THE CORRESPONDENCE BETWEEN SILTING OBJECTS AND T-STRUCTURES FOR NON-POSITIVE DG ALGEBRAS

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Abstract. For a finite-dimensional algebra Λ, Koenig and Yang established the bijection between silting objects of $K^b(\mathbf{proj}\Lambda)$ and simple-minded collection of $D^b(\mathbf{mod}\Lambda)$ [KoY], which is called ST-correspondence. The dg endomorphism of silting objects are non-positive dg algebra, and the dg endomorphism of simple-minded collections are positive dg algebra. We establish Koszul dualities between locally finite non-positive dg algebras and locally finite positive dg algebras. As a direct application of the theory of Koszul dualities, we establish the bijection between silting objects and simple-minded collections, and characterize length hearts via functorially finiteness.

1. Introduction

Let A be a dg algebra over a field k. We denote by $D(A)$ the derived category of A. The thick closure of A is denoted by $\mathbf{per}(A)$, and called *perfect derived category* of A. We denote by $\mathbf{pvd}(A)$ the subcategory of $D(A)$ consisting of A-module whose underlying complex is perfect over k. Note that we have $\mathbf{per}(\Lambda) = K^b(\mathbf{proj}\Lambda)$ and $\mathbf{pvd}(\Lambda) = D^b(\mathbf{mod}\Lambda)$ for every finite-dimensional algebra Λ. It is one of the main subject of representation theory that classify t-structures of $\mathbf{pvd}(A)$. The following theorem due to Koenig and Yang is a very important.

Theorem 1. [KoY] *Let Λ be a finite-dimensional algebra. Then the following sets correspond bijectively:*

1. *{iso classes of basic silting objects of $K^b(\mathbf{proj}\Lambda)$},*
2. *{bounded co-t-structures on $K^b(\mathbf{proj}\Lambda)$},*
3. *{iso classes of simple-minded collections of $D^b(\mathbf{mod}\Lambda)$},*
4. *{algebraic t-structures on $D^b(\mathbf{mod}\Lambda)$}.*

The above theorem was generalized to the case of proper non-positve dg algebras by Su and Yang [SY], and proved for the case of locally finite non-positive homologically smooth dg algebras by Keller and Nicolás [KeN2]. In this article, we prove this result to every locally finite non-positive dg algebras by applying the theory of Koszul dualities.

2. Koszul dualities

In this section, we establish the Koszul duality between non-positive dg algebras and positive dg algebras.

Definition 2. Let A be a dg algebra.

1. A is called *locally finite* if $H^i(A)$ is finite-dimensional for every $i \in \mathbb{Z}$.

The detailed version of this paper will be submitted for publication elsewhere.

(2) A is called *non-positive* if $H^{>0}(A) = 0$.
(3) A is called *positive* if $H^{<0}(A) = 0$ and $H^0(A)$ is a semi-simple algebra.

Definition 3 (Koszul dual)**.**

(1) Let A be a locally finite non-positive dg algebra. Then $A^! := REnd_A(topH^0A)$ is called the *Koszul dual* of A.
(2) Let A be a locally finite positive dg algebra. Then $A^! := REnd_A(H^0A)$ is called the *Koszul dual* of A.

The Koszul dual of locally finite non-positive dg algebra is locally finite positive dg algebra, and the Koszul dual of positive dg algebra is non-positive. However, the Koszul dual of locally finite positive dg algebra is not necessary locally finite. Therefore, we first determine the subclass of locally finite positive dg algebras whose Koszul dual is locally finite.

Proposition 4. [F2, Theorem 3.23] *Let A be a locally finite positive dg algebra. Then the following conditions are equivalent:*

(1) *The minimal subcategory of $D(A)$ containing A and closed under taking extensions, finite coproducts and direct summands has a projective generator,*
(2) $\mathbf{pvd}(A)$ *is Hom-finite,*
(3) $A^!$ *is locally finite.*

A locally finite positive dg algebra A is called *pvd-finite* if A satisfies the equivalent conditions in the above proposition. The following theorem is the main theorem of this section.

Theorem 5. [F2, Theorem 4.4]

(1) *Let A be a locally finite non-positive dg algebra. Then $A^!$ is a pvd-finite locally finite positive dg algebra, and $RHom_A(-, topH^0A)$ induces*

$$\begin{array}{ccc}
\mathbf{per}(A) & \xrightarrow{\simeq} & \mathbf{pvd}((A^!)^{\mathbf{op}})^{\mathbf{op}} \\
\cap & & \cap \\
D^-_{fd}(A) & \xrightarrow{\simeq} & D^+_{fd}((A^!)^{\mathbf{op}})^{\mathbf{op}} \\
\cup & & \cup \\
\mathbf{pvd}(A) & \xrightarrow{\simeq} & \mathbf{per}((A^!)^{\mathbf{op}})^{\mathbf{op}}.
\end{array}$$

(2) *Let A be a pvd-finite locally finite positive dg algebra. Then $A^!$ is a locally finite non-positive dg algebra, and $RHom_A(-, H^0A)$ induces*

$$\begin{array}{ccc}
\mathbf{per}(A) & \xrightarrow{\simeq} & \mathbf{pvd}((A^!)^{\mathbf{op}})^{\mathbf{op}} \\
\cap & & \cap \\
D^+_{fd}(A) & \xrightarrow{\simeq} & D^-_{fd}((A^!)^{\mathbf{op}})^{\mathbf{op}} \\
\cup & & \cup \\
\mathbf{pvd}(A) & \xrightarrow{\simeq} & \mathbf{per}((A^!)^{\mathbf{op}})^{\mathbf{op}}.
\end{array}$$

The above theorem refines results in [LPWZ].

3. Applications

By Theorem 5 and [KeN1], we can establish the following ST-correspondence.

Theorem 6. [F1, Theorem 3.5] *Let A be a locally finite non-positive dg algebra. Then the following sets correspond bijectively:*

(1) $\{$*iso classes of basic silting objects of* $\mathbf{per}(A)\}$,
(2) $\{$*bounded co-t-structures on* $\mathbf{per}(A)\}$,
(3) $\{$*iso classes of simple-minded collections of* $\mathbf{pvd}(A)\}$,
(4) $\{$*algebraic t-structures on* $\mathbf{pvd}(A)\}$.

The above theorem simultaneously generalize the results of [KeV, KoY, KeN2, SY]. Next, we state an application of Koszul dualities and the above theorem to the functorially finiteness of hearts.

Theorem 7. [F2, Theorem 5.12] *Let $\mathcal{T}$ be a Hom-finite algebraic triangulated category. Let $\mathcal{L}$ be a simple-minded collection of $\mathcal{T}$ such that $Filt\mathcal{L}$ has a projective generator.*

(1) $\mathcal{T} \simeq \mathbf{pvd}(A)$ *for some locally finite non-positive dg algebra A.*
(2) *For a heart $\mathcal{H}$ of a bounded t-structure on $\mathcal{T}$, the following conditions are equivalent:*
  (i) $\mathcal{H}$ *is contravariantly finite in* $\mathcal{T}$,
  (i)' $\mathcal{H}$ *is covariantly finite in* $\mathcal{T}$,
  (ii) $\mathcal{H}$ *is functorially finite in* $\mathcal{T}$,
  (iii) $\mathcal{H}$ *is a length category,*
  (iv) $\mathcal{H}$ *has a projective generator,*
  (v) $\mathcal{H} \simeq \mathbf{mod}\Lambda$ *for some finite-dimensional algebra* Λ.

If A is isomorphic to a finite-dimensional algebra Λ whose global dimension is finite, the above theorem has been shown in [SPP]. Note that the first part of the above theorem gives a characterization of perfect valued derived category of locally finite non-positive dg algebras among triangulated categories.

References

[AI] T. Aihara, O. Iyama, *Silting mutation in triangulated categories*, J. London. Math. Soc. (2) **85** (2012), no. 3, 633–668.

[BBD] P. Deligne, A. Beilinson, J. Bernstein, *Faisceaux pervers*, Astérisque, Vol.100, (1983), 1-172.

[F1] R. Fushimi, *The correspondence between silting objects and t-structures for non-positive dg algebras*, arXiv:2312.17597.

[F2] R. Fushimi, *Contravariant Koszul duality between non-positive and positive dg algebras*, arXiv:2409.08842.

[Ke] B. Keller, *Deriving DG categories*, Ann. Sci. Ecole Norm. Sup. (4) **27** (1994), no. 1, 63–102.

[KeN1] B. Keller, P. Nicolás, *Weight structures and simple dg modules for positive dg algebras*, Int. Math. Res. Not. IMRN 2013, no. 5, 1028–1078.

[KeN2] B. Keller, P. Nicolás, *Cluster hearts and cluster tilting objects*, work in preparation.

[KeV] B. Keller, D. Vossieck, *Aisles in derived categories*, Bull. Soc. Math. Belg. Sér. A **40** (1988), no. 2, 239–253.

[KoY] S. Koenig, D. Yang, *Silting objects, simple-minded collections, t-structures and co-t-structures for finite-length algebras*, Doc. Math. **19** (2014), 403–438.

[LPWZ] D. M. Lu, J. H. Palmieri, Q. S. Wu, and J. J. Zhang.*Koszul equivalences in A_∞-algebras*, New York J. Math, 14:325–378, 2008.

[MSSS] O. Mendoza, E. C. S´aenz, V. Santiago, M. J. Souto Salonia. *Auslander–Buchweitz Context and Co-tstructures*, Applied Categorical Structurs, vol.21, 417–440 (2013).

[R] J. Rickard, *Equivalences of derived categories for symmetric algebras*, Journal of pure and applied Algebra 61.3 (1989): 303-317.

[SPP] R. Coelho Simões, D. Pauksztello, and D. Ploog, *Functorially finite hearts, simple-minded systems in negative cluster categories, and noncrossing partitions*, Compositio Mathematica 158.1 (2022): 211-243.

[SY] H. Su, D. Yang, *From simple-minded collections to silting objects via Koszul duality.* Algebras and Representation Theory, vol.22, 219–238 (2019).

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