Auslander-Reiten's Cohen-Macaulay algebras and contracted preprojective algebras

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This is a joint work with Aaron Chan and Rene Marczinzik [CIM].

Auslander and Reiten [AR1] called a finite dimensional algebra A over a field k *Cohen-Macaulay* (*CM* for short) if there is an A-bimodule W which gives equivalences

$$- \otimes_A W : \mathcal{P}^{<\infty}(A) \simeq \mathcal{I}^{<\infty}(A) : \mathrm{Hom}_A(W, -),$$

where $\mathcal{P}^{<\infty}(A) := \{X \in \mathsf{mod} A \mid \mathrm{proj.dim}\, X < \infty\}$ and $\mathcal{I}^{<\infty}(A) := \{X \in \mathsf{mod} A \mid \mathrm{inj.dim}\, X < \infty\}$. Such W is called a *dualizing A-module*, and the following equalities are satisfied:

$$\mathrm{fin.dim}\, A = \mathrm{inj.dim}\, W_A = \mathrm{inj.dim}_A W = \mathrm{fin.dim}\, A^{\mathrm{op}}.$$

Dualizing modules are characterized in terms of tilting theory: Recall that an A-module T is called *cotilting* if the k-dual DT is a tilting A^{op}-module. The set $\mathrm{cotilt}\, A$ of additive equivalence classes of cotilting A-modules has a natural partial order given by $T \geq U \Leftrightarrow \mathrm{Ext}^i_A(T, U) = 0$ for all $i \geq 1$.

Proposition 1. [AR1, 1.3] *An A-bimodule W is a dualizing A-module if and only if the following conditions are satisfied.*

- *The A-module W gives a maximal element in* $\mathrm{cotilt}\, A$.
- *The A^{op}-module W gives a maximal element in* $\mathrm{cotilt}\, A^{\mathrm{op}}$.
- *The natural map $A \to \mathrm{End}_A(W)$ is an isomorphism.*

For example, Iwanaga-Gorenstein algebras are precisely CM algebras with $W = A$, and algebras with finitistic dimension zero on both sides are precisely CM algebras with $W = DA$. Moreover, tensor products of CM algebras are again CM. They seem to be all of the known examples of CM algebras.

In this talk, we give the first non-trivial class of CM algebras. For a quiver Q, its double $\overline{Q}$ is defined by adding a new arrow $a^* : j \to i$ for each arrow $a : i \to j$ in Q. The *preprojective algebra* of Q is the factor algebra of the path algebra $k\overline{Q}$ given by

$$\Pi := k\overline{Q}/\langle \sum_{a \in Q_1} (aa^* - a^*a) \rangle.$$

A *contracted preprojective algebra* of Q is the subalgebra $e\Pi e$ of Π, where e is an arbitrary idempotent of Π [IW]. Our first main result is the following.

Theorem 2. *Each contracted preprojective algebra A of Dynkin type is a Cohen-Macaulay algebra. Moreover,* $\mathrm{fin.dim}\, A$ *is either 0 or 2.*

For a CM algebra A with dualizing module W, the category of *Cohen-Macaulay A-modules* is defined as $\mathsf{CM}\, A := {}^{\perp_{>0}}W$. Clearly $\mathsf{CM}\, A \supset \Omega^d(\mathsf{mod}\, A)$ holds for $d := \mathrm{fin.dim}\, A$. Moreover, the equality holds if A is Iwanaga-Gorenstein. Auslander and Reiten posed a question if the converse holds for $d \geq 1$. We show that a family of contracted preprojective algebras gives a negative answer to this question. In fact, if A is a CM algebra that is additionally d-Gorenstein for $d := \mathrm{fin.dim}\, A$ in the sense of [AR2], then $\mathsf{CM}\, A = \Omega^d(\mathsf{mod}\, A)$ always holds. This is an analogue of a well-known equality for isolated singularities.

References

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