SINGULARITY CATEGORIES OF RATIONAL DOUBLE POINTS IN ARBITRARY CHARACTERISTIC

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Abstract. We consider whether there is a one-to-one correspondence between the singularity categories of rational double points and the simply-laced Dynkin graphs in arbitrary characteristic.

1. Introduction

A finite-dimensional separated Noetherian scheme X over an algebraically closed field k is *ELF* if X has enough locally free sheaves of finite rank (i.e. any coherent sheaf on X is a quotient of a locally free sheaf of finite rank). The *singularity category* $\mathbf{D}^{\mathrm{sg}}(X)$ is defined to be the Verdier quotient of the derived category $\mathbf{D}^{\mathrm{b}}(\mathrm{Coh}\, X)$ by the full subcategory $\mathbf{Perf}(X)$ of perfect complexes. Since $\mathbf{D}^{\mathrm{sg}}(X)$ is trivial if and only if X is smooth, $\mathbf{D}^{\mathrm{sg}}(X)$ can be thought of what measures complexity of singularities. We are interested in the idempotent-completion $\overline{\mathbf{D}^{\mathrm{sg}}(X)}$, which turns out to be triangulated equivalent to $\mathbf{D}^{\mathrm{sg}}(\widehat{\mathcal{O}}_{X,p})$ if X has only one isolated Gorenstein singular point $p \in X$, rather than $\mathbf{D}^{\mathrm{sg}}(X)$ itself. In fact, two ELF k-schemes X and Y whose formal completions $\widehat{X}$ and $\widehat{Y}$ along singular loci are isomorphic may have non-equivalent singularity categories $\mathbf{D}^{\mathrm{sg}}(X)$ and $\mathbf{D}^{\mathrm{sg}}(Y)$, whereas their idempotent-completions $\overline{\mathbf{D}^{\mathrm{sg}}(X)}$ and $\overline{\mathbf{D}^{\mathrm{sg}}(Y)}$ are triangulated equivalent. Conversely, does a triangulated equivalence $\overline{\mathbf{D}^{\mathrm{sg}}(X)} \simeq \overline{\mathbf{D}^{\mathrm{sg}}(Y)}$ induce an isomorphism $\widehat{X} \cong \widehat{Y}$? The answer is no in general because of Knörrer's periodicity ([11] and [16]). On the other hand, the answer is yes for the rational double points in characteristic 0 (cf. [1, Proposition 5.8]). The proof depends on their structures of quotient singularities and tautness. Notable facts in comparison with the case in characteristic 0 that the rational double points in positive characteristic are neither quotient singularities ([12, Theorem 9.2]) nor taut ([2, Section 3]) in general lead us to ask what happens in positive characteristic. In this paper, we consider the next question.

Question 1 (Question 15)**.** *Set*

$$\begin{aligned}\mathsf{Cat} &:= \{\mathbf{D}^{\mathrm{sg}}(\widehat{\mathcal{O}}_{X,p}) \mid (X,p) \text{ is a rational double point over } k\},\\ \mathsf{Dyn} &:= \{\Delta \mid \Delta \text{ is a simply-laced Dynkin graph}\}.\end{aligned}$$

Then a correspondence

$$\begin{array}{ccc}\mathsf{Cat}/(\text{triangulated equivalence}) & \to & \mathsf{Dyn}\\ \mathbf{D}^{\mathrm{sg}}(\widehat{\mathcal{O}}_{X,p}) & \mapsto & \Delta(X,p)\end{array}$$

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is a well-defined bijection, where each $\Delta(X,p)$ is the dual graph of the exceptional prime divisors of the minimal resolution of a rational double point (X,p). In particular, if the characteristic of k is 2, 3 or 5, then there exist two rational double points which are not analytically isomorphic but whose singularity categories are triangulated equivalent.

If this is true, we can construct counter-examples (Corollary 18) in positive characteristic of the next theorem (1) (and hence (2)).

Theorem 2 ([8, Theorem 5.9], cf. [9, Theorem 1.4])**.** *Let $R = \mathbb{C}[\![x_0, x_1, \ldots, x_n]\!] / \langle f \rangle$ be an isolated hypersurface singularity.*

(1) *The 0-th Hochschild cohomology of the dg singularity category $\mathbf{D}^{\mathrm{sg}}_{\mathrm{dg}}(R)$ is isomorphic to the Tyurina algebra of f.*
(2) *Let $S = \mathbb{C}[\![x_0, x_1, \ldots, x_n]\!] / \langle g \rangle$ be an isolated hypersurface singularity. If the dg singularity category $\mathbf{D}^{\mathrm{sg}}_{\mathrm{dg}}(S)$ is quasi-equivalent to $\mathbf{D}^{\mathrm{sg}}_{\mathrm{dg}}(R)$, then S is isomorphic to R.*

Conventions. (1) k denotes an algebraically closed field.
(2) Any functor between k-linear categories is assumed to be k-linear.
(3) Let $\mathcal{A}$ be a Krull–Schmidt category. $\mathrm{add}(L)$ is the full subcategory of $\mathcal{A}$ consisting of the summands of finite direct sums of an object L of $\mathcal{A}$. $\mathrm{ind}(\mathcal{A})$ stands for the category of isomorphism classes of indecomposable objects of $\mathcal{A}$. The Auslander algebra of $\mathcal{A}$ is an endomorphism ring $\mathrm{Aus}(\mathcal{A}) := \mathrm{End}_{\mathcal{A}}(M)$, where $M \in \mathcal{A}$ is the direct sum of representatives for $\mathrm{ind}(\mathcal{A})$. This is uniquely determined by $\mathcal{A}$ up to isomorphism. Provided $\mathcal{A}$ constitutes a Frobenius category, the stable category is denoted by $\underline{\mathcal{A}}$ and any representative $N \in \mathcal{A}$ for $\mathrm{ind}(\underline{\mathcal{A}})$ is assumed to be indecomposable.
(4) Let Γ be a quiver. $k(\Gamma)$ (resp. $k[\Gamma]$) denotes the path category (resp. the path algebra) of Γ.
(5) Let (Γ, τ) be a finite translation quiver. $k(\Gamma, \tau)$ (resp. $k[\Gamma, \tau]$) denotes the mesh category (resp. the mesh algebra) of (Γ, τ), which is the quotient of the path category (resp. the path algebra) of Γ by the mesh ideal.
(6) Let A be a ring. $\mathrm{mod}\, A$ stands for the category of finitely generated left A-modules, and $\mathrm{proj}\, A$ is a full subcategory of $\mathrm{mod}\, A$ consisting of the projective left A-modules. $M^\vee$ denotes the dual $\mathrm{Hom}_A(M, A)$ of a left A-module M.
(7) Let X be a normal surface with only one isolated singular point p. The dual graph of the exceptional prime divisors of the minimal resolution is referred to as the dual graph of the singularity.

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2. Preliminaries

Let X be a quasi-projective k-variety with only one isolated Gorenstein singular point p and R denote $\widehat{\mathcal{O}}_{X,p}$.

Theorem 3 ([5, Chapter 4])**.** (1) *The category* $\mathrm{MCM}(R)$ *of maximal Cohen–Macaulay R-modules constitutes a Frobenius subcategory of* $\mathrm{mod}\, R$*.*

(2) *A functor*

$$\underline{\mathrm{MCM}}(R) \to \mathbf{D}^{\mathrm{sg}}(R)$$

induced by composition $\mathrm{MCM}(R) \hookrightarrow \mathrm{mod}\, R \hookrightarrow \mathbf{D}^{\mathrm{sg}}(R)$ *is a triangulated equivalence.*

By virtue of Theorem 3, it suffices to consider $\underline{\mathrm{MCM}}(R)$ in order to study the singularity category of an isolated Gorenstein singularity (X, p). We use Auslander–Reiten theory of maximal Cohen–Macaulay R-modules. This theory provides us cohomological methods which enable us to systematically investigate the structures of $\mathrm{MCM}(R)$ and $\underline{\mathrm{MCM}}(R)$ via their Auslander–Reiten quivers.

Lemma 4. (1) $\underline{\mathrm{MCM}}(R)$ *is a* Hom*-finite k-linear Krull–Schmidt category.*

(2) *Let $(\mathit{\Gamma}, \tau)$ be the Auslander–Reiten quiver of* $\underline{\mathrm{MCM}}(R)$*. Then for any representatives* $M, N \in \mathrm{MCM}(R)$ *for* $\mathrm{ind}(\underline{\mathrm{MCM}}(R))$*,*

$$\mathrm{rk}_k \underline{\mathrm{Hom}}_R(M, N) = \mathrm{rk}_k \mathrm{Ext}^1_R(N, \tau(M)).$$

Definition 5. (X, p) is simple if the following conditions hold.

- (X, p) is a hypersurface singularity;
- $\# \, \mathrm{ind}(\mathrm{MCM}(R)) < \infty$.

Theorem 6 ([2, Section 3] and [6, Theorem 1.4])**.** *The following statements are equivalent.*

(1) (X, p) *is a rational double point.*
(2) (X, p) *is simple of dimension* 2*.*
(3) *R is isomorphic to* $k\,[\![x, y, z]\!] \,/\, \langle f \rangle$ *with f one of the forms in Table 1.*

Table 1: The rational double points.

In characteristic 0 or not less than 7		
A_n	$z^{n+1} + xy$	*for* $n \geq 1$
D_n	$z^2 + x^2 y + y^{n-1}$	*for* $n \geq 4$
E_6	$z^2 + x^3 + y^4$	
E_7	$z^2 + x^3 + xy^3$	
E_8	$z^2 + x^3 + y^5$	
In characteristic 5		
A_n	$z^{n+1} + xy$	*for* $n \geq 1$
D_n	$z^2 + x^2 y + y^{n-1}$	*for* $n \geq 4$
E_6	$z^2 + x^3 + y^4$	
E_7	$z^2 + x^3 + xy^3$	
E_8^0	$z^2 + x^3 + y^5$	

E_8^1	$z^2+x^3+y^5+xy^4$	

<u>*In characteristic* 3</u>

A_n	$z^{n+1}+xy$	*for* $n\geq 1$
D_n	$z^2+x^2y+y^{n-1}$	*for* $n\geq 4$
E_6^0	$z^2+x^3+y^4$	
E_6^1	$z^2+x^3+y^4+x^2y^2$	
E_7^0	$z^2+x^3+xy^3$	
E_7^1	$z^2+x^3+xy^3+x^2y^2$	
E_8^0	$z^2+x^3+y^5$	
E_8^1	$z^2+x^3+y^5+x^2y^3$	
E_8^2	$z^2+x^3+y^5+x^2y^2$	

<u>*In characteristic* 2</u>

A_n	$z^{n+1}+xy$	*for* $n\geq 1$
D_{2n}^0	$z^2+x^2y+xy^n$	*for* $n\geq 2$
D_{2n}^r	$z^2+x^2y+xy^n+xy^{n-r}z$	*for* $n\geq 2$ *and* $1\leq r<n$
D_{2n+1}^0	$z^2+x^2y+y^nz$	*for* $n\geq 2$
D_{2n+1}^r	$z^2+x^2y+y^nz+xy^{n-r}z$	*for* $n\geq 2$ *and* $1\leq r<n$
E_6^0	$z^2+x^3+y^2z$	
E_6^1	$z^2+x^3+y^2z+xyz$	
E_7^0	$z^2+x^3+xy^3$	
E_7^1	$z^2+x^3+xy^3+x^2yz$	
E_7^2	$z^2+x^3+xy^3+y^3z$	
E_7^3	$z^2+x^3+xy^3+xyz$	
E_8^0	$z^2+x^3+y^5$	
E_8^1	$z^2+x^3+y^5+xy^3z$	
E_8^2	$z^2+x^3+y^5+xy^2z$	
E_8^3	$z^2+x^3+y^5+y^3z$	
E_8^4	$z^2+x^3+y^5+xyz$	

Remark 7. In Theorem 6 (3), the label of f in Table 1 illustrates what the dual graph of the rational double point is. For example if f is the form of E_8^1 in characteristic 2, the dual graph is the Dynkin graph of type E_8. On the other hand, for any f and g in Table 1, k-algebras $k\,[\![x,y,z]\!]\,/\,\langle f\rangle$ and $k\,[\![x,y,z]\!]\,/\,\langle g\rangle$ are isomorphic if and only if $f=g$. In particular, we cannot distinguish rational double points by their dual graphs if the characteristic of k is 2, 3 or 5.

Proposition 8 ([4, Theorem 1])**.** *Assume (X,p) be a rational double point. The Auslander–Reiten quiver (Γ,τ) of $\underline{\mathrm{MCM}}(R)$ is a double quiver of the dual graph of (X,p), where the Auslander–Reiten translation τ is the identity on Γ_0.*

In a suitable condition, the categorical structure of a Hom-finite k-linear Krull–Schmidt category can be recovered from its Auslander–Reiten quiver. The key is a notion of standardness of categories we introduce here.

Definition 9. A Hom-finite k-linear Krull–Schmidt category $\mathcal{A}$ is standard if

$$k(\Gamma,\tau) \simeq \mathrm{ind}(\mathcal{A}),$$

where (Γ,τ) is the Auslander–Reien quiver.

Remark 10. Let $\mathcal{A}$ be a Hom-finite k-linear Krull–Schmidt category satisfying $\#\,\mathrm{ind}(\mathcal{A}) < \infty$ and (Γ,τ) denote the Auslander–Reiten quiver. Then $\mathcal{A}$ is standard if and only if $k[\Gamma,\tau] \cong \mathrm{Aus}(\mathcal{A})$.

Proposition 11 ([7, Theorem 3.3], cf. [10, Corollary 2])**.** *Set*

$$\mathsf{Cat} := \left\{ \mathcal{A} \;\middle|\; \begin{array}{l} \mathcal{A} \text{ is a standard Hom-finite } k\text{-linear Krull–Schmidt} \\ \text{algebraic triangulated category satisfying } \#\,\mathrm{ind}(\mathcal{A}) < \infty \end{array} \right\},$$

$$\mathsf{Alg} := \{ k[\Gamma,\tau] \mid (\Gamma,\tau) \text{ is a finite translation quiver satisfying } \mathrm{rk}_k\, k[\Gamma,\tau] < \infty \}.$$

Then a correspondence

$$\begin{array}{ccc} \mathsf{Cat}/(\text{triangulated equivalence}) & \rightleftarrows & \mathsf{Alg}/\cong \\ \mathcal{A} & \mapsto & \mathrm{Aus}(\mathcal{A}) \\ \mathrm{proj}\, k[\Gamma,\tau] & \leftarrow\!\shortmid & k[\Gamma,\tau] \end{array}$$

is a well-defined bijection.

3. Main result

Let R denote $\widehat{\mathcal{O}}_{X,p}$ for a rational double point (X,p) over k, $\pi\colon \widetilde{X} \to \mathrm{Spec}\, R$ the minimal resolution, $E_1, E_2, \ldots, E_n$ the exceptional prime divisors and T_n the Dynkin type of the dual graph of (X,p) with $T \in \{A, D, E\}$. We identify a maximal Cohen–Macaulay R-module M with the associated coherent sheaf, and set

$$\widetilde{M} := (\pi^* M)/(\text{torsion}).$$

Let (Γ,τ) denote the Auslander–Reiten quiver of $\underline{\mathrm{MCM}}(R)$, and $M_1, M_2, \ldots, M_n \in \mathrm{MCM}(R)$ be the representatives for $\mathrm{ind}(\underline{\mathrm{MCM}}(R))$.

Question 12. *A* Hom-*finite k-linear Krull–Schmidt algebraic triangulated category* $\underline{\mathrm{MCM}}(R)$ *is standard if*

$$\mathrm{rk}_k\, k[\Gamma,\tau] = \mathrm{rk}_k\, \mathrm{Aus}(\underline{\mathrm{MCM}}(R)).$$

Lemma 13. *For any maximal Cohen–Macaulay R-modules M and N,*

$$\mathrm{rk}_k\, \mathrm{Ext}^1_R(M,N) = c_1(\widetilde{M}) \cdot c_1(\widetilde{N^\vee}^\vee / \widetilde{N}).$$

Proposition 14.

$$\mathrm{rk}_k\, k[\Gamma,\tau] = \mathrm{rk}_k\, \mathrm{Aus}(\underline{\mathrm{MCM}}(R)).$$

Proof. We proceed in two steps.

Step 1. We first calculate $\mathrm{rk}_k\, \mathrm{Aus}(\underline{\mathrm{MCM}}(R))$. By [3, Theorem 1.11], there exist irreducible curves $\widetilde{D}_1, \widetilde{D}_2, \ldots, \widetilde{D}_n$ on $\widetilde{X}$ which represent 1st Chern classes $c_1(\widetilde{M_1}), c_1(\widetilde{M_2}), \ldots, c_1(\widetilde{M_n}) \in \mathrm{Cl}(\widetilde{X})$ respectively and after reordering $M_1, M_2, \ldots, M_n$ satisfy

$$c_1(\widetilde{M_i}) \cdot E_j = \widetilde{D}_i \cdot E_j = \delta_{i,j} \quad \text{for } i,j = 1,2,\ldots,n.$$

Set

$$D_j \coloneqq \pi_* \widetilde{D}_j \quad \text{for } j = 1, 2, \ldots, n.$$

Then the rational pullbacks are

$$\pi^* D_j = \widetilde{D}_j + \begin{pmatrix} E_1 & E_2 & \cdots & E_n \end{pmatrix} \boldsymbol{v}_j = c_1(\widetilde{M}_j) + \begin{pmatrix} E_1 & E_2 & \cdots & E_n \end{pmatrix} \boldsymbol{v}_j \quad \text{for } j = 1, 2, \ldots, n,$$

where

$$\begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{pmatrix} \coloneqq (v_{i,j})_{i,j} \coloneqq -(E_i \cdot E_j)^{-1}_{i,j}$$

is the inverse of the Cartan matrix of type T_n. Each $M_j^\vee$ is also an indecomposable maximal Cohen-Macaulay R-module and there exists a permutation $\sigma \in \mathfrak{S}_n$ such that $M_j^\vee \cong M_{\sigma(j)}$ for any $j = 1, 2, \ldots, n$. Furthermore, considering $\operatorname{Supp}\left(\pi_*(\widetilde{M_j^\vee}^\vee / \widetilde{M}_j)\right) \subset \{p\}$, we have

$$0 = \pi_* \left(c_1(\widetilde{M_j^\vee}^\vee / \widetilde{M}_j) \right) = \pi_*(-\widetilde{D}_{\sigma(j)} - \widetilde{D}_j) = -D_{\sigma(j)} - D_j \quad \text{for } j = 1, 2, \ldots, n.$$

From the above,

$$\begin{aligned}
&\operatorname{rk}_k \operatorname{Aus}(\underline{\mathrm{MCM}}(R)) \\
&= \sum_{i,j} \operatorname{rk}_k \operatorname{Ext}^1_R(M_i, M_j) && \text{by Lemma 4} \\
&= \sum_{i,j} c_1(\widetilde{M}_i) \cdot \left(-c_1(\widetilde{M_{\sigma(j)}}) - c_1(\widetilde{M}_j) \right) && \text{by Lemma 13} \\
&= \sum_{i,j} c_1(\widetilde{M}_i) \cdot \left(-\pi^*(D_{\sigma(j)} + D_j) + \begin{pmatrix} E_1 & E_2 & \cdots & E_n \end{pmatrix} (\boldsymbol{v}_{\sigma(j)} + \boldsymbol{v}_j) \right) \\
&= \sum_{i,j} c_1(\widetilde{M}_i) \cdot \begin{pmatrix} E_1 & E_2 & \cdots & E_n \end{pmatrix} (\boldsymbol{v}_{\sigma(j)} + \boldsymbol{v}_j) \\
&= 2 \sum_{i,j} v_{i,j}.
\end{aligned}$$

Summing up the entries of the inverse (i.e. $(v_{i,j})_{i,j}$) of the Cartan matrix of type T_n, we get $\operatorname{rk}_k \operatorname{Aus}(\underline{\mathrm{MCM}}(R))$ as in Table 2.

TABLE 2. The rank of $\operatorname{Aus}(\underline{\mathrm{MCM}}(R))$ over k.

T_n	A_n	D_n	E_6	E_7	E_8
rk_k	$\frac{1}{6}n(n+1)(n+2)$	$\frac{1}{3}n(n-1)(2n-1)$	156	399	1240

Step 2. Next, we calculate $\operatorname{rk}_k k[\varGamma, \tau]$. Let Q be a quiver whose underlying graph is the Dynkin graph of type T_n. Since $k[\varGamma, \tau]$ is the preprojective algebra associated to the Dynkin graph of type T_n, this is the direct sum of indecomposable left $k[Q]$-modules.

Hence by Gabriel's Theorem,

$$\begin{aligned}
&\operatorname{rk}_k k[\Gamma, \tau] \\
&= \sum_{\alpha:\text{ a positive root in type } T_n} (\text{the height of } \alpha) \\
&= \sum_{e:\text{ an exponent in type } T_n} (1 + 2 + \cdots + e).
\end{aligned}$$

We can verify $\operatorname{rk}_k k[\Gamma, \tau]$ is equal to $\operatorname{rk}_k \operatorname{Aus}(\underline{\mathrm{MCM}}(R))$. □

If Question 12 is true, combining Proposition 14 and Proposition 11, we can conclude that $\underline{\mathrm{MCM}}(R) \simeq \mathbf{D}^{\mathrm{sg}}(R)$ depends only on the Dynkin type T_n but not the analytic class of R.

Question 15. *Set*

$$\begin{aligned}
\mathsf{Cat} &:= \{\mathbf{D}^{\mathrm{sg}}(\widehat{\mathcal{O}}_{X,p}) \mid (X,p) \text{ is a rational double point over } k\}, \\
\mathsf{Dyn} &:= \{\Delta \mid \Delta \text{ is a simply-laced Dynkin graph}\}.
\end{aligned}$$

Then a correspondence

$$\begin{array}{ccc}
\mathsf{Cat}/(\text{triangulated equivalence}) & \to & \mathsf{Dyn} \\
\mathbf{D}^{\mathrm{sg}}(\widehat{\mathcal{O}}_{X,p}) & \mapsto & \text{the dual graph of } (X,p)
\end{array}$$

is a well-defined bijection. In particular, if the characteristic of k is 2, 3 or 5, then there exist two rational double points which are not analytically isomorphic but whose singularity categories are triangulated equivalent.

4. Application

If Question 15 is true, we can construct counter-examples (Question 18) in positive characteristic of Theorem 2 (1) (and hence (2)). Let $\mathbf{Perf}_{\mathrm{dg}}(X)$ and $\mathbf{D}^{\mathrm{b}}_{\mathrm{dg}}(\operatorname{Coh} X)$ denote the unique dg enhancements ([13, Theorem 7.9 and Theorem 8.13]) of $\mathbf{Perf}(X)$ and $\mathbf{D}^{\mathrm{b}}(\operatorname{Coh} X)$ respectively for an ELF k-scheme X. The *dg singularity category* $\mathbf{D}^{\mathrm{sg}}_{\mathrm{dg}}(X)$ is defined to be the dg quotient of $\mathbf{D}^{\mathrm{b}}_{\mathrm{dg}}(\operatorname{Coh} X)$ by $\mathbf{Perf}_{\mathrm{dg}}(X)$. By construction, $\mathbf{D}^{\mathrm{sg}}_{\mathrm{dg}}(X)$ is a dg enhancement of $\mathbf{D}^{\mathrm{sg}}(X)$.

Theorem 16 ([15])**.** *Let $\mathcal{A}$ be a* Hom*-finite k-linear Krull–Schmidt algebraic triangulated category. If $\mathcal{A}$ has an additive generator, then $\mathcal{A}$ admits a unique dg enhancement.*

Corollary 17. *Let (X,p) be a simple singularity. Then the singularity category $\mathbf{D}^{\mathrm{sg}}(\widehat{\mathcal{O}}_{X,p})$ admits a unique dg enhancement.*

Proof. The direct sum $M \in \underline{\mathrm{MCM}}(\widehat{\mathcal{O}}_{X,p})$ of representatives for $\operatorname{ind}(\underline{\mathrm{MCM}}(\widehat{\mathcal{O}}_{X,p}))$ is an additive generator of $\underline{\mathrm{MCM}}(\widehat{\mathcal{O}}_{X,p}) \simeq \mathbf{D}^{\mathrm{sg}}(\widehat{\mathcal{O}}_{X,p})$. □

Corollary 18. *Assume the characteristic of k is 2, 3 or 5 and Question 15 is true. For any two positive integers m and n, there exist even dimensional simple singularities $k[\![x_0, x_1, \ldots, x_{2m}]\!] / \langle f \rangle$ and $k[\![x_0, x_1, \ldots, x_{2n}]\!] / \langle g \rangle$ such that their dg singularity categories are quasi-equivalent but the Tyurina algebras of f and g are not isomorphic.*

Proof. By Knörrer's periodicity and [6], it suffices to show for $m = n = 1$. Let x, y and z denote x_0, x_1 and x_2 respectively and take f and g from Table 1 such that their Dynkin types coincide but $f \neq g$. Considering Question 15 and Corollary 17, we get a quasi-equivalence

$$\mathbf{D}^{\mathrm{sg}}_{\mathrm{dg}}(k\,[\![x,y,z]\!]\,/\,\langle f\rangle) \simeq \mathbf{D}^{\mathrm{sg}}_{\mathrm{dg}}(k\,[\![x,y,z]\!]\,/\,\langle g\rangle).$$

On the other hand, the Tyurina numbers of f and g are different (cf. [14, Part III 1.3]). ☐

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