g-FANS OF RANK 2

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Abstract. For a finite dimensional algebra A over a field $\Bbbk$, the set of (isomorphic classes of) basic 2-term presilting objects of A gives a nonsingular fan $\Sigma(A)$ in the real Grothendieck group $K_0(\mathsf{proj}A)_{\mathbb{R}}$ called the g-fan of A. In this paper, we give a classification of complete g-fans of rank 2.

1. Introduction

The notion of tilting objects is central to control equivalences of derived categories. The set of partial tilting modules for a finite dimensional algebra has a structure of a simplicial complex, and gives rise to a fan in the real Grothendieck group. Their structure has been studied by a number of authors including [12, 13, 8].

The class of silting objects gives a completion of the class of tilting objects in terms of mutation [10, 2]. Silting objects correspond bijectively with other important objects in the derived category, including (co-)t-structures and simple-minded collections [1, 9, 11]. For each finite dimensional algebra A, the collection of (basic) 2-term presilting objects forms a simplicial complex. Furthermore, this gives rise to a nonsingular fan in the real Grothendieck group of A [5], which we call the g-fan of A. g-fans have information about the mutation and also have a property called sign-coherence [5, 3].

Definition 1. A *sign-coherent fan* is a pair (Σ, σ_+) satisfying the following conditions.

(a) Σ is a nonsingular fan in $\mathbb{R}^n$, and $\sigma_+, -\sigma_+$ are cones of dimension n contained in Σ.
(b) Take $e_1, \ldots, e_n \in \mathbb{R}^n$ such that $\sigma_+ = \mathsf{cone}\{e_i \mid 1 \leq i \leq n\}$. Then for each $\sigma \in \Sigma$, there exists $\epsilon_1, \ldots, \epsilon_n \in \{1, -1\}$ such that $\sigma \subseteq \mathsf{cone}\{\epsilon_1 e_1, \ldots, \epsilon_n e_n\}$.
(c) Each cone of dimension $n-1$ is contained in precisely two cones of dimension n.
(d) Each maximal cone has dimension n.

Then the following problem is central in the study of g-fans.

- Characterize complete sign-coherent fans in $\mathbb{R}^n$ which can be realized as g-fans of some finite dimensional algebras.

As a complete answer to this problem for the case $n = 2$, we have the follwoing statement.

Theorem 2. *Complete g-fans of rank 2 are precisely sign-coherent fans of rank 2 with $\sigma_+ = \mathsf{cone}\{\left[\begin{smallmatrix}1 & 0\end{smallmatrix}\right], \left[\begin{smallmatrix}0 & 1\end{smallmatrix}\right]\}$*

In this paper, we explain our method to prove Theorem 2.

The detailed version of this paper will be submitted for publication elsewhere.

Notations. Let A be a finite dimensional algebra over a field $\Bbbk$. $\mathsf{proj}A$ denotes the category of finitely generated projective right A-modules and $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}A)$ denotes the bounded homotopy category of $\mathsf{proj}A$. For an object X of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}A)$, $|X|$ denotes the number of non-isomorphic indecomposable direct summands of X.

2. 2-term silting objects, g-vectors, and g-fans

In this section, we recall the definition of 2-term silting objects, g-vectors, and g-fans.

2.1. 2-term silting objects.

First we recall the definition of 2-term silting objects.

Definition 3 (2-term (pre)silting objects)**.** Let T be a 2-term object in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}A)$.

(1) T is 2-term *presilting* if $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}A)}(T, T[1]) = 0$.
(2) T is 2-term *silting* if T is presilting and $|T| = |A|$.

In the rest of this paper, we use the following notations:

- $\mathsf{2\text{-}psilt}A$ is the set of basic 2-term presilting object of A and $\mathsf{2\text{-}silt}A$ is the set of basic 2-term silting object of A. Furthermore, for a non-negative integer d, we define $\mathsf{2\text{-}psilt}^d A := \{U \in \mathsf{2\text{-}psilt}A \mid |U| = d\}$.
- For a morphism $f : P' \to P$ with $P, P' \in \mathsf{proj}A$, we denote by P_f the 2-term object in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}A)$ given by f.

We give an example.

Example 4. Let $A = k[1 \xrightarrow{a} 2]$. Then we have the following statements:

$$\begin{aligned}
\mathsf{2\text{-}psilt}^0 A &= \{0\} \\
\mathsf{2\text{-}psilt}^1 A &= \{P_1, P_2, P_a, P_1[1], P_2[1]\} \\
\mathsf{2\text{-}silt}A = \mathsf{2\text{-}psilt}^2 A &= \{P_1 \oplus P_2, P_1 \oplus P_a, P_a \oplus P_2[1], P_1[1] \oplus P_2, P_1[1] \oplus P_2[2]\}
\end{aligned}$$

2.2. g-vectors.

In this subsection, we recall the definition and important properties of g-vectors. Fix a complete set of indecomposable projective modules $P_1, \dots, P_n$ of A. We set

$$\begin{aligned}
K_0(\mathsf{proj}A) &:= \text{Grothendieck group of } \mathsf{K}^{\mathrm{b}}(\mathsf{proj}A) \\
K_0(\mathsf{proj}A)_{\mathbb{R}} &:= K_0(\mathsf{proj}A) \otimes_{\mathbb{Z}} \mathbb{R} : \text{ Real Grothendieck group of } \mathsf{K}^{\mathrm{b}}(\mathsf{proj}A).
\end{aligned}$$

Since $([P_1], \dots, [P_n])$ is a $\mathbb{Z}$-basis of $K_0(\mathsf{proj}A)$, we have

$$K_0(\mathsf{proj}A) = \mathbb{Z}^n, \; K_0(\mathsf{proj}A)_{\mathbb{R}} = \mathbb{R}^n$$

Definition 5 (g-vectors)**.** For $T \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}A)$, we denote by $[T]$ the corresponding element in $K_0(\mathsf{proj}A) = \mathbb{Z}^n$. We say that $[T]$ is the *g-vector* of T.

If $T = \left[\bigoplus_i P_i^{\oplus b_i} \to \bigoplus_i P_i^{\oplus a_i}\right]$, then we have

$$[T] = [a_1 - b_1 \; a_2 - b_2 \; \cdots \; a_n - b_n].$$

The following results are important properties of g-vectors.

Theorem 6 ([1])**.** *If $[T] = [T']$ holds for 2-term presilting objects T and T', then $T \simeq T'$.*

Theorem 7 ([2])**.** *Let $T = T_1 \oplus \cdots \oplus T_n \in \mathsf{2\text{-}silt}A$. $\{[T_1], \dots, [T_n]\}$ forms a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.*

Example 8. Let $A = k[1 \xrightarrow{a} 2]$. We have

$$[P_1] = [\,1\;0\,],\; [P_2] = [\,0\;1\,],\; [P_a] = [\,1\;{-1}\,],\; [P_1[1]] = [\,{-1}\;0\,],\; [P_2[1]] = [\,0\;{-1}\,].$$

2.3. g**-fans.** In this subsection, we recall the definition of g-fans. A *convex polyhedral cone* σ is a set of the form $\sigma = \{\sum_{i=1}^{s} r_i v_i \mid r_i \geq 0\}$, where $v_1, \dots, v_s \in \mathbb{R}^n$. We denote it by $\sigma = \mathsf{cone}\{v_1, \dots, v_s\}$. Note that $\{0\}$ is regarded as a convex polyhedral cone. We collect some notions concerning convex polyhedral cones. Let σ be a convex polyhedral cone.

(a) The dimension of σ is the dimension of the linear space generated by σ.
(b) We say that σ is *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$ holds.
(c) We call σ *rational* if each v_i can be taken from $\mathbb{Q}^n$.
(d) We denote by $\langle \cdot, \cdot \rangle$ the usual inner product. A *supporting hyperplane* of σ is a hyperplane $\{v \in \sigma \mid \langle u, v \rangle = 0\}$ in $\mathbb{R}^n$ given by some $u \in \mathbb{R}^d$ satisfying $\sigma \subset \{v \in \mathbb{R}^d \mid \langle u, v \rangle \geq 0\}$.
(e) A *face* τ of σ is the intersection of σ with a supporting hyperplane of σ.

In the rest of this paper, a cone means a strongly convex rational polyhedral cone for short.

Definition 9. A *fan* Σ in $\mathbb{R}^n$ is a collection of cones in $\mathbb{R}^n$ such that

(a) each face of a cone in Σ is also contained in Σ, and
(b) the intersection of two cones in Σ is a face of each of those two cones.

For each $d \geq 0$, we denote by Σ_d the subset of cones of dimension d. We call an element of Σ_1 (resp. Σ_n) a ray (resp. a maximal cone) of Σ.

For a basic presilting object U with an indecomposable decomposition $U = X_1 \oplus \cdots \oplus X_d$, we define a cone $C(U)$ in $\mathbb{R}^n$ as follows:

$$C(U) := \mathsf{cone}\{[X_1], \dots, [X_d]\}$$

Theorem 10 (g-fans [5])**.** *Let $\Sigma(A) := \{C(U) \mid U \in \mathsf{2\text{-}psilt}A\}$. Then $\Sigma(A)$ is a fan in $\mathbb{R}^n$. We call it the g-fan of A.*

Let Σ be a fan in $\mathbb{R}^n$.

- Σ is said to be *complete* if $\cup_{\sigma \in \Sigma} \sigma = \mathbb{R}^n$ holds.
- Σ is said to be *nonsingular* if each cone $\sigma \in \Sigma$ is generated by $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

By Theorem 7, $\Sigma(A)$ is a nonsingular fan. Furthermore, we have the following correspondence:

$\Sigma(A)$	$\mathsf{2\text{-}psilt}A$
$\Sigma(A)_d$	$\mathsf{2\text{-}psilt}^d A$
Faces of $C(U)$	Direct summands of U
$C(U) \cap C(U')$	The maximal common direct summand of U and U'

Example 11. Let $A = k[1 \xrightarrow{a} 2]$. We have the following statement:

$$\begin{aligned} \Sigma(A)_0 &= \{0\} \\ \Sigma(A)_1 &= \{C(P_1), C(P_2), C(P_a), C(P_1[1]), C(P_2[1])\} \\ \Sigma(A)_2 &= \{C(P_1 \oplus P_2), C(P_1 \oplus P_a), C(P_a \oplus P_2[1]), C(P_1[1] \oplus P_2), C(P_1[1] \oplus P_2[1])\} \end{aligned}$$

3. Sign-coherent fans of rank 2

In this section, we introduce some terminologies of sign-coherent fans of rank 2, and discuss some fundamental properties.

We denote by $\mathsf{c\text{-}Fan}_{\mathrm{sc}}(2)$ the set of all complete sign-coherent fans (Definition 1) of rank 2 with positive and negative cones

$$\sigma_+ := \mathsf{cone}\{[\,1\ 0\,], [\,0\ 1\,]\} \text{ and } \sigma_- := \mathsf{cone}\{[\,-1\ 0\,], [\,0\ -1\,]\} \text{ respectively.}$$

We note that $\Sigma(A) \in \mathsf{c\text{-}Fan}_{\mathrm{sc}}(2)$ if and only if $|A| = 2$ and $\#\mathsf{2\text{-}psiltA} < \infty$ hold ([4, Theorem 4.7]).

3.1. **Quiddity sequences of fans.** Let $\Sigma \in \mathsf{c\text{-}Fan}_{\mathrm{sc}}(2)$. We denote rays of Σ in a clockwise orientation by

$$\mathsf{cone}\{v_1\}, \mathsf{cone}\{v_2\}, \dots, \mathsf{cone}\{v_{n-1}\}, \mathsf{cone}\{v_n\} = \mathsf{cone}\{v_0\},$$

where $v_0 = v_n = [\,0\ 1\,]$, $v_1 = [\,1\ 0\,]$, and $\{v_i, v_{i+1}\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^2$ for each i. For each $1 \le i \le n$, since Σ is nonsingular, there exists an integer a_i satisfying

$$a_i v_i = v_{i-1} + v_{i+1} \text{ for each } 1 \le i \le n.$$

We note that there exists $2 \le \ell \le n-2$ such that $v_\ell = (0, -1)$ and $v_{\ell+1} = (-1, 0)$. In this case, following the terminology of frieze by [6, 7], we call the sequence of integers

$$\mathsf{s}(\Sigma) = (a_1, \dots a_\ell; a_n, a_{n-1}, \dots, a_{\ell+1}) \tag{3.1}$$

the *quiddity sequence* of Σ. $\Sigma \in \mathsf{c\text{-}Fan}_{\mathrm{sc}}(2)$ is uniquely determined by its quiddity sequence. Then a fan with quiddity sequence $(a_1, \dots a_\ell; a_n, a_{n-1}, \dots, a_{\ell+1})$ is denoted by

$$\Sigma(a_1, \dots, a_\ell; a_n, a_{n-1}, \dots, a_{\ell+1}).$$

Example 12. Assume that $|A| = 2$ and $\mathsf{2\text{-}silt}A$ has precisely n elements. Let $P_2 = X_0 = Y_n, P_1 = X_1 = Y_{n+1}$. Then we have two mutation sequences

$$X_0 \oplus X_1 \to X_1 \oplus X_2 \to \cdots \to X_{\ell-1} \oplus X_\ell \text{ and } Y_{n+1} \oplus Y_n \to Y_n \oplus Y_{n-1} \to \cdots \to Y_{\ell+1} \oplus Y_\ell.$$

Let $X_{i-1} \to X_i^{\oplus a_i} \to X_{i+1} \to X_{i-1}[1]$ and $X_{j-1} \to Y_j^{\oplus a_j} \to Y_{j+1} \to Y_{j+1}[1]$ be exchange triangles. Then we have

$$\Sigma(A) = \Sigma(a_1, \dots, a_\ell; a_n, a_{n-1}, \dots, a_{\ell+1})$$

(1) Let $A = k[1 \xrightarrow{a} 2]$. Then $\Sigma(A) = \Sigma(1, 1, 1; 0, 0)$

(2) Let $A = k[1 \underset{b}{\overset{a}{\rightleftarrows}} 2]/\langle ab, ba\rangle$. Then $\Sigma(A) = \Sigma(1, 1, 1; 1, 1, 1)$

3.2. **Gluing, Rotation, and Subdivision of fans.** In this subsection, we introduce three operations for complete sign-coherent fans of rank 2. We define subsets $\mathsf{c\text{-}Fan}_{\mathrm{sc}}^{+-}(2)$ and $\mathsf{c\text{-}Fan}_{\mathrm{sc}}^{+-}(2)$ of $\mathsf{c\text{-}Fan}_{\mathrm{sc}}(2)$ as follows:

$$\mathsf{c\text{-}Fan}_{\mathrm{sc}}^{+-}(2) := \{\Sigma \mid \mathsf{s}(\Sigma) = (-; 0, 0)\},\ \mathsf{c\text{-}Fan}_{\mathrm{sc}}^{-+}(2) := \{\Sigma \mid \mathsf{s}(\Sigma) = (0, 0; -)\}$$

Definition 13 (Gluing of fans)**.** For $\Sigma = \Sigma(a_1, \dots, a_\ell; 0, 0) \in \mathsf{c\text{-}Fan}_{\mathrm{sc}}^{+-}(2)$ and $\Sigma' = \Sigma(0, 0; a_n, \dots, a_{\ell+1}) \in \mathsf{c\text{-}Fan}_{\mathrm{sc}}^{-+}(2)$, we can define a new fan $\Sigma * \Sigma' \in \mathsf{c\text{-}Fan}_{\mathrm{sc}}(2)$ as follows:

$$\Sigma * \Sigma' := \Sigma(a_1, \dots, a_\ell; a_n, \dots, a_{\ell+1})$$

Definition 14 (Rotation of fans)**.** For $\Sigma = \Sigma(a_1, \ldots, a_\ell; 0, 0) \in \mathsf{c\text{-}Fan}_{\mathrm{sc}}^{+-}(2)$, we can define a new fan $\rho(\Sigma) \in \mathsf{c\text{-}Fan}_{\mathrm{sc}}^{+-}(2)$ as follows:

$$\rho(\Sigma) := \Sigma(a_2, \ldots, a_\ell, a_1; 0, 0)$$

Similarly, we define $\rho(\Sigma')$ for each $\Sigma' \in \mathsf{c\text{-}Fan}_{\mathrm{sc}}^{-+}(2)$.

Definition 15 (Subdivision of fans)**.** For $\Sigma = \Sigma(a_1, \ldots, a_\ell; 0, 0) \in \mathsf{c\text{-}Fan}_{\mathrm{sc}}^{+-}(2)$ and $i \in \{1, \ldots, \ell - 1\}$, we can define a new fan $D_i(\Sigma) \in \mathsf{c\text{-}Fan}_{\mathrm{sc}}^{+-}(2)$ as follows:

$$D_i(\Sigma) := \Sigma(a_1, \ldots, a_{i-1}, a_i + 1, 1, a_{i+1} + 1, a_{i+2}, \ldots, a_\ell; 0, 0)$$

For $\Sigma \in \mathsf{c\text{-}Fan}_{\mathrm{sc}}^{+-}(2)$ having precisely $\ell + 2$ maximal cones, we set

$$D(\Sigma) := D_{\ell-1}(\Sigma).$$

Then the following equation holds for $\Sigma = \Sigma(a_1, \ldots, a_\ell; 0, 0) \in \mathsf{c\text{-}Fan}_{\mathrm{sc}}^{+-}(2)$:

$$D_i(\Sigma) = \rho^{\ell+1-i} \circ D \circ \rho^i(\Sigma)$$

Furthermore, we have the following key statement.

Proposition 16. (1) *We have*

$$\mathsf{c\text{-}Fan}_{\mathrm{sc}}(2) = \mathsf{c\text{-}Fan}_{\mathrm{sc}}^{+-}(2) * \mathsf{c\text{-}Fan}_{\mathrm{sc}}^{-+}(2)$$

(2) *Each fan in* $\mathsf{c\text{-}Fan}_{\mathrm{sc}}^{+-}(2)$ *can be obtained by applying a finite composition of* ρ *and* D *to* $\Sigma(0, 0; 0, 0)$.

4. Gluing Theorem, Rotation Theorem, and Subdivision Theorem

In this section, we realize Gluing, Rotation, and Subdivision on fans in the level of finite dimensional algebras. To simplify our statements, we assume $\Bbbk$ is algebraically closed.

4.1. **Settings.** Let A_1 and A_2 be local algebras and X be a A_1-A_2 bimodule. Then we set $A := \left[\begin{smallmatrix} A_1 & X \\ 0 & A_2 \end{smallmatrix}\right]$, $e_1 := \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $e_2 := \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, and $P_i = e_i A$ $(i = 1, 2)$. We note that A is a basic finite dimensional algebra such that $\Sigma(A) \in \mathsf{c\text{-}Fan}_{\mathrm{sc}}^{+-}(2)$. Similarly, we set a basic finite dimensional algebra $B := \left[\begin{smallmatrix} B_1 & 0 \\ Y & B_2 \end{smallmatrix}\right]$ such that $\Sigma(A) \in \mathsf{c\text{-}Fan}_{\mathrm{sc}}^{-+}(2)$.

Remark 17. We have $\Sigma(\left[\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right]) = \Sigma(0, 0; 0, 0)$.

4.2. **Gluing Theorem.** Keep the settings in subsection 4.1. We define an algebra $A * B$ as follows: Let $A_i \times_\Bbbk B_i$ be the fiber product with respect to canonical surjections $A_i \to \Bbbk$ and $B_i \to \Bbbk$. By defining the action as $(a_1, b_1) \cdot x \cdot (a_2, b_2) := a_1 \cdot x \cdot b_2$, X becomes a $(A_1 \times_\Bbbk B_1)$-$(A_2 \times_\Bbbk B_2)$ bimodule. Similarly, Y is a $(A_2 \times_\Bbbk B_2)$-$(A_1 \times_\Bbbk B_1)$ bimodule. We now define

$$A * B := \left[\begin{smallmatrix} A_1 \times_k B_1 & X \\ Y & A_2 \times_k B_2 \end{smallmatrix}\right],$$

where the multiplication of the elements of X and those of Y are defined to be zero. Then we have the following statement.

Theorem 18 (Gluing Theorem)**.** *We have* $\Sigma(A * B) = \Sigma(A) * \Sigma(B)$.

Example 19. Let $A = \left[\begin{smallmatrix} \Bbbk & \Bbbk a \\ 0 & \Bbbk \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} \Bbbk & 0 \\ \Bbbk b & \Bbbk \end{smallmatrix}\right]$. We have

$$A * B = \left[\begin{smallmatrix} \Bbbk & \Bbbk a \\ \Bbbk b & \Bbbk \end{smallmatrix}\right],$$

where the multiplication of a and b are defined to be zero. Then the following equation holds:

$$\Sigma(A * B) = \Sigma(A) * \Sigma(B) = \Sigma(1,1,1;0,0) * \Sigma(0,0;1,1,1) = \Sigma(1,1,1;1,1,1).$$

4.3. **Rotation Theorem.** Keep the settings in subsection 4.1. We define an algebra $\rho(A)$ as follows: Let $T = U \oplus P_1$ be a basic 2-term silting object given by the left mutation of A at P_2. We set f_1 (resp. f_2) the idempotent of $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}A)}(T)$ corresponding to U (resp. P_1). We now define

$$\rho(A) := \begin{bmatrix} f_1 \operatorname{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}A)}(T) f_1 & f_1 \operatorname{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}A)}(T) f_2 \\ 0 & f_2 \operatorname{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}A)}(T) f_2 \end{bmatrix}.$$

Then we have the following statement.

Theorem 20 (Rotation Theorem)**.** *We have* $\Sigma(\rho(A)) = \rho\left(\Sigma(A)\right)$.

4.4. **Subdivision Theorem.** Keep the settings in subsection 4.1. Let J_i be the Jacobson radical of A_i and

$$\overline{X} := X/XJ_2.$$

Then the $\Bbbk$-dual $(\overline{X})^*$ of $\overline{X}$ is an A_1-module, and we regard it as an A^{op}-module by using the action of $\Bbbk$ through the natural surjection $A_1 \to \Bbbk$. Let

$$C_1 := A_1 \oplus (\overline{X})^*$$

be the trivial extension algebra of A_1 by $(\overline{X})^*$. Let

$$\overline{(\cdot)} : A \to \Bbbk, \ \overline{(\cdot)} : B \to \Bbbk, \quad \text{and} \quad \overline{(\cdot)} : X \to \overline{X}$$

be canonical surjections. We regard

$$Z := \left[\begin{smallmatrix} \Bbbk \\ X \end{smallmatrix}\right]$$

as a C_1-A_2 bimodule by

$$(a_1, f) \cdot \left[\begin{smallmatrix} \alpha \\ x \end{smallmatrix}\right] \cdot a_2 := \left[\begin{smallmatrix} \overline{a_1}\alpha\overline{a_2} + f(\overline{x})\overline{a_2} \\ a_1 x a_2 \end{smallmatrix}\right]$$

We now define

$$D(A) := \left[\begin{smallmatrix} C_1 & Z \\ 0 & A_2 \end{smallmatrix}\right].$$

Then we have the following statement.

Theorem 21 (Subdivision Theorem)**.** *We have* $\Sigma(D(A)) = D\left(\Sigma(A)\right)$.

Example 22. Let $A = \left[\begin{smallmatrix} \Bbbk & \Bbbk \\ 0 & \Bbbk \end{smallmatrix}\right]$. We have

$$D(A) \cong \left[\begin{smallmatrix} \Bbbk[x]/\langle x^2 \rangle & \Bbbk[x]/\langle x^2 \rangle \\ 0 & \Bbbk \end{smallmatrix}\right]$$

Then the following statement holds:

$$\Sigma(D(A)) = D(\Sigma(A)) = D(\Sigma(1,1,1;0,0)) = \Sigma(1,2,1,2;0,0)$$

4.5. **Proof of Theorem 2.** By Proposition 16, Remark 17, Gluing Theorem, Rotation Theorem, and Subdivision Theorem, we can show Theorem 2.

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