sets:*

$$\begin{array}{rcl} \mathrm{Mold}_{3,5}^{\mathrm{S}_{13}} & \cong & \mathrm{GL}_3 \times_{\mathrm{P}_{1,2}} \mathrm{Mold}_{2,2}^{\mathrm{D}_2}, \\ \mathrm{Mold}_{3,5}^{\mathrm{S}_{12}} & \cong & \mathrm{GL}_3 \times_{\mathrm{P}_{1,2}} \mathrm{Mold}_{2,2}^{\mathrm{N}_2}. \end{array}$$

Theorem 17. *The morphism* $\Psi'_{2,2} : \mathrm{GL}_3 \times_{\mathrm{P}_{2,1}} \mathrm{Mold}_{2,2} \to \mathrm{Mold}_{3,5}$ *induces an isomorphism between* $\mathrm{GL}_3 \times_{\mathrm{P}_{2,1}} \mathrm{Mold}_{2,2}$ *and* $\overline{\mathrm{Mold}_{3,5}^{\mathrm{S}_{14}}}$*. Moreover, we have the following correspondences as sets:*

$$\begin{array}{rcl} \mathrm{Mold}_{3,5}^{\mathrm{S}_{14}} & \cong & \mathrm{GL}_3 \times_{\mathrm{P}_{2,1}} \mathrm{Mold}_{2,2}^{\mathrm{D}_2}, \\ \mathrm{Mold}_{3,5}^{\mathrm{S}_{10}} & \cong & \mathrm{GL}_3 \times_{\mathrm{P}_{2,1}} \mathrm{Mold}_{2,2}^{\mathrm{N}_2}. \end{array}$$

For $(L, W) \in \mathbb{P}_*(V) \times \mathbb{P}^*(V)$, set

$$A_{(L,W)} = \left\{ f \in \mathrm{End}(V) \cong \mathrm{M}_3(k) \;\middle|\; \begin{array}{c} f(L) \subset L, f(W) \subset W \text{ such that} \\ L \cong V/W \text{ as } k[f]\text{-modules} \end{array} \right\}.$$

Let us define a morphism

$$\begin{array}{rcccl} \Phi_{\mathrm{M}_2 \times \mathrm{D}_1} & : & \mathbb{P}_*(V) \times \mathbb{P}^*(V) & \to & \mathrm{Mold}_{3,5} \\ & & (L, W) & \mapsto & A_{(L,W)}. \end{array}$$

Theorem 18. *The image of* $\Phi_{\mathrm{M}_2 \times \mathrm{D}_1}$ *is open and closed in* $\mathrm{Mold}_{3,5}$*. Moreover,* $\Phi_{\mathrm{M}_2 \times \mathrm{D}_1}$ *gives an isomorphism between* $\mathbb{P}_*(V) \times \mathbb{P}^*(V)$ *and the closure* $\overline{\mathrm{Mold}_{3,5}^{\mathrm{M}_2 \times \mathrm{D}_1}}$ *of* $\mathrm{Mold}_{3,5}^{\mathrm{M}_2 \times \mathrm{D}_1}$*. Moreover, we have the following correspondences as sets:*

$$\begin{array}{rcl} \mathrm{Mold}_{3,5}^{\mathrm{M}_2 \times \mathrm{D}_1} & \cong & \{(L, W) \in \mathbb{P}_*(V) \times \mathbb{P}^*(V) \mid L \not\subset W\}, \\ \mathrm{Mold}_{3,5}^{\mathrm{S}_{11}} & \cong & \mathrm{Flag}_3 = \{(L, W) \in \mathbb{P}_*(V) \times \mathbb{P}^*(V) \mid L \subset W\}. \end{array}$$

By the results above, we have:

Theorem 19 ([5])**.** *We have an irreducible decomposition*

$$\mathrm{Mold}_{3,5} = \overline{\mathrm{Mold}_{3,5}^{\mathrm{M}_2\times\mathrm{D}_1}} \coprod \overline{\mathrm{Mold}_{3,5}^{\mathrm{S}_{13}}} \coprod \overline{\mathrm{Mold}_{3,5}^{\mathrm{S}_{14}}},$$

whose irreducible components are all connected components. The relative dimensions of $\overline{\mathrm{Mold}_{3,4}^{\mathrm{M}_2\times\mathrm{D}_1}}$, $\overline{\mathrm{Mold}_{3,4}^{\mathrm{S}_{13}}}$, *and* $\overline{\mathrm{Mold}_{3,4}^{\mathrm{S}_{14}}}$ *over* $\mathbb{Z}$ *are all* 4*. Moreover,*

$$\begin{aligned}\overline{\mathrm{Mold}_{3,5}^{\mathrm{M}_2\times\mathrm{D}_1}} &= \mathrm{Mold}_{3,5}^{\mathrm{M}_2\times\mathrm{D}_1} \cup \mathrm{Mold}_{3,5}^{\mathrm{S}_{11}},\\ \overline{\mathrm{Mold}_{3,5}^{\mathrm{S}_{13}}} &= \mathrm{Mold}_{3,5}^{\mathrm{S}_{13}} \cup \mathrm{Mold}_{3,5}^{\mathrm{S}_{12}},\\ \overline{\mathrm{Mold}_{3,5}^{\mathrm{S}_{14}}} &= \mathrm{Mold}_{3,5}^{\mathrm{S}_{14}} \cup \mathrm{Mold}_{3,5}^{\mathrm{S}_{10}}.\end{aligned}$$

Summarizing the results on $\mathrm{Mold}_{3,d}$ $(1 \le d \le 9)$, we obtain the following corollary.

Theorem 20. *Let A and B be d-dimensional subalgebras of $\mathrm{M}_3(k)$ over an arbitrary field k. Assume that $d \neq 3, 5$. If $A \otimes_k \overline{k} \sim B \otimes_k \overline{k}$, then $A \sim B$.*

Theorem 20 does not hold in the case $d = 3$ or 5, as shown by the following examples.

Example 21 (The case $d = 5$)**.** Let $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \in \mathrm{M}_2(\mathbb{F}_2)$. Note that the characteristic polynomial $\det(xI_2 - A)$ of A is $x^2 + x + 1$ whose smallest splitting field is $\mathbb{F}_4$. Set

$$\begin{aligned}\mathcal{A}_1 &= \left\{ \begin{pmatrix} * & * & * \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{X}} \\ 0 & & \end{pmatrix} \in \mathrm{M}_3(\mathbb{F}_2) \;\middle|\; X \in \mathbb{F}_2 I_2 + \mathbb{F}_2 A \right\},\\ \mathcal{A}_2 &= \left\{ \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(\mathbb{F}_2) \right\}.\end{aligned}$$

Then $\mathcal{A}_1 \not\sim \mathcal{A}_2$, while $\mathcal{A}_1 \otimes_{\mathbb{F}_2} \mathbb{F}_4 \sim \mathcal{A}_2 \otimes_{\mathbb{F}_2} \mathbb{F}_4$.

Example 22 (The case $d = 3$)**.** Let $A = \begin{pmatrix} 0 & 0 & -c_3 \\ 1 & 0 & -c_2 \\ 0 & 1 & -c_1 \end{pmatrix} \in \mathrm{M}_3(k)$ over a field k. Assume that the characteristic polynomial $\det(xI_3 - A) = x^3 + c_1 x^2 + c_2 x + c_3$ of A is irreducible over k and has distinct roots. Set

$$\begin{aligned}\mathcal{A}_1 &= kI_3 + kA + kA^2,\\ \mathcal{A}_2 &= \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}.\end{aligned}$$

Then $\mathcal{A}_1 \not\sim \mathcal{A}_2$, while $\mathcal{A}_1 \otimes_k \overline{k} \sim \mathcal{A}_2 \otimes_k \overline{k}$.

References

[1] P. Gabriel, Finite representation type is open, Proceedings of the International Conference on Representations of Algebras (Carleton Univ., Ottawa, Ont., 1974), Paper No. 10, 23 pp. Carleton Math. Lecture Notes, No. 9, Carleton Univ., Ottawa, Ont., 1974.

[2] K. Nakamoto, *The moduli of representations with Borel mold*, Internat. J. Math. **25** (2014), no.7, 1450067, 31 pp.

[3] K. Nakamoto and T. Torii, *Applications of Hochschild cohomology to the moduli of subalgebras of the full matrix ring*, J. Pure Appl. Algebra **227** (2023), no.11, Paper No. 107426, 59 pp.

[4] ———, *The moduli of* 4*-dimensional subalgebras of the full matrix ring of degree* 3, Proceedings of the 55th Symposium on Ring Theory and Representation Theory, 66–73, Symposium on Ring Theory and Representation Theory Organizing Committee, Matsumoto, 2024.

[5] ———, *On the classification of subalgebras of the full matrix ring of degree* 3, in preparation.

Center for Medical Education and Sciences
Faculty of Medicine
University of Yamanashi
1110 Shimokato, Chuo, Yamanashi 409-3898, Japan
Email address: nakamoto@yamanashi.ac.jp

Department of Mathematics
Okayama University
Okayama 700-8530 JAPAN
Email address: torii@math.okayama-u.ac.jp