THE STABLE CATEGORY OF GORENSTEIN-PROJECTIVE MODULES OVER A MONOMIAL ALGEBRA

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Abstract. Let Λ be an arbitrary monomial algebra. We investigate the stable category $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda$ of graded Gorenstein-projective Λ-modules and the orbit category $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda/(1)$ induced by $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda$ and the degree shift functor (1). We prove that $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda$ is triangle equivalent to the bounded derived category of a path algebra of Dynkin type $\mathbb{A}$ and that $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda/(1)$ is triangle equivalent to the stable module category of a self-injective Nakayama algebra. The latter result provides an explicit description of the stable category of (ungraded) Gorenstein-projective Λ-modules.

1. Introduction

Throughout this paper, let K be a field. By an algebra, we mean a finite dimensional associative K-algebra with a unit (except when considering the path algebra KQ of a finite quiver Q that contains a cycle). Further, a module means a finitely generated right module. For a Krull-Schmidt category $\mathcal{C}$, we denote by $\operatorname{ind}\mathcal{C}$ the set of indecomposable objects of $\mathcal{C}$ up to isomorphism. We always assume that any full subcategory of an additive category is closed under isomorphisms.

The notion of Gorenstein-projective Λ-modules was originally introduced by Auslander and Bridger [2], and it is well known that the stable category $\underline{\mathrm{Gproj}}\Lambda$ of Gorenstein-projective Λ-modules carries a structure of a triangulated category. Many authors work with Iwanaga-Gorenstein algebras because of a theorem of Buchweitz [4, Theorem 4.4.1] and Happel [10, 4.6], which says that if Λ is an Iwanaga-Gorenstein algebra, then $\underline{\mathrm{Gproj}}\Lambda$ is triangle equivalent to the singularity category $\mathcal{D}_{\mathrm{sg}}(\operatorname{mod}\Lambda)$ of Λ.

On the other hand, for more general algebras, Gorenstein-projective modules over monomial algebras Λ have been intensively studied. Indeed, Ringel [14] showed that when Λ is a connected Nakayama algebra without simple projective modules, $\underline{\mathrm{Gproj}}\Lambda$ is triangle equivalent to the stable module category of some connected self-injective Nakayama algebra. Further, Chen, Shen and Zhou [6] proved that $\underline{\mathrm{Gproj}}\Lambda$ is triangle equivalent to the stable module category of some radical square zero self-injective Nakayama algebra when Λ is a monomial algebra with a certain condition. Moreover, Kalck [11] and Lu and Zhu [13] obtained a similar result for gentle algebras and 1-Iwanaga-Gorenstein monomial algebras, respectively.

In this paper, we study $\underline{\mathrm{Gproj}}\Lambda$ for arbitrary monomial algebras $\Lambda = KQ/I$ and, by applying the covering theory developed in [1], we show that $\underline{\mathrm{Gproj}}\Lambda$ is triangle equivalent to the stable module category of some self-injective Nakayama algebra.

The detailed version of this paper will be submitted for publication elsewhere.

2. Preliminaries

In this section, we recall the definition of Gorenstien-projective modules and related notions and facts used in this paper.

2.1. **Gorenstein-projective modules.** Throughout this subsection, we let Λ be an algebra. A cochain complex $P^\bullet : \cdots \to P^{i-1} \xrightarrow{d^{i-1}} P^i \xrightarrow{d^i} P^{i+1} \to \cdots$ of projective Λ-modules is called *totally acyclic* [3] if both $P^\bullet$ and the Hom complex $\mathrm{Hom}_\Lambda(P^\bullet, \Lambda)$ are acyclic. We say that a Λ-module M is *Gorenstein-projective* [7] if there exists a totally acyclic complex $P^\bullet$ such that $\mathrm{Ker}\, d^0$ is isomorphic to M in $\mathrm{mod}\,\Lambda$. We refer to [5] for their basic properties. Let $\mathrm{Gproj}\,\Lambda$ be the full subcategory of $\mathrm{mod}\,\Lambda$ consisting of Gorenstien-projective Λ-modules. Since projective modules are Gorenstein-projective, we have that $\mathrm{proj}\,\Lambda \subseteq \mathrm{Gproj}\,\Lambda \subseteq \mathrm{mod}\,\Lambda$. We say that Λ is *CM-free* if $\mathrm{proj}\,\Lambda = \mathrm{Gproj}\,\Lambda$. For example, algebras of finite global dimension are CM-free. We say that Λ is *CM-finite* if there are only finitely many pairwise non-isomorphic indecomposable Gorenstein-projective Λ-modules. CM-free algebras and representation-finite algebras are both examples of CM-finite algebras. Also, it is easily seen that $\mathrm{Gproj}\,\Lambda = \mathrm{mod}\,\Lambda$ if and only if Λ is self-injective.

Recall that the *stable category* $\underline{\mathrm{mod}}\Lambda$ of $\mathrm{mod}\,\Lambda$ is defined as the category whose objects are the same as $\mathrm{mod}\,\Lambda$ and whose morphisms are given by

$$\underline{\mathrm{Hom}}_\Lambda(M,N) := \mathrm{Hom}_{\underline{\mathrm{mod}}\Lambda}(M,N) = \mathrm{Hom}_\Lambda(M,N)/\mathcal{P}(M,N) \tag{2.1}$$

for any M and $N \in \underline{\mathrm{mod}}\Lambda$, where $\mathcal{P}(M,N)$ denotes the space of morphisms from M to N that factor through a projective Λ-module. Let $\underline{\mathrm{Gproj}}\Lambda$ be the full subcategory of $\underline{\mathrm{mod}}\Lambda$ consisting of Gorenstein-projective Λ-modules. The category $\mathrm{Gproj}\,\Lambda$ is known to be a Frobenius exact category whose projective objects are precisely projective Λ-modules, so that the stable category $\underline{\mathrm{Gproj}}\Lambda$ carries a structure of a triangulated category; see [9].

Recall that Λ is called *d-Iwanaga-Gorenstein* (or simply *Iwanaga-Gorenstein*) if both $\mathrm{id}_\Lambda\,\Lambda$ and $\mathrm{id}\,\Lambda_\Lambda$ are finite and at most d, where $\mathrm{id}_\Lambda\,M$ denotes the injective dimension of M in $\mathrm{mod}\,\Lambda$. In case Λ is Iwanaga-Gorenstein, there exists a triangle equivalence from the stable category $\underline{\mathrm{Gproj}}\Lambda$ to the *singularity category* $\mathsf{D}_{\mathrm{sg}}(\mathrm{mod}\,\Lambda)$ of Λ, where $\mathsf{D}_{\mathrm{sg}}(\mathrm{mod}\,\Lambda)$ is defined to be the Verdier quotient of the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$ of Λ by the perfect derived category $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$; see [4].

2.2. **Monomial algebras and their Gorenstein-projective modules.** In this subsection, we review some notations and facts related to monomial algebras, including an explicit description of Gorenstein-projective modules by Chen, Shen and Zhou [6].

Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver. A *path of length* n in Q is a sequence $p = a_1 a_2 \cdots a_n$ of arrows $a_i \in Q_1$ with $t(a_i) = s(a_{i+1})$ for all $1 \le i \le n-1$. We define its source $s(p) := s(a_1)$ and its target $t(p) := t(a_n)$. The length of a path p is denoted by $l(p)$. For a vertex v in Q, we associate a *trivial path* e_v of length zero with $s(e_v) = v = t(e_v)$. We denote by $\mathcal{B}$ the set of paths, by $\mathcal{B}_i$ the set of paths of length i and by $\mathcal{B}_{\ge i}$ the set of paths of length $\ge i$. We put $\mathcal{B}_{>0} := \mathcal{B}_{\ge 1}$. The concatenation of two paths p and q with $t(p) = s(q)$ is denoted by pq. For any p and $q \in \mathcal{B}$, we say that q is a *subpath* of p if $p = p'qp''$ for some $p', p'' \in \mathcal{B}$. When p' is trivial, we refer to the subpath q as a *left divisor* of p. Dually, when p'' is trivial, we refer to the subpath q as a *right divisor* of p. A subpath q of p is called *proper* if $q \neq p$. A *cycle* in Q is a path c such that $s(c) = t(c)$. We say that two cycles c and c' are *equivalent* if c coincides with c' up to cyclic permutation. This is equivalent to saying that c is a subpath of $(c')^m$ and c' is a subpath of c^n for some positive integers m, n. Let S be a subset of $\mathcal{B}$. We say that $p \in S$ is *left minimal in S* if there exists no proper left divisor of p that belongs to S (i.e. there exists no path $s \in S$ such that $p = sp'$ for some $p' \in \mathcal{B}_{>0}$). Dually, one can define right minimal path in S. We say that $p \in S$ is *minimal in S* if there exists no proper subpath of p that belongs to S.

Recall that a bound quiver algebra KQ/I is called *monomial* if the admissible ideal I is generated by paths. In the rest of this section, let $\Lambda = KQ/I$ be a monomial algebra. We denote by $\mathbb{F}$ the set of minimal paths in the set $\{\text{paths belonging to } I\}$. Note that $\mathbb{F}$ generates I as an ideal of KQ. In particular, $\mathbb{F}$ is a finite set. We say that $p \in \mathcal{B}$ is *non-zero in* Λ if the canonical image $p + I$ in Λ is non-zero, or equivalently, p does not contain any path of $\mathbb{F}$ as a subpath. The non-zero paths form a K-basis of Λ. For each $p \in \mathcal{B}$, we denote the canonical image $p + I$ in Λ by p. We write $p = 0$ in Λ when p lies in I. For a non-zero non-trivial path p, we define $L(p)$ as the set of right minimal paths in the set $\{\text{non-zero paths } q \mid t(q) = s(p) \text{ and } qp = 0 \text{ in } \Lambda\}$. Dually, we define $R(p)$ as the set of left minimal paths in the set $\{\text{non-zero paths } q \mid t(p) = s(q) \text{ and } pq = 0 \text{ in } \Lambda\}$.

Definition 1 (cf. [6, Definitions 3.3 and 3.7]). (1) A pair (p, q) of non-zero paths in Λ is said to be *perfect* if the following conditions are satisfied:

(P1) p and q are both non-trivial and satisfy $t(p) = s(q)$ and $pq = 0$ in Λ;

(P2) If $pq' = 0$ for a non-zero path q' with $t(p) = s(q')$, then q is a left divisor of q' (in other words, $R(p) = \{q\}$);

(P3) If $p'q = 0$ for a non-zero path p' with $t(p') = s(q)$, then p is a right divisor of p' (in other words, $L(q) = \{p\}$).

(2) A sequence $(p_1, \ldots, p_n, p_{n+1} = p_1)$ of non-zero paths in Λ is called a *perfect path sequence* if (p_i, p_{i+1}) is a perfect pair for all $1 \leq i \leq n$. A path appearing in a perfect path sequence is called a *perfect path*.

(3) A perfect path sequence $(p_1, \ldots, p_n, p_{n+1} = p_1)$ is called *minimal* if $p_i \neq p_j$ for any $1 \leq i \neq j \leq n$.

Let $\mathbb{P}_\Lambda$ denote the set of perfect paths. The following result describes indecomposable non-projective Gorenstein-projective Λ-modules. Note that $\operatorname{ind} \underline{\operatorname{Gproj}} \Lambda$ can be identified with the set of the isomorphism classes of indecomposable non-projective Gorenstein-projective Λ-modules.

Theorem 2 ([6, Theorem 4.1]). *Let Λ be a monomial algebra. Then there is a bijection*

$$\mathbb{P}_\Lambda \xleftrightarrow{1:1} \operatorname{ind} \underline{\operatorname{Gproj}} \Lambda,$$

where a perfect path p is sent to the cyclic Λ-module $p\Lambda$.

Remark 3. It follows from the theorem that monomial algebras Λ are always CM-finite. It also follows that Λ is CM-free precisely when $\mathbb{P}_\Lambda$ is empty.

2.3. **Positively graded algebras.** This subsection is devoted to recalling some basic facts in the representation theory of finite dimensional positively graded algebras from [8, 13]. Throughout this subsection, let $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$ be a positively graded algebra (i.e. a $\mathbb{Z}$-graded algebra satisfying $\Lambda_i = 0$ for $i < 0$).

Let $\operatorname{mod}^{\mathbb{Z}} \Lambda$ be the category of graded Λ-modules and $\operatorname{proj}^{\mathbb{Z}} \Lambda$ its full subcategory consisting of graded projective Λ-modules. Recall that the space of morphisms from M to N in $\operatorname{mod}^{\mathbb{Z}} \Lambda$ is defined by

$$\operatorname{Hom}_{\Lambda}^{\mathbb{Z}}(M,N) := \operatorname{Hom}_{\operatorname{mod}^{\mathbb{Z}} \Lambda}(M,N) = \{ f \in \operatorname{Hom}_{\Lambda}(M,N) \mid f(M_i) \subseteq N_i \text{ for all } i \in \mathbb{Z} \}.$$

Let $\underline{\operatorname{mod}}^{\mathbb{Z}}\Lambda$ be the stable category of $\operatorname{mod}^{\mathbb{Z}} \Lambda$, defined in a way similar to the case of $\underline{\operatorname{mod}}\Lambda$. For any $M, N \in \underline{\operatorname{mod}}^{\mathbb{Z}}\Lambda$, we denote $\underline{\operatorname{Hom}}_{\Lambda}^{\mathbb{Z}}(M,N) := \operatorname{Hom}_{\underline{\operatorname{mod}}^{\mathbb{Z}}\Lambda}(M,N)$. For a graded Λ-module $M = \bigoplus_{i\in\mathbb{Z}} M_i$ and an integer j, we define the degree shift $M(j) \in \operatorname{mod}^{\mathbb{Z}} \Lambda$ by $M(j)_i = M_{i+j}$ for $i \in \mathbb{Z}$. This operation induces an automorphism $(j) : \operatorname{mod}^{\mathbb{Z}} \Lambda \to \operatorname{mod}^{\mathbb{Z}} \Lambda$.

Replacing $\operatorname{mod}\Lambda$ by $\operatorname{mod}^{\mathbb{Z}} \Lambda$, one can define the notion of *graded Gorenstein-projective Λ-modules*. We denote by $\operatorname{Gproj}^{\mathbb{Z}} \Lambda$ (resp. $\underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda$) the full subcategory of $\operatorname{mod}^{\mathbb{Z}} \Lambda$ (resp. $\underline{\operatorname{mod}}^{\mathbb{Z}}\Lambda$) consisting of graded Gorenstein-projective Λ-modules. Then we have $\operatorname{proj}^{\mathbb{Z}} \Lambda \subseteq \operatorname{Gproj}^{\mathbb{Z}} \Lambda \subseteq \operatorname{mod}^{\mathbb{Z}} \Lambda$. We say that Λ is *graded CM-free* if $\operatorname{proj}^{\mathbb{Z}} \Lambda = \operatorname{Gproj}^{\mathbb{Z}} \Lambda$. The algebra Λ is said to be *graded CM-finite* if the number of the isomorphism classes of indecomposable graded Gorenstein-projective Λ-modules is finite up to degree shift. By definition, graded CM-freeness implies graded CM-finiteness. Let $F : \operatorname{mod}^{\mathbb{Z}} \Lambda \to \operatorname{mod}\Lambda$ be the forgetful functor. It follows from [8, Theorem 3.2] (resp. [8, Theorem 3.3]) that a graded Λ-module M is indecomposable (resp. projective) in $\operatorname{mod}^{\mathbb{Z}} \Lambda$ if and only if FM is indecomposable (resp. projective) in $\operatorname{mod}\Lambda$. Therefore, if Λ is CM-free (resp. CM-finite) as an ungraded algebra, then Λ is graded CM-free (resp. graded CM-finite). On the other hand, as in the ungraded case, $\operatorname{Gproj}^{\mathbb{Z}} \Lambda$ is a Frobenius category whose projective objects are precisely graded projective Λ-modules. Hence, the stable category $\underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda$ carries a structure of a triangulated category.

We say that Λ is *graded Iwanaga-Gorenstein* if $\operatorname{gr.id}_{\Lambda}\Lambda < \infty$ and $\operatorname{gr.id}\Lambda_{\Lambda} < \infty$, where $\operatorname{gr.id}_{\Lambda}M$ denotes the injective dimension of M in $\operatorname{mod}^{\mathbb{Z}} \Lambda$. As mentioned in [13, Section 2.1], the positively graded algebra Λ is graded Iwanaga-Gorenstein if and only if Λ is Iwanaga-Gorenstein as an ungraded algebra. Thus we do not distinguish between being graded Iwanaga-Gorenstein and being Iwanaga-Gorenstein. As in the ungraded case, if Λ is Iwanaga-Gorenstein, then $\underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda$ is triangle equivalent to the *graded singularity category* $\mathsf{D}_{\mathrm{sg}}(\operatorname{mod}^{\mathbb{Z}} \Lambda)$ of Λ, where $\mathsf{D}_{\mathrm{sg}}(\operatorname{mod}^{\mathbb{Z}} \Lambda)$ is defined by the Verdier quotient of $\mathsf{D}^{\mathrm{b}}(\operatorname{mod}^{\mathbb{Z}} \Lambda)$ by $\mathsf{K}^{\mathrm{b}}(\operatorname{proj}^{\mathbb{Z}} \Lambda)$.

3. Stable categories of graded Gorenstien-projective modules

In the rest of this paper, let $\Lambda = KQ/I$ be a monomial algebra and $\mathbb{F}$ the set of minimal paths in I. Further, we always think of Λ as a positively graded algebra by setting the degree of each arrow to one. We know from [13, Section 4.1] that $\operatorname{ind}\underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda = \{ p\Lambda(i) \mid p \in \mathbb{P}_{\Lambda}, i \in \mathbb{Z}\}$, where we regard $p\Lambda$ as a graded Λ-module whose top is concentrated in degree $l(p)$. In particular, $p\Lambda = \bigoplus_{i\in\mathbb{Z}} p\Lambda_i$ satisfies that $p\Lambda_i$ is spanned by the non-zero paths of the form px with $x \in \mathcal{B}_{i-l(p)}$ if such non-zero paths exist and otherwise $p\Lambda_i = 0$.

In this section, we apply tilting theory to obtain a triangle equivalence between the stable category $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda$ and the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, KQ)$ of a path algebra KQ, where Q is a disjoint union of Dynkin quivers of type $\mathbb{A}$.

We need some preparation to proceed. A perfect path sequence $(p_1, \ldots, p_n, p_{n+1} = p_1)$ gives rise to a cycle $p_1 \cdots p_n$. We refer to the shortest cycle c such that $p_1 \cdots p_n = c^l$ for some $l > 0$ as the *underlying cycle associated with the perfect path* p_1 and denote it by c_{p_1}. Cyclic permutations define an equivalence relation on the set of underlying cycles. We denote by $\mathcal{C}(\Lambda)$ the set of the equivalence classes of underlying cycles modulo this equivalence relation. On the other hand, for two perfect paths p and q, we write $p \preceq q$ if p is a left divisor of q. It is easy to check that the pair $(\mathbb{P}_\Lambda, \preceq)$ is a partially ordered set, and that the Hasse quiver $H(\mathbb{P}_\Lambda, \preceq)$ is a disjoint union of linear quivers. We say that a perfect path p is *co-elementary* if p is a sink in $H(\mathbb{P}_\Lambda, \preceq)$. The following observation asserts that any underlying cycle is the concatenation of co-elementary paths.

Proposition-Definition 4. *For $c \in \mathcal{C}(\Lambda)$, there uniquely exist finitely many co-elementary paths $r_1, \ldots, r_n$ such that $c = r_1 \cdots r_n$. We denote $|c| := n$.*

For each underlying cycle $c = r_1 \cdots r_n$ in Λ with each r_i co-elementary, we set

$$\mathbb{P}_\Lambda(c) := \{p \in \mathbb{P}_\Lambda \mid r_1 \preceq p\} \quad \text{and} \quad T_c := \bigoplus_{p \in \mathbb{P}_\Lambda(c)} p\Lambda.$$

Then we define an object T of $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda$ as

$$T := \bigoplus_{c \in \mathcal{C}(\Lambda)} \bigoplus_{0 \le i < l(c)} T_c(i).$$

From the definition, T depends on the choice of underlying cycles and is basic in the sense that any two distinct indecomposable summands of T are non-isomorphic.

Let $\mathcal{T}$ be a triangulated category with suspension functor Σ. We denote by $\mathcal{T}^{(k)}$ a direct product of k copies of $\mathcal{T}$. For a class $\mathcal{X}$ of objects of $\mathcal{T}$, we denote by $\mathrm{thick}_{\mathcal{T}}\, \mathcal{X}$ the smallest thick subcategory of $\mathcal{T}$ that contains $\mathcal{X}$. When $\mathcal{X}$ consists of a single object X, we write $\mathrm{thick}_{\mathcal{T}}\, X$ instead of $\mathrm{thick}_{\mathcal{T}}\{X\}$. In what follows, we drop the index $\mathcal{T}$ in $\mathrm{thick}_{\mathcal{T}}\, \mathcal{X}$ when $\mathcal{T}$ is clear from the context. Recall that a object T of $\mathcal{T}$ is called a *tilting object* if the following two conditions are satisfied:

(i) $\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i T) = 0$ for all $i \neq 0$.
(ii) $\mathrm{thick}\, T = \mathcal{T}$.

We are now ready to state the main result of this section is the following.

Theorem 5. *The following statements hold.*

(1) *The object T is a tilting object of $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda$.*
(2) *There exists an algebra isomorphism*

$$\underline{\mathrm{End}}^{\mathbb{Z}}_{\Lambda} T \;\cong\; \prod_{c \in \mathcal{C}(\Lambda)} (K\mathbb{A}_c)^{(l(c))},$$

where $\mathbb{A}_c$ is the following linear quiver

$$\mathbb{A}_c : 1 \to 2 \to \cdots \to |\mathbb{P}_\Lambda(c)|,$$

and $(K\mathbb{A}_c)^{(l(c))}$ *is the direct product of* $l(c)$ *copies of the path algebra* $K\mathbb{A}_c$.

(3) *There exists a triangle equivalence*

$$\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda \cong \prod_{c\in\mathcal{C}(\Lambda)} \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_c)^{(l(c))},$$

where $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_c)^{(l(c))}$ *is the direct product of* $l(c)$ *copies of the bounded derived category* $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_c)$ *of the path algebra* $K\mathbb{A}_c$.

Remark 6. The theorem explicitly describes the graded singularity category $\mathsf{D}_{\mathrm{sg}}(\mathrm{mod}^{\mathbb{Z}}\,\Lambda)$ of an Iwanaga-Gorenstein monomial algebra Λ and in particular improves a result of Lu-Zhu [13, Theorem 5.2.2] for Iwanaga-Gorenstein monomial algebras.

4. Stable categories of Gorenstien-projective modules

The aim of this section is to realize $\underline{\mathrm{Gproj}}\Lambda$ as the stable module category of a self-injective Nakayama algebra. For this, we apply the covering theory developed in [1]. We follow the terminologies in [1]. We rely on the following proposition, essentially proved by Lu and Zhu [13, Lemma 4.2.1]. Recall that for any $i \in \mathbb{Z}$, the automorphism $(i) : \mathrm{mod}^{\mathbb{Z}}\,\Lambda \to \mathrm{mod}^{\mathbb{Z}}\,\Lambda$ and the forgetful functor $F : \mathrm{mod}^{\mathbb{Z}}\,\Lambda \to \mathrm{mod}\,\Lambda$ satisfy that $F = F \circ (i)$.

Proposition 7. *The forgetful functor* $F : \mathrm{mod}^{\mathbb{Z}}\,\Lambda \to \mathrm{mod}\,\Lambda$ *induces a* G*-covering*

$$\tilde{F}_G : \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda \to \underline{\mathrm{Gproj}}\Lambda$$

in the sense of [1], *where* G *is the cyclic group generated by the automorphism* $(1) : \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda \to \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda$, *and the invariance adjuster of* $\tilde{F}_G$ *is given by the induced formula* $\tilde{F}_G = \tilde{F}_G \circ (i)$ *with* $i \in \mathbb{Z}$.

By [1, Theorem 2.9], there exists an equivalence $H : \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda/(1) \xrightarrow{\sim} \underline{\mathrm{Gproj}}\Lambda$ of additive categories such that $\tilde{F}_G = HP$ (as G-invariant functors), where $P : \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda \to \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda/(1)$ is the canonical functor:

$$\begin{array}{ccc} \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda & \xrightarrow{\tilde{F}_G} & \underline{\mathrm{Gproj}}\Lambda \\ & {}_{P}\searrow \quad \nearrow^{\sim}_{H} & \\ & \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda/(1) & \end{array}$$

Thus the orbit category $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda/(1)$ becomes a triangulated category whose triangulated structure is derived from that of the triangulated category $\underline{\mathrm{Gproj}}\Lambda$ via the equivalence H. Finally, since $\tilde{F}_G$ and H are both triangulated functors, the canonical functor P is also triangulated. On the other hand, we know from Theorem 5 that

$$\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda = \prod_{c\in\mathcal{C}(\Lambda)} \prod_{0\leq i<l(c)} \mathrm{thick}\, T_c(i) \quad \text{with} \quad \mathrm{thick}\, T_c(i) \cong \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_c).$$

For a class $\mathcal{X}$ of objects of $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda$, we denote by $P(\mathcal{X})$ the full subcategory of $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda/(1)$ given by $P(\mathcal{X}) = \{P(X) \mid X \in \mathcal{X}\}$. Then for $c \in \mathcal{C}(\Lambda)$ and $i \in \mathbb{Z}$, we have

$$P(\mathrm{thick}\, T_c(i)) = P((\mathrm{thick}\, T_c)(i)) = P(\mathrm{thick}\, T_c),$$

so that we obtain a decomposition into additive categories

$$\underline{\mathrm{Gproj}}\Lambda \cong \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda/(1) = \prod_{c\in\mathcal{C}(\Lambda)} \prod_{0\le i<l(c)} P(\mathrm{thick}\, T_c(i)) = \prod_{c\in\mathcal{C}(\Lambda)} P(\mathrm{thick}\, T_c).$$

This is a decomposition into triangulated categories, since $P(\mathrm{thick}\, T_c) = \mathrm{thick}\, P(T_c)$ for all $c \in \mathcal{C}(\Lambda)$ and $i \in \mathbb{Z}$. The following lemma investigates each triangulated subcategory $P(\mathrm{thick}\, T_c)$.

Proposition 8. *We have the following statements.*

(1) *For $c \in \mathcal{C}(\Lambda)$ and $i, j \in \mathbb{Z}$, we have that* $\mathrm{thick}\, T_c(i) = \mathrm{thick}\, T_c(j)$ *in* $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda$ *if and only if* $i \equiv j \mod l(c)$.

(2) *For $c \in \mathcal{C}(\Lambda)$, the restriction of the canonical functor* $P : \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda \to \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda/(1)$ *to* $\mathrm{thick}\, T_c$ *induces a G_c-covering*

$$P_c : \mathrm{thick}\, T_c \to P(\mathrm{thick}\, T_c)$$

where G_c is the cyclic group generated by the induced automorphism $(l(c)) : \mathrm{thick}\, T_c \to \mathrm{thick}\, T_c$.

(3) *For $c \in \mathcal{C}(\Lambda)$, we have a triangle equivalence*

$$P(\mathrm{thick}\, T_c) \cong \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_c)/\tau^{|c|},$$

where $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_c)/\tau^{|c|}$ *denotes the triangulated orbit category induced by* $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_c)$ *and its Auslander-Reiten translation τ in the sense of* [12].

The following consequence of the proposition is the main result of this section.

Theorem 9. *Let Λ be a monomial algebra over a field K. Then we have the following triangle equivalences*

$$\begin{aligned}\underline{\mathrm{Gproj}}\Lambda &\cong \prod_{c\in\mathcal{C}(\Lambda)} \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_c)/\tau^{|c|} \\ &\cong \prod_{c\in\mathcal{C}(\Lambda)} \underline{\mathrm{mod}}\, K\Big(1 \rightleftarrows 2 \to \cdots \to |c|\Big) \Big/ R^{|\mathbb{P}_\Lambda(c)|+1},\end{aligned}$$

where $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_c)/\tau^{|c|}$ *is the triangulated orbit category induced by* $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_c)$ *and its Auslander-Reiten translation τ in the sense of* [12]*, and R is the arrow ideal of the path algebra* $K\Big(1 \rightleftarrows 2 \to \cdots \to |c|\Big)$.

Remark 10. The theorem explicitly describes the singularity categories $\mathsf{D}_{\mathrm{sg}}(\mathrm{mod}\,\Lambda)$ of Iwanaga-Gorenstein monomial algebras Λ. Moreover, it recovers the results of Chen, Shen and Zhou [6], Kalck [11], Lu and Zhu [13] and Ringel [14].

References

[1] H. Asashiba, *A generalization of Gabriel's Galois covering functors and derived equivalences*, J. Algebra, **334(1)** (2011), 109–149.

[2] M. Auslander and M. Bridger, *Stable module theory*, Memoirs of the American Mathematical Society, No. 94. American Mathematical Society, Providence, R.I., 1969.

[3] L. L. Avramov and A. Martsinkovsky, *Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension*, Proc. London Math. Soc. (3) **85(2)** (2002), 393–440.
[4] R.-O. Buchweitz, *Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings*, unpublished manuscript, (1986). available at http://hdl.handle.net/1807/16682.
[5] X.-W. Chen, *Gorenstein Homological Algebra of Artin Algebras*, (2017). arXiv:1712.04587.
[6] X.-W. Chen, D. Shen, and G. Zhou, *The Gorenstein-projective modules over a monomial algebra*, Proc. Roy. Soc. Edinburgh Sect. A **148(6)** (2018), 1115–1134.
[7] E. E. Enochs and O. M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220(4)** (1995), 611–633.
[8] R. Gordon and E. L. Green, *Graded Artin algebras*, J. Algebra **76(1)** (1982), 111–137.
[9] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, **119** London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, (1988).
[10] ———, *On Gorenstein algebras*, In Representation Theory of Finite Groups and Finite-Dimensional Algebras, **95** Progress in Math. Basel: Birkhäuser Verlag (1991), 389–404.
[11] M. Kalck, *Singularity categories of gentle algebras*, Bull. London Math. Soc. **47** (2015), 65–74.
[12] B. Keller, *On triangulated orbit categories*, Doc. Math. **10** (2005), 551–581.
[13] M. Lu and B. Zhu, *Singularity categories of Gorenstein monomial algebras*, J. Pure Appl. Algebra **225(8)** (2021), Paper No. 106651, 39.
[14] C. M. Ringel, *The Gorenstein projective modules for the Nakayama algebras. I*, J. Algebra **385** (2013), 241–261.

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