THE ATIYAH CLASS OF A DG MODULE AND NAÏVE LIFTINGS

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Abstract. In this report, we explain necessary and sufficient conditions for a semi-free DG module N to have the classical Atiyah class of N vanish. In order to do this, we introduce the generalized connection on DG modules.

1. Naïve lifting of DG modules

Throughout this report, let R be a commutative ring and $A \to B$ be a DG R-algebra homomorphism between strongly commutative, non-negatively graded DG R-algebras. Let B^e denote the enveloping DG algebra $B^o \otimes_A B$ of B over A, and J denote the diagonal ideal of B^e. Unless otherwise stated, DG modules are right DG modules. The main objects in the report are DG algebras and DG modules. We refer the reader to [1, 2, 3, 4] on these subjects.

First of all, we give the definition of naïve liftablity of DG modules.

Definition 1. Let N be a semi-free DG B-module. We say that N is *naïvely liftable to* B if the DG B-homomorphism $\pi_N : N \otimes_A B \to N$ defined by $\pi_N(n \otimes b) = nb$ is a split DG B-module epimorphism.

The notion of naïve lifting of DG modules was introduced by the authors for the purpose of studying the Auslander-Reiten conjecture. For our series of studies on naïvely liftable DG modules, please refer to [5, 6, 7, 8, 9, 10, 11].

Next, we explain an explicit description of the obstruction to the naïve lifting. Let N be a semi-free DG B-module with a semi-free basis $E = \{e_\lambda\}_{\lambda \in \Lambda}$, where Λ is a well-ordered set. For $e_\lambda \in E$, let $\partial^N(e_\lambda) = \sum_{\mu<\lambda} e_\mu b_{\mu\lambda}$. We define a homogeneous B-module homomorphism $\Delta_N : N \to N \otimes_B J(-1)$ by

$$\Delta_N(e_\lambda) = \sum_{\mu<\lambda} e_\mu \otimes \delta(b_{\mu\lambda})$$

where $\delta(b_{\mu\lambda}) = b_{\mu\lambda} \otimes 1 - 1 \otimes b_{\mu\lambda}$. We see that Δ_N is a DG B-module homomorphism. See [8] for details. Then Δ_N defines a cohomology class in $\mathrm{Ext}^1_B(N, N \otimes_B J)$, which we denote by $[\Delta_N]$. We call $[\Delta_N]$ *the Atiyah class of* N.

Theorem 2. [8, Proposition 3.10] *Let N be a semifree DG B-module N. Then the following are equivalent:*

(1) *N is naïvely liftable to A;*
(2) $[\Delta_N] = 0$*, that is, Δ_N is null-homotopic.*

The detailed version of this paper will be submitted for publication elsewhere.

Let $\widetilde{\Delta}_N : N \to N \otimes_B J/J^2$ denote a composition of $\Delta_N : N \to N \otimes_B J$ and the natural projection $p : N \otimes_B J \to N \otimes_B J/J^2$, that is, $\widetilde{\Delta}_N = p \circ \Delta_N$. Note that $\widetilde{\Delta}_N : N \to N \otimes_B J/J^2$ is a DG B-module homomorphism. The cohomology class $[\widetilde{\Delta}_N]$ in $\mathrm{Ext}^1_B(N, N \otimes_B J/J^2)$, defined by $\widetilde{\Delta}_N$, is called *the classcal Atiyah class of* N. The purpose of this report is to state equivalent conditions to $[\widetilde{\Delta}_N] = 0$.

2. MAIN RESULT

To state the main result, we introduce (generalized) connections on DG modules.

For DG B-modules M, N, we set

$$ {}^*\mathrm{Hom}_B(M, N) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_B(M, N)_i $$

where $\mathrm{Hom}_B(M, N)_i$ is the set of homogeneous B-module homomorphisms from M to $N(i)$. Note that ${}^*\mathrm{Hom}_B(M, N)$ has a DG module structure as follows;

$$ (fb)(n) := f(n)b \qquad \text{and} \qquad \partial^{{}^*\mathrm{Hom}(M,N)}(f) := \partial^N \circ f - (-1)^i f \circ \partial^M $$

for $f \in \mathrm{Hom}_B(M, N)_i$, $b \in B$, and $n \in N$.

Definition 3. Let L be a DG B^e-modules and n be an integer. A homogeneous A-module homomorphism $D : B \to L(n)$ is called an A-derivation (of degree n) if

$$ D(b_1 b_2) = D(b_1)b_2 + (-1)^{n|b_1|} b_1 D(b_2) $$

holds for $b_1, b_2 \in B$. Set

$$ \mathrm{Der}_A(B, L)_n := \{D : B \to L(n) \mid D \text{ is an } A\text{ -derivation of degree } n\} $$

and

$$ {}^*\mathrm{Der}_A(B, L) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Der}_A(B, L)_n. $$

The right DG B-module structure on ${}^*\mathrm{Der}_A(B, L)$ is given as follows:

$$ (Db)(c) := (-1)^{|c||b|} D(c)b \qquad \text{and} \qquad \partial^{{}^*\mathrm{Der}_A(B,L)}(D) = \partial^L \circ D - (-1)^i D \circ d^B $$

for $D \in \mathrm{Der}_A(B, L)_i$, and $b, c \in B$.

Example 4. Let $\delta : B \to J$ be the universal derivation that is defined by $\delta(b) = b \otimes 1 - 1 \otimes b$ for $b \in B$. Remark that $\delta \in \mathrm{Der}_A(B, J)_0$. Set $\overline{\delta} = \pi \circ \delta$ where $\pi : J \to J/J^2$ is the natural projection. Note that $\overline{\delta} \in \mathrm{Der}_A(B, J/J^2)_0$.

Definition 5. Let N be a semi-free DG B-module. Let $D : B \to L(n)$ be an A-derivation of degree n. A homogeneous A-module homomorphism $\psi : N \to N \otimes_B L(n)$ is called a *D-connection on* N if it satisfies

$$ \psi(xb) = \psi(x)b + (-1)^{|x|n} x \otimes D(b) $$

for $x \in N$ and $b \in B$.

The definition of D-connections was inspired by the classical connection in non-commutative geometry introduced by A.Connes. The following two lemmas are basic facts for our connections.

Lemma 6. *Let N be a semi-free DG B-module with a semi-free basis $E = \{e_\lambda\}_{\lambda\in\Lambda}$ and L be a DG B^e-module. For $D \in \mathrm{Der}_A(B, L)_n$, the mapping $\varphi(D) : N \to N \otimes_B L(n)$ is defined by*

$$\varphi(D)\left(\sum_{i=1}^{r} e_{\lambda_i} b_{\lambda_i}\right) = \sum_{i=1}^{r} e_{\lambda_i} \otimes D(b_{\lambda_i})$$

where $b_{\lambda_i} \in B$, $e_{\lambda_i} \in E\,(i = 1, \cdots r)$. Then $\varphi(D)$ is a D-connection.

Lemma 7. *Let N be a semi-free DG B-module and L be a DG B^e-module. For $D, D' \in \mathrm{Der}_A(B, L)_i$, let $\psi : N \to N \otimes_B L(i)$ be a D-connection and $\psi' : N \to N \otimes_B L(i)$ be a D'-connection. Then the assertions hold.*

(1) *Let $b \in B$ be a homogeneous element. Define $(\psi b)(n) := (-1)^{|n||b|}\psi(n)b$ for a homogeneous element $n \in N$. Then ψb is a (Db)-connection.*
(2) *Define $(\psi+\psi')(n) := \psi(n)+\psi'(n)$ for $n \in N$. Then $\psi+\psi'$ is a $(D+D')$-connection.*
(3) *The mapping $\partial^{N\otimes_B L} \circ \psi - (-1)^i \psi \circ \partial^N$ is a $\partial^{{}^*\mathrm{Der}_A(B,L)}(D)$-connection.*

Definition 8. Let N be a semi-free DG B-module and L be a DG B^e-module. For a integer n, we denote by $\mathrm{Conn}(N, N \otimes_B L)_n$ the set of all connections on N of degree n, that is,

$$\mathrm{Conn}(N, N \otimes_B L)_n := \{\psi \,|\, \psi \text{ is a } D\text{-connection for some } D \in \mathrm{Der}_A(B, L)_n\}.$$

We define

$${}^*\mathrm{Conn}(N, N \otimes_B L) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Conn}(N, N \otimes_B L)_n.$$

It follows from Lemma 7 that ${}^*\mathrm{Conn}(N, N \otimes_B L)$ has a DG B-module structure.

Let N be a semi-free DG B-module and L be a DG B^e-module. There is the natural inclusion

$$\iota : {}^*\mathrm{Hom}_B(N, N \otimes_B L) \to {}^*\mathrm{Conn}(N, N \otimes_B L)$$

since all B-module homomorphism $N \to N \otimes_B L$ are 0-connections. There is also the natural mapping

$$\nu : {}^*\mathrm{Conn}(N, N \otimes_B L) \to {}^*\mathrm{Der}_A(B, L)\,; \quad \text{a } D\text{-connection } \psi \mapsto D.$$

Theorem 9. [9] *Let N be a semi-free DG B-module and L be a DG B^e-module. There is a short exact sequence*

$$\mathcal{E}_N(L) : 0 \to {}^*\mathrm{Hom}_B(N, N \otimes_B L) \xrightarrow{\iota} {}^*\mathrm{Conn}(N, N \otimes_B L) \xrightarrow{\nu} {}^*\mathrm{Der}_A(B, L) \to 0$$

of DG B-modules. We call $\mathcal{E}_N(L)$ the fundamental exact sequence of connections on N along L.

The next theorem is the main result of this report.

Theorem 10. [9] *Let N be a semi-free DG B-module. Then the following conditions are equivalent:*

(1) $[\widetilde{\Delta}_N] = 0$ *in* $\mathrm{Ext}^1_B(N, N \otimes_B J/J^2)$*;*
(2) *There is a $\overline{\delta}$-connection ψ in ${}^*\mathrm{Conn}(N, N\otimes_B J/J^2)$ such that $\partial^{{}^*\mathrm{Conn}(N,N\otimes_B J/J^2)}(\psi) = 0$;*

(3) *The fundamental exact sequence* $\mathcal{E}_N(J/J^2)$ *of connections on* N *along* J/J^2 *is a split exact sequence of DG* B*-modules.*

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