CONSTRUCTION OF TWO-SIDED TILTING COMPLEXES FOR GENERALIZED BRAUER TREE ALGEBRAS

SHUJI FUJINO, YUTA KOZAKAI AND KOHEI TAKAMURA

Abstract. We construct a two-sided tilting complex that corresponds to a Membrillo-Hernández's tree-to-star tilting complex for a generalized Brauer tree algebra. This complex is a generalization of a Kozakai–Kunugi's two-sided tilting complex that corresponds to a Rickard's tree-to-star tilting complex for a Brauer tree algebra.

1. Introduction

Rickard [9] and Keller [3] showed that there exists a two-sided tilting complex that corresponds to an arbitrary tilting complex for a finite dimensional algebra. Constructing such a two-sided tilting complex is difficult for practical calculations. Therefore, Kozakai–Kunugi [4] provided an explicit description of a two-sided tilting complex that corresponds to a Rickard's tree-to-star tilting complex built in [8] for a Brauer tree algebra. Since Rickard's complexes for Brauer tree algebras are generalized to Membrilio-Hernández's complexes for generalized Brauer tree algebras, we anticipate that we can construct a two-sided tilting complex that corresponds to a Membrilio-Hernández's complex. In this paper, we explicitly construct a two-sided tilting complex that corresponds to a Membrilio-Hernández's tree-to-star tilting complex for generalized Brauer tree algebras. This note is based on [2].

The strategy to construct the two-sided tilting complex is the same as the one used in Kozakai–Kunugi [4], as follows: taking an indecomposable bimodule inducing a stable equivalence of Morita type that corresponds to the Membrilio-Hernández's tree-to-star tilting complex, taking a minimal projective resolution of the bimodule, and deleting some direct summands of each term of the resolution.

However, it does not work in parallel to prove that a constructed complex is indeed a two-sided tilting complex. The second syzygy of a simple module is a simple module for an endomorphism algebra of a Rickard's tilting complex for a Brauer tree algebra, but is not for an endomorphism algebra of a Membrillo-Hernández's tilting complex for a generalized Brauer tree algebra in general. Then we use perverse equivalences to show that the constructed complex is indeed a two-sided tilting complex.

In the paper, let k be an algebraically closed field, Γ and Λ finitely generated symmetric k-algebras, $D^b(\Gamma) := D^b(\operatorname{mod}\Gamma)$ the bounded derived category of finitely generated right Γ-modules, $K^b(\operatorname{proj}\Gamma)$ the bounded homotopy category of finitely generated projective right Γ-modules, and $\underline{\operatorname{mod}}\,\Gamma$ the stable module category. We assume that modules are finitely generated right modules unless otherwise stated.

The detailed version of this paper will be submitted for publication elsewhere.

Definition 1. We say that a bounded complex T of projective Γ-modules is a tilting complex if the following conditions are satisfied:

- $\mathrm{Hom}_{D^b(\Gamma)}(T, T[n]) = 0$ for any non-zero integer n.
- The algebra Γ is obtained by applying a finite sequence of operations, including taking direct sums, direct summands, mapping cones, and shifts of T.

The following theorem by Rickard ties tilting complexes with derived equivalences.

Theorem 2 ([7])**.** *The following conditions are equivalent.*

- *The bounded categories $D^b(\Gamma)$ and $D^b(\Lambda)$ are equivalent as triangulated categories.*
- *There exists a tilting complex T over the algebra Γ such that its endomorphism algebra is isomorphic to the algebra Λ.*

Definition 3 ([9])**.** Let C be a bounded complex of (Λ, Γ)-bimodules which are projective as Λ^{op}-modules and Γ-modules and D a bounded complex of (Γ, Λ)-bimodules which are projective as Γ^{op}-modules and Λ-modules. We say that C is a two-sided tilting complex if the following hold.

$$D \otimes^{\mathbb{L}}_{\Lambda} C \cong \Gamma \quad \text{in} \quad D^b\left(\Gamma^{\mathrm{op}} \otimes_k \Gamma\right) \qquad \text{and} \qquad C \otimes^{\mathbb{L}}_{\Gamma} D \cong \Lambda \quad \text{in} \quad D^b\left(\Lambda^{\mathrm{op}} \otimes_k \Lambda\right).$$

Rickard and Keller showed that we can construct a two-sided tilting complex from a tilting complex theoretically.

Theorem 4 ([9, 3])**.** *If T is a tilting complex of Γ-modules such that $\mathrm{End}_{D^b(\Gamma)}(T) \cong \Lambda$, then there exists a two-sided tilting complex C of (Λ, Γ)-bimodules whose restriction to Γ is isomorphic to T in $D^b(\Gamma)$.*

In [4], they construct a two-sided tilting complex over a Brauer tree algebra that corresponds to a Rickard tilting complex by lifting the stable equivalence of Morita type. We recall the definition of the equivalence introduced by Broué.

Definition 5. We say that Γ and Λ are *stably equivalent of Morita type* if there exist a (Λ, Γ)-bimodule M and a (Γ, Λ)-bimodule N such that the following conditions are satisfied.

- The bimodules M and N are projective as left modules and right modules.
- $N \otimes_{\Lambda} M \cong \Gamma \oplus P$ as (Γ, Γ)-bimodules for some projective (Γ, Γ)-bimodule P.
- $M \otimes_{\Gamma} N \cong \Lambda \oplus Q$ as (Λ, Λ)-bimodules for some projective (Λ, Λ)-bimodule Q.

We say that M *induces a stable equivalence of Morita type.*

The following lemma implicates that if two symmetric algebras are derived equivalent then they are stably equivalent of Morita type.

Proposition 6 ([9, 5])**.** *If $F : D^b(\Gamma) \to D^b(\Lambda)$ is a derived equivalence, then there is an indecomposable (Λ, Γ)-bimodule M inducing a stable equivalence of Morita type which* *commutes with the following diagram.*

$$\begin{array}{ccc}
D^b(\Lambda) & \xrightarrow{F^{-1}} & D^b(\Gamma) \\
\downarrow & & \downarrow \\
D^b(\Lambda)/\,K^b(\operatorname{proj}\Lambda) & & D^b(\Gamma)/\,K^b(\operatorname{proj}\Gamma) \\
\wr\!\| & & \wr\!\| \\
\underline{\operatorname{mod}}\,\Lambda & \xrightarrow{-\otimes_\Lambda M} & \underline{\operatorname{mod}}\,\Gamma
\end{array} .$$

We remark that the existence of a module M satisfying the conditions in the above proposition follows from [9, Corollary 5.5] and its indecomposability follows from [5, Proposition 2.4].

The following proposition gives some expression of a projective cover of a bimodule.

Proposition 7 ([10])**.** *Let M be a (Λ, Γ)-bimodule, which is projective as a Λ^{op}-module and a Γ-module. Then a projective cover of M is isomorphic to*

$$\bigoplus_{V \in \mathcal{S}'} Q(V)^* \otimes_k P(V \otimes_\Lambda M),$$

where $P(-)$ (resp. $Q(-)$) denotes a projective cover of a Γ (resp. Λ)-module, $-^*$ a k-dual of a module, and $\mathcal{S}'$ a complete set of representatives of isomorphism classes of simple Λ-modules.

Since the kernel of a projective cover of M is $\Omega\, M$, projective as a Λ^{op}-module and a Γ-module, we can express each term of a minimal projective resolution of M.

2. Perverse equivalences

Perverse equivalences are special derived equivalences introduced by Chuang–Rouquier in [1]. In this chapter, we recall the notion of perverse equivalences. Let Γ and Λ be symmetric k-algebras. Let $\mathcal{S}$ be a complete set of representatives of isomorphism classes of simple Γ-modules, r a positive integer, $q : \{0, \dots, r-1\} \to \mathbb{Z}$ a map, and $\mathcal{S}_\bullet$ a filtration of $\mathcal{S}$ satisfying

$$\emptyset = \mathcal{S}_{-1} \subseteq \mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_{r-1} = \mathcal{S}.$$

Similarly, let $\mathcal{S}'$ be a complete set of representatives of isomorphism classes of simple Λ-modules and $\mathcal{S}'_\bullet$ a filtration of $\mathcal{S}'$ satisfying

$$\emptyset = \mathcal{S}'_{-1} \subseteq \mathcal{S}'_0 \subseteq \mathcal{S}'_1 \subseteq \cdots \subseteq \mathcal{S}'_{r-1} = \mathcal{S}'.$$

Definition 8. An equivalence $F : D^b(\Gamma) \to D^b(\Lambda)$ is perverse relative to $(\mathcal{S}_\bullet, \mathcal{S}'_\bullet, q)$ if for every i, the following hold:

- For $S \in \mathcal{S}_i - \mathcal{S}_{i-1}$ and $\ell \neq -q(i)$, all composition factors of the ℓth cohomology of $F(S)$, denoted by $H^\ell(F(S))$ are in S'_{i-1}.
- There exist submodules ${}^\exists L_1 \leq {}^\exists L_2 \leq H^{-q(i)}(F(S))$ such that all composition factors of L_1 and $H^{-q(i)}(F(S))/L_2$ are in $\mathcal{S}'_{i-1}$ and $L_2/L_1 \in \mathcal{S}'_i - \mathcal{S}'_{i-1}$.
- The map $S \mapsto L_2/L_1$ induces a bijection $\mathcal{S}_i - \mathcal{S}_{i-1} \to \mathcal{S}'_i - \mathcal{S}'_{i-1}$.

We remark that the definition of perverse equivalences above is equivalent to the one established by Chuang–Rouquier, as supported by [1, Lemma 4.19]. The following proposition holds.

Proposition 9 ([1])**.** *Let $\mathcal{S}_\bullet$, $\mathcal{S}'_\bullet$ and $\mathcal{S}''_\bullet$ denote filtrations of sets of representatives of isomorphism classes of simple Γ, Γ', and Γ''-modules, respectively, having a common length. Let q and q' be maps from the indices of filtrations to integers. Let $F : D^b(\Gamma) \xrightarrow{\sim} D^b(\Gamma')$ and $G : D^b(\Gamma') \xrightarrow{\sim} D^b(\Gamma'')$ be perverse equivalences relative to $(\mathcal{S}_\bullet, \mathcal{S}'_\bullet, q)$ and $(\mathcal{S}'_\bullet, \mathcal{S}''_\bullet, q')$, respectively. The following hold:*

(1) *F^{-1} is perverse relative to $(\mathcal{S}'_\bullet, \mathcal{S}_\bullet, -q)$.*
(2) *$G \circ F$ is perverse relative to $(\mathcal{S}_\bullet, \mathcal{S}''_\bullet, q + q')$.*
(3) *If $q = 0$, then F restricts to a Morita equivalence from Γ to Γ'.*

For symmetric algebras, Chuang–Rouquier provided decreasing perversities, the computational framework for perverse equivalences relative to $(\mathcal{S}_\bullet, \mathcal{S}'_\bullet, q)$ satisfying $q(i) = -i$ and established images of simple modules through the equivalences. We can describe how to construct such derived equivalences and the images of simple modules by the following way [1].

For $i \in \{0, \ldots, r-1\}$ and $S \in \mathcal{S}_i - \mathcal{S}_{i-1}$, we construct $T_S = (T_S^\ell, d^\ell)_{\ell \in \mathbb{Z}}$ a complex with zero terms in degrees other than $-r+1, \ldots, -i$, as follows. Put $T_S^{-i} = P(S)$. Having constructed T_S^u and d^u for all $u \in \{-j, \ldots, -i\}$ for $j \in \{i, \ldots, r-2\}$, let M^{-j} be the smallest submodule of $K^{-j} := \mathrm{Ker}(d^{-j} : T_S^{-j} \to T_S^{1-j})$ such that all composition factors of K^{-j}/M^{-j} lie in $\mathcal{S}_j$. Define $d^{-j-1} : T_S^{-j-1} \to T_S^{-j}$ to be the composition of a projective cover $T_S^{-j-1} \to M^{-j}$ with the inclusion of M^{-j} into T_S^{-j}. Then the following proposition holds.

Proposition 10 ([1, Proposition 5.7])**.** *The complex $T = \bigoplus_{S \in \mathcal{S}} T_S$ is a tilting complex and the equivalence $F = \mathrm{Hom}^\bullet_\Gamma(T, -) : D^b(\Gamma) \to D^b(\mathrm{End}_{K^b(\mathrm{proj}\,\Gamma)}(T))$ is perverse relative to $(\mathcal{S}_\bullet, \mathcal{S}'_\bullet, q)$, where q is given by $q(i) = -i$.*

Let $\Lambda = \mathrm{End}_{K^b(\mathrm{proj}\,\Gamma)}(T)$. Given $S \in \mathcal{S}$, then $F(T_S)$ is isomorphic to an indecomposable projective Λ-module whose simple quotient we denote by S'. For $S \in \mathcal{S}_i - \mathcal{S}_{i-1}$, we construct a complex $Y_S = (Y_S^\ell, d^\ell)_{\ell \in \mathbb{Z}}$ with zero terms in degrees other than $-i, \ldots, 0$. If $i = 0$, we put $Y_S = S$. Otherwise we start by putting $Y_S^{-i} = P(S)$. If $i = 1$, we define $d^{-i} : Y_S^{-i} \to Y_S^{1-i}$ to be the quotient map from $P(S)$ to $P(S)/N^{1-i}$, where we define N^{1-i} to be the largest submodule of $P(S)$ such that all composition factors of N^{1-i}/S are in $\mathcal{S}_{i-1}$. Otherwise, we define $d^{-i} : Y_S^{-i} \to Y_S^{1-i}$ to be the composition of the quotient map $P(S) \to P(S)/N^{1-i}$ with an injective hull $P(S)/N^{1-i} \to Y_S^{1-i}$.

Having constructed Y_S^u and d_S^{-1+u} for all $u \in \{-i, \ldots, -j\}$ for $-j \in \{1-i, \ldots, -1\}$, let N^{1-j} be the largest quotient of C^{1-j}:=Coker$(d^{-1-j} : Y_S^{-1-j} \to Y_S^{-j})$ such that all composition factor of $\mathrm{Ker}(C^{1-j} \to N^{1-j})$ are in $\mathcal{S}_{j-1}$. Then let $d^{-j} : Y_S^{-j} \to Y_S^{1-j}$ be the composition of the canonical epimorphism $Y_S^{-j} \to N^{1-j}$ with an injective hull $N^{1-j} \to Y_S^{1-j}$. When $j = 1$, the constructions of C^{1-j} and N^{1-j} are the same, but $d^{-j} : Y_S^{-j} \to Y_S^{1-j}$ is the canonical epimorphism $Y_S^{-j} \to N^{1-j} = Y_S^{1-j}$. The following proposition holds:

Proposition 11 ([1, Lemma 5.9.])**.** *We have $Y_S \cong F^{-1}(S')$ for $S \in \mathcal{S}$.*

3. Generalized Brauer tree algebras

In this section, we recall a generalized Brauer tree algebra.

Definition 12. A generalized Brauer tree $\mathcal{T}$ consists of a quadruple (V, E, m, ρ), where V is a finite set, E is a subset of the set consisting just two elements of the power set of V such that (V, E) is a connected tree, m is a function from V to positive integers, and ρ_v is a cyclic order on the set $\{e \in E \mid v \in e\}$ for $v \in V$. We remark that when the set $\{e \in E \mid v \in e\}$ consists of a unique element e_0, we do not define the cyclic order for v if the multiplicity of v is 1, on the other hand, we define the cyclic order on v as $e_0 < e_0$ if the multiplicity of v is greater than 1. We call an element of V a *vertex*, an element of E an *edge*, and an image of $v \in E$ by m denoted by m_v the *multiplicity* of v.

Let $\mathcal{T} = (V, E, m, \rho)$ be a generalized Brauer tree, $Q = (Q_0, Q_1)$ be a quiver and I a relation ideal of path algebra kQ. *A bound quiver algebra $A \cong kQ/I$ is a generalized Brauer tree algebra associated to* $\mathcal{T}$ if the following hold.

The set Q_0 is the set of edges E. For $e, e' \in E$, we draw a unique arrow in Q_1 from e to e' if and only if there exists a vertex v such that e' is the following edge of e respective to the cyclic order ρ_v. For $\alpha \in Q_1$ from e to e', we denote $\pi(\alpha)$ to be a unique vertex $v \in V$ such that e' is the following edge of e respective to the cyclic order ρ_v. The ideal I is generated by

- $\alpha\beta$ if $\pi(\alpha) \neq \pi(\beta)$ for $\alpha, \beta \in Q_1$.
- $\alpha_1\alpha_2 \dots \alpha_s\alpha_{s+1}$ for $\pi(\alpha_i) = v$ for all $1 \leq i \leq s$ and $s = m_v|\{e \in E \mid v \in e\}|$.
- $\alpha_1\alpha_2 \dots \alpha_s - \beta_1\beta_2 \dots \beta_t$ for $\pi(\alpha_i) = v$ for all $1 \leq i \leq s$ and for $\pi(\beta_j) = w$ for all $1 \leq j \leq t$ and $s = m_v|\{e \in E \mid v \in e\}|$ and $t = m_w|\{e \in E \mid w \in e\}|$ if $\{v, w\} \in E$ and the sources of α_1 and β_1 are equivalent to the edge $\{v, w\}$.

Remark 13. Generalized Brauer tree algebras are Brauer graph algebras associated to tree-shaped Brauer graphs and indecomposable symmetric special biserial algebras of tame representation type. Moreover, a generalized Brauer tree algebra is of finite representation type if and only if it is a Brauer tree algebra.

4. Previous work

Let $\mathcal{T} = (V, E, m, \rho)$ be a generalized Brauer tree and A a generalized Brauer tree algebra associated to $\mathcal{T}$. We denote an indecomposable projective A-module associated to an edge n by P_n and simple module by S_n. We fix a vertex v_0 of $\mathcal{T}$. For each edge n, we take the path starting from n and ending at v_0 as

$$(p_n^0, p_n^1, \dots, p_n^{d(n)-2}, p_n^{d(n)-1}),$$

where we denote the length of path from n to v_0 by $d(n)$. We also denote the maximum of $d(n)$ for $n \in E$ by r. Let T_n denote a bounded complex of A-modules:

$$\begin{array}{ccccccc} (-r+1)\text{st} & & (-r+2)\text{nd} & & & & (-r+d(n))\text{th} \\ P_{p_n^{d(n)-1}} & \longrightarrow & P_{p_n^{d(n)-2}} & \longrightarrow \quad \cdots \quad \longrightarrow & P_{p_n^1} & \longrightarrow & P_{p_n^0} \end{array},$$

where each differential morphism of the complex is right minimal. Put $T = \bigoplus_{n \in E} T_n$. Membrillo-Hernández showed that the complex T is a tilting complex for a generalized

Brauer tree algebra which is a generalization of a Rickard's tilting complex for a Brauer tree algebra.

Theorem 14 ([6]). *The complex T is a tilting complex. The endomorphism algebra $B = \mathrm{End}(T)$ is a generalized Brauer tree algebra associated to a star-shaped generalized Brauer tree.*

For Brauer tree algebras, Kozakai–Kunugi constructed a two-sided tilting complex corresponding to T by lifting a bimodule inducing a stable equivalence of Morita type. In the following section we see the same holds for generalized Brauer tree algebras.

5. Main results

We use the hypotheses and notation made in Section 4. Let $\mathcal{S}$ denote the set $\{S_n \mid n \in E\}$. The set $\mathcal{S}_i = \{S_n \in \mathcal{S} \mid d(n) \geq r - i\}$ gives a filtration of the simple A-modules.

$$\mathcal{S}_\bullet : \emptyset = \mathcal{S}_{-1} \subseteq \mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_{r-1} = \mathcal{S}$$

Let V_n denote a simple B-module that corresponds to the indecomposable projective module $\mathrm{Hom}(T, T_n)$ for an edge n of $\mathcal{T}$, and $\mathcal{S}'$ the set $\{V_n \mid n \in E\}$. The set $\mathcal{S}'_i = \{V_n \in \mathcal{S}' \mid S_n \in \mathcal{S}_i\}$ gives a filtration of the set $\mathcal{S}'$.

$$\mathcal{S}'_\bullet : \emptyset = \mathcal{S}'_{-1} \subseteq \mathcal{S}'_0 \subseteq \mathcal{S}'_1 \subseteq \cdots \subseteq \mathcal{S}'_{r-1} = \mathcal{S}'$$

By applying the construction method of tilting complex described before Proposition 10 and the induction on the index of simple filtrations, we have the following lemma:

Lemma 15 ([2]). *The derived equivalence F induced by the Membrillo-Hernández's tilting complex T is perverse equivalence relative to $(\mathcal{S}_\bullet, \mathcal{S}'_\bullet, q)$ for $q(i) = -i$.*

For each edge $n \in E$, we denote the source of the path starting from n and ending at $v_0 \in V$ by v_n. We denote the following edge of n associated to the cyclic order ρ_{v_n} around v_n by n'. We denote the smallest length uniserial module to have S_n for the bottom and $S_{n'}$ for the top by L_n. By applying the construction method described before Proposition 11 and the induction on the index of simple filtrations, we have the following lemma:

Lemma 16 ([2]). *For each edge n of $\mathcal{T}$, the $(r-d(n))$-shift of the module L_n is isomorphic to a complex of A-modules $F^{-1}(V_n)$ in the bounded derived A-module category.*

We denote an indecomposable projective B-module $\mathrm{Hom}(T, T_n)$ by Q_n for each edge n. We denote by $P(-)$ a projective cover of a module and by $\Omega(-)$ a syzygy of a module. We denote an indecomposable (B, A)-bimodule giving stable equivalence of Morita type induced by F^{-1} by M. By Proposition 6, we have $V_n \otimes_B M$ is isomorphic to $\Omega^{d(n)-r} L_n$ in $\mathrm{mod}\, A$.

Let $P^\bullet$ be a minimal projective resolution of $B^{\mathrm{op}} \otimes_k A$-module M. By Proposition 7, we can express each term of $P^\bullet$ as follows.

$$P^{-t} = \begin{cases} M & (t = 0), \\ \bigoplus_{n \in E} Q_n^* \otimes_k P(\Omega^{t-1+d(n)-r} L_n) & (t > 0), \\ 0 & (t < 0). \end{cases}$$

We delete some direct summands of each term and get a subcomplex C of $P^\bullet$ as follows.

$$C^{-t} = \begin{cases} M & (t=0), \\ \bigoplus_{n \in E,\ d(n) \le r-t} Q_n^* \otimes_k P(\Omega^{t-1+d(n)-r} L_n) & (t>0), \\ 0 & (t<0). \end{cases}$$

Each differential map of C is restriction of the differential map of $P^\bullet$.

Theorem 17 ([2])**.** *Let A be a generalized Brauer tree k-algebra, T a tilting complex of A-modules given in* [6], *$B := \mathrm{End}(T)$ denote the generalized Brauer star algebra, M an indecomposable $B^{\mathrm{op}} \otimes_k A$-module inducing a stable equivalence of Morita type corresponding to T, and C a complex given by deleting some direct summands of a minimal projective resolution of M as above. Then following statements hold.*

(1) *The complex C is a two-sided tilting complex of $B^{\mathrm{op}} \otimes_k A$-modules.*
(2) *The restriction of the complex C of $B^{\mathrm{op}} \otimes_k A$-modules to A-modules is isomorphic to the tilting complex T in $D^b(A)$.*

Sketch of proof. Since $V_n \otimes_B Q_\ell^* = \mathrm{Hom}_B(Q_\ell, V_n) = \delta_{n,\ell} k$ for $n, \ell \in E$, where δ means Kronecker's delta, we have

$$V_n \otimes_B C^{-t} = \begin{cases} V_n \otimes_B M \cong \Omega^{d(n)-r} L_n & (t=0), \\ P(\Omega^{t-1+d(n)-r} L_n) & (t>0 \quad \text{and} \quad d(n) \le r-t), \\ 0 & (t<0 \quad \text{and} \quad d(n) > r-t). \end{cases}$$

Since each differential map of C from C^{-t} to C^{-t+1} for $d(n) - r \le -t < 0$ is minimal and the cohomology at the 0th term of $V_n \otimes_B C$ is 0, the complex $V_n \otimes_B C$ is isomorphic to the $(r - d(n))$-shift of a minimal projective resolution of L_n. Therefore $V_n \otimes_B C$ is isomorphic to $L_n[r - d(n)]$ in the bounded derived A-module category. By Lemma 16, we have $V_n \otimes_B C \cong F^{-1}(V_n)$. Hence, the following isomorphisms hold for $n, \ell \in E$ and u an integer.

$$\begin{aligned} & \mathrm{Hom}_{D^b(B^{\mathrm{op}} \otimes_k B)}(C \otimes_A C^*, V_n^* \otimes_k V_\ell[-u]) \\ \cong & \mathrm{Hom}_{D^b(A)}(V_n \otimes_B C, V_\ell \otimes_B C[-u]) \\ \cong & \mathrm{Hom}_{D^b(A)}(F^{-1}(V_n), F^{-1}(V_\ell)[-u]) \\ \cong & \mathrm{Hom}_{D^b(B)}(V_n, V_\ell[-u]) \\ \cong & \delta_{n\ell} \delta_{u0} k. \end{aligned}$$

Therefore, for $i \neq 0$, $C \otimes_A C^*$ has no cohomology in the ith term. Since the ith term of $C \otimes_A C^*$ is projective, $C \otimes_A C^*$ is homotopy equivalent to the 0th term. By [11, Lemma 10.2.5], the 0th cohomology of $C \otimes_A C^*$ has a direct summand B. Therefore, we have $C \otimes_A C^*$ is homotopy equivalent to B. By [11, Lemma 10.2.4], the complex C is a two-sided tilting complex. Hence we have C is a two-sided tilting complex and (1) holds. Moreover, $- \otimes_B C$ is perverse equivalence with respect to $(\mathcal{S}'_\bullet, \mathcal{S}_\bullet, q)$ for a perverse function $q(i) = i$. Since $F(- \otimes_B C)$ induces a Morita equivalence by Proposition 9, we have

$$F(B \otimes_B C) \cong B \cong F(T).$$

By applying the functor F^{-1}, we have (2). □

This theorem generalizes a Kozakai–Kunugi's two-sided tilting complex that corresponds to a Rickard's tilting complex for a Brauer tree algebra to a two-sided tilting complex that corresponds to a Membrillo-Hernández's tilting complex for a generalized Brauer tree algebra.

They used the concept of perverse equivalences to show that the restriction of a constructed two-sided tilting complex is equivalent to the Rickard's tilting complex. On the other hand, we used perverse equivalences to show that not only the restriction of a constructed two-sided tilting complex is equivalent to the Membrillo-Hernández's tilting complex but also the constructed complex is indeed a two-sided tilting complex.

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Shuji Fujino

Department of Mathematics
Graduate School of Science
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601 JAPAN

Email address: `1124702@ed.tus.ac.jp`

Yuta Kozakai

Department of Mathematics
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601 JAPAN

Email address: `kozakai@rs.tus.ac.jp`

Kohei Takamura

Department of Mathematics
Graduate School of Science
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601 JAPAN

Email address: `1122514@alumni.tus.ac.jp`