FINITENESS OF ORLOV SPECTRA OF SINGULARITY CATEGORIES

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Abstract. A uniformly dominant local ring is defined as a commutative noetherian local ring with an integer r such that the residue field is built out of any nonzero object in the singularity category by direct summands, shifts and at most r mapping cones. In this article, we provide sufficient conditions for uniform dominance, by which it turns out that Burch rings and local rings with quasi-decomposable maximal ideal are uniformly dominant. For a uniformly dominant excellent equicharacteristic isolated singularity, we also give an upper bound of the Orlov spectrum of the singularity category.

Throughout the present article, let R be a commutative noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. All subcategories are assumed to be strictly full.

First of all, we want to state a celebrated theorem of Ballard, Favero and Katzarkov [2]. For this purpose, we need to recall several definitions.

Definition 1. (1) Assume that R is complete and equicharacteristic (i.e., the characteristics of R and k are equal). Then by Cohen's structure theorem R is isomorphic to a factor ring

$$S = k[\![x_1, \dots, x_n]\!]/(f_1, \dots, f_c)$$

of a formal power series ring over the field k. Then the *Jacobian ideal* of R is defined as the preimage in R of the ideal of S generated by $h \times h$ minors of the Jacobian matrix

$$\frac{\partial(f_1, \dots, f_c)}{\partial(x_1, \dots, x_n)} = \left(\frac{\partial f_i}{\partial x_j}\right)_{1 \leqslant i \leqslant n,\, 1 \leqslant j \leqslant n}$$

where $h = \operatorname{ht}(f_1, \dots, f_c) = n - \dim S$. We denote this ideal of R by $\operatorname{jac} R$.

(2) We say that R is a *hypersurface* if the ($\mathfrak{m}$-adic) completion of R is isomorphic to a factor ring $S/(f)$ of a regular local ring S by a principal ideal (f).

The detailed version [8] of this article has been submitted for publication elsewhere.

(3) We say that R has an *isolated singularity* if for every prime ideal $\mathfrak{p}$ of R that is different from $\mathfrak{m}$, the localization $R_{\mathfrak{p}}$ of R at $\mathfrak{p}$ is a regular local ring.
(4) For a finitely generated R-module M, we denote by $\ell\ell(M)$ the *Loewy length* of M, that is,

$$\ell\ell(M) = \inf\{n \in \mathbb{N} \mid \mathfrak{m}^n M = 0\}.$$

(5) The *singularity category* $\mathrm{D}^{\mathrm{sg}}(R)$ of R is defined to be the Verdier quotient of the bounded derived category $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} R)$ of finitely generated R-modules by the bounded homotopy category $\mathrm{K}^{\mathrm{b}}(\operatorname{proj} R)$ of finitely generated projective R-modoules. Thus, by definition, the singularity category $\mathrm{D}^{\mathrm{sg}}(R)$ is a triangulated category.
(6) Let $\mathcal{T}$ be a triangulated category.
   (a) For two subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{T}$ we denote by $\mathcal{X} * \mathcal{Y}$ the subcategory of $\mathcal{T}$ consisting of all objects E that fits into an exact triangle of the form

$$X \to E \to Y \to X[1],$$

   where X and Y are objects in $\mathcal{X}$ and $\mathcal{Y}$, respectively.
   (b) Let $\mathcal{C}$ be a subcategory of $\mathcal{T}$. We set $\langle \mathcal{C} \rangle_0 = 0$ and denote by $\langle \mathcal{C} \rangle$ the smallest subcategory of $\mathcal{T}$ which contains all objects in $\mathcal{C}$ and is closed under taking finite direct sums, direct summands and shifts. For an integer $r \geqslant 1$ we set

$$\langle \mathcal{C} \rangle_r = \langle \langle \mathcal{C} \rangle_{r-1} * \langle \mathcal{C} \rangle \rangle.$$

   When $\mathcal{C}$ consists of a single object X, we simply write $\langle X \rangle_r$ instead of $\langle \mathcal{C} \rangle_r$.
   (c) For an object T of $\mathcal{T}$, we set

$$\operatorname{gt} T = \inf\{n \in \mathbb{Z}_{\geqslant -1} \mid \langle T \rangle_{n+1} = \mathcal{T}\},$$

   and call it the *generation time* of T in $\mathcal{T}$.
   (d) We set

$$\operatorname{Ospec} \mathcal{T} = \{\operatorname{gt} T \mid T \in \mathcal{T},\ \operatorname{gt} T < \infty\},$$

   and call it the *Orlov spectrum* of $\mathcal{T}$.
   (e) The *(Rouquier) dimension* $\dim \mathcal{T}$ of $\mathcal{T}$ and the *ultimate dimension* $\operatorname{udim} \mathcal{T}$ of $\mathcal{T}$ are defined by:

$$\dim \mathcal{T} = \inf(\operatorname{Ospec} \mathcal{T}), \qquad \operatorname{udim} \mathcal{T} = \sup(\operatorname{Ospec} \mathcal{T}).$$

Now we can state the theorem of Ballard, Favero and Katzarkov which is mentioned above.

Theorem 2 (Ballard–Favero–Katzarkov [2]). *Let R be a complete equicharacteristic local hypersurface of (Krull) dimension d such that k is algebraically closed and of characteristic 0. Suppose that R has an isolated singularity. Put $J = \operatorname{jac} R$ and*

$l = \ell\ell(R/J)$. *Then, for all the nonzero objects X of the singularity category $\mathrm{D^{sg}}(R)$, one has*

$$\operatorname{gt} X \leqslant 2(d+2)l - 1,$$

that is, the equality $\mathrm{D^{sg}}(R) = \langle X \rangle_{2(d+2)l}$ holds. In particular, one has

$$\operatorname{udim} \mathrm{D^{sg}}(R) \leqslant 2(d+2)l - 1,$$

and $\operatorname{Ospec} \mathrm{D^{sg}}(R)$ is a finite set.

Here we introduce the notion of a uniformly dominant local ring.

Definition 3. We put

$$\operatorname{dx}(R) = \inf\{n \in \mathbb{Z}_{\geqslant -1} \mid k \in \langle X \rangle_{n+1} \text{ for all nonzero objects } X \text{ of } \mathrm{D^{sg}}(R)\}$$

and call this the *dominant index* of R. We say that R is a *uniformly dominant* local ring if it has finite dominant index.

When the local ring R is uniformly dominant, it is a dominant local ring in the sense of [7], whence R is Tor-friendly and Ext-friendly in the sense of [1], in particular, the Auslander–Reiten conjecture holds for R, and furthermore, the thick subcategories of $\mathrm{D^{sg}}(R)$ are classified completely under some assumptions. We refer the reader to [7] for the details.

To state our main result, we need to recall some more definitions.

Definition 4. (1) Let M be a finitely generated R-module.

(a) We denote by $\operatorname{depth} M$ the *depth* of M, namely,

$$\operatorname{depth} M = \inf\{n \in \mathbb{N} \mid \operatorname{Ext}_R^n(k, M) \neq 0\}.$$

(b) We denote by $\nu(M)$ the *minimal number of generators* of M, namely,

$$\nu(M) = \dim_k(M/\mathfrak{m}M).$$

(c) For each $n \in \mathbb{N}$, we denote by $\Omega^n M$ the *nth syzygy* of M in a minimal free resolution of M. Note that $\Omega^n M$ is uniquely determined up to isomorphism, since so is a minimal free resolution of M.

(2) We denote by $\operatorname{edim} R$ the *embedding dimension* of R, that is to say,

$$\operatorname{edim} R = \nu(\mathfrak{m}) = \dim_k(\mathfrak{m}/\mathfrak{m}^2).$$

(3) We denote by $\operatorname{ann} \mathrm{D^{sg}}(R)$ the *annihilator* of $\mathrm{D^{sg}}(R)$, that is,

$$\operatorname{ann} \mathrm{D^{sg}}(R) = \left\{ a \in R \;\middle|\; \begin{array}{c} \text{the multiplication morphism} \\ X \xrightarrow{a} X \text{ is zero in } \mathrm{D^{sg}}(R) \end{array} \right\}.$$

It is easy to observe that $\operatorname{ann} \mathrm{D^{sg}}(R)$ is an ideal of R.

Now we can state the main result of this article.

Theorem 5. *Let R be a local ring of depth t. Put*

$$s = \begin{cases} 1 & \text{if } t = 0, \\ 2^{\operatorname{edim} R} & \text{if } t > 0. \end{cases}$$

Then the following statements hold true.

(1) (a) *If the syzygy $\Omega^{t+1}k$ is a direct summand of a finite direct sum of copies of $\Omega^{t+2}k$, then R is a uniformly dominant local ring with*

$$\operatorname{dx}(R) \leqslant s(2t+3) - 1.$$

(b) *If the syzygy $\Omega^{t}k$ is a direct summand of a finite direct sum of copies of $\Omega^{t+2}k$, then R is a uniformly dominant local ring with*

$$\operatorname{dx}(R) \leqslant s(2t+4) - 1.$$

(2) *Assume that R is excellent (e.g., complete), equicharacteristic and has an isolated singularity. Let J be an ideal of R such that*

$$\mathfrak{m}^r \subseteq J \subseteq \operatorname{ann} \mathrm{D}^{\mathrm{sg}}(R)$$

for some integer $r > 0$. Put

$$m = \nu(J), \qquad l = \ell\ell(R/J).$$

Suppose that R is uniformly dominant with dominant index n. Then every nonzero object X of $\mathrm{D}^{\mathrm{sg}}(R)$ is such that

$$\operatorname{gt} X \leqslant (n+1)(m-t+1)l - 1.$$

In particular, the Orlov spectrum $\operatorname{Ospec} \mathrm{D}^{\mathrm{sg}}(R)$ is a finite set, and

$$\operatorname{udim} \mathrm{D}^{\mathrm{sg}}(R) \leqslant (n+1)(m-t+1)l - 1.$$

Here it is necessary to recall two definitions.

Definition 6. (1) We say that R is a *Burch ring* if there exist an $\widehat{R}$-regular sequence $\boldsymbol{x} = x_1, \dots, x_n$, a regular local ring S with maximal ideal $\mathfrak{n}$, and an ideal I of S such that

$$\widehat{R}/(\boldsymbol{x}) \cong S/I, \qquad \mathfrak{n}(I : \mathfrak{n}) \neq \mathfrak{n}I.$$

Here, $\widehat{R}$ stands for the completion of R, while $I : \mathfrak{n} = \{a \in S \mid \mathfrak{n}a \subseteq I\}$.

(2) We say that $\mathfrak{m}$ is *quasi-decomposable* if there exists an R-regular sequence $\boldsymbol{x} = x_1, \dots, x_n$ in $\mathfrak{m}$ such that $\mathfrak{m}/(\boldsymbol{x})$ is decomposable as an R-module.

It is a basic fact that if a local ring R of depth t is a singular (i.e., non-regular) hypersurface, then $\Omega^t k$ is isomorphic to $\Omega^{t+2}k$. Relating to this, Dao, Kobayashi and Takahashi [3] show that if R is a hypersurface, then it is Burch and that if R is a singular Burch ring, then $\Omega^t k$ is a direct summand of $\Omega^{t+2}k$. One can also show that $\Omega^{t+1}k$ is a direct summand of $\Omega^{t+2}k$ if $\mathfrak{m}$ is quasi-decomposable. Thus, the assumption of Theorem 5(1a) is satisfied if $\mathfrak{m}$ is quasi-decomposable, and that of Theorem 5(1b) is satisfied if R is a singular Burch ring, paticularly if R is a singular hypersurface. As such an ideal J as in Theorem 5(2), one can always take $\operatorname{ann} \mathrm{D}^{\mathrm{sg}}(R)$, and can even take $\operatorname{jac} R$ if R is a complete Cohen–Macaulay local ring and k is perfect. Therefore, Theorem 5 considerably extends Theorem 2 in terms of providing a finite uniform bound of the generation times of nonzero objects of the singularity category (the bound is itself looser).

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