TILTING FOR ARTIN-SCHELTER GORENSTEIN ALGEBRAS OF DIMENSION ONE

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Abstract. In this article, we study the stable categories of graded Cohen-Macaulay modules over Artin-Schelter Gorenstein algebras. We give a characterization of the existence of tilting objects in the stable categories. Gorenstein parameters of such algebras play an important role. Gorenstein tiled orders are typical examples of Artin-Schelter Gorenstein algebras of dimension one. We give an explicit description of the endomorphism algebra of a tilting object over a Gorenstein tiled order.

1. Introduction

One of the main objects of representation theory of a Cohen-Macaulay ring A is the category $\mathsf{CM}A$ of maximal Cohen-Macaulay modules (CM modules for short). By many results on this category, such as the study of Auslander-Reiten sequcence, the structure of the category is gradually becoming clearer. Moreover, if A is Gorenstein, then the situation is much nicer. In fact, the stable category $\underline{\mathsf{CM}}A$ is a triangulated category, and it is equivalent to the singularity category of A by the result of Buchweitz [2].

In this study, we focus on a $\mathbb{N}$-graded Artin-Schelter Gorenstein algebra A, which is one of the main objects in noncommutative algebraic geometry. We study tilting objects of the stable category $\underline{\mathsf{CM}}^{\mathbb{Z}}A$ of graded CM modules. We refer [3, 5, 6, 8] for recent related works.

Notation. Throughout this article k is a field. For an $\mathbb{N}$-graded ring A, we denote by $\mathsf{mod}^{\mathbb{Z}}A$ the category of finitely generated $\mathbb{Z}$-graded right A-modules. Let $D = \operatorname{Hom}_k(-, k)$ the k-dual of k-vector spaces.

2. Artin-Schelter Gorenstein algebras

In this section, we introduce Artin-Schelter Gorenstein algebras. Let $A = \bigoplus_{i\in\mathbb{N}} A_i$ be an $\mathbb{N}$-graded k-algebra. We say that A is *locally finite* if $\dim_k A_i$ is finite for each $i \geq 0$. It is easy to see that if A is locally finite, then the category $\mathsf{mod}^{\mathbb{Z}}A$ is Hom-finite and Krull-Schmidt.

Assume that A is locally finite. So A_0 is a finite dimensional k-algebra. Let $1 = e_1+e_2+\cdots+e_n$ be primitive orthogonal idempotents of A_0. Then e_iA is an indecomposable projective A-module for each i. Note that A_0 is basic if and only if A is basic, that is, $e_iA \simeq e_jA$ as (ungraded) A-modules implies $i = j$. By taking graded Morita equivalences, we may assume that A is basic. From now on to the end of this article, we assume that a locally finite algebra A is basic.

The detailed version of this paper will be submitted for publication elsewhere.

Let $\mathbb{I} := \{1, 2, \ldots, n\}$. For a locally finite algebra A, we denote by $\mathsf{sim}A := \{S_i := \mathrm{top}(e_i A_0) \mid i \in \mathbb{I}\}$ the set of all simple A_0-modules. This set $\mathsf{sim}A$ is the set of all $\mathbb{Z}$-graded simple A-modules concentrated in degree zero. Let $\mathsf{sim}A^{\mathrm{op}} := \{S'_i := D(S_i) \mid i \in \mathbb{I}\}$ be the set of all $\mathbb{Z}$-graded simple left A-modules concentrated in degree zero.

Let $d \geq 0$ be an integer. We say that A is *d-Iwanaga-Gorenstein* if A is Noetherian and $\mathrm{inj.dim}(A_A) = \mathrm{inj.dim}({}_A A) = d$ holds.

We give a definition of Artin-Schelter Gorenstein algebras.

Definition 1. Let A be a basic, Noetherian and locally finite $\mathbb{N}$-graded k-algebra. Let $d \geq 0$ be an integer. We say that A is an *Artin-Schelter Gorenstein algebra of dimension d* (AS-Gorenstein for short) if it satisfies the following properties.

(1) A is a d-Iwanaga-Gorenstein algebra.
(2) There exist a permutation $\nu : \mathbb{I} \to \mathbb{I}$ and integers p_i ($i \in \mathbb{I}$) such that the following isomorphism holds for each $i \in \mathbb{I}$;

$$\mathrm{Ext}^j_A(S_i, A) \simeq \begin{cases} S'_{\nu(i)}(p_i) & j = d, \\ 0 & \text{else.} \end{cases}$$

We call ν the *Nakayama permutation*, and call $(p_i)_{i \in \mathbb{I}}$ *Gorenstein parameters* of A.

Note that Definition 1(2) is a natural generalization of the Gorenstein parameter of a commutative Noetherian Gorenstein local ring.

Example 2. We give one typical example. See the next section to the definition of $\mathsf{CM}R$.

(1) Let R be an $\mathbb{N}$-graded commutative Noetherian Gorenstein k-algebra with Krull dimension $\dim R = d$. An $\mathbb{N}$-graded R-algebra A is called a *Gorenstein R-order* if $A_R \in \mathsf{CM}R$ and $\mathrm{Hom}_R(A_A, R) \in \mathsf{proj}({}_A A)$ hold. We can show that a Gorenstein R-order is an AS-Gorenstein algebra of dimension d.
(2) More concretely, let $R = k[x]$ with $\deg\ x = 1$ and $\mathfrak{m} = (x)$. Then for $a, b \in \mathbb{Z}_{\geq 0}$ with $a + b > 0$,

$$A = \begin{bmatrix} R & \mathfrak{m}^a \\ \mathfrak{m}^b & R \end{bmatrix}$$

is a Gorenstein R-order. So A is an AS-Gorenstein algebra of dimension 1 by (1). Let $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in A$. Then $\nu = (1\ 2)$ and $(p_1, p_2) = (1 - b,\ 1 - a)$ hold.

3. The category of Cohen-Macaulay modules and our results

Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be an Iwanaga-Gorenstein algebra, and let

$$\mathsf{CM}^{\mathbb{Z}} A := \{M \in \mathsf{mod}^{\mathbb{Z}} A \mid \mathrm{Ext}^i_A(M, A) = 0\ \forall i > 0\}$$

be the category of *Cohen-Macaulay A-modules* (CM modules for short). It is known that $\mathsf{CM}^{\mathbb{Z}} A$ is a Frobenius category [2]. Thus the projective stable category $\underline{\mathsf{CM}}^{\mathbb{Z}} A$ is a triangulated category.

From now on, we consider an AS-Gorenstein algebra A of dimension 1. Denote by $\mathsf{mod}^{\mathbb{Z}}_0 A$ the category of finite dimensional $\mathbb{Z}$-graded A-modules. We consider the Serre quotient category

$$\mathsf{qgr} A := \mathsf{mod}^{\mathbb{Z}} A / \mathsf{mod}^{\mathbb{Z}}_0 A$$

which is traditionally called the *noncommutative projective scheme* [1]. Define the *graded total quotient ring* $Q := \bigoplus_{i\in\mathbb{Z}} \mathrm{End}_{\mathsf{qgr}A}(A, A(i))$. Moreover, let

$$\mathsf{CM}_0^{\mathbb{Z}}A = \{M \in \mathsf{CM}^{\mathbb{Z}}A \mid M \otimes_A Q \text{ is a graded projective } Q\text{-module}\}.$$

This is a Frobenius subcategory of $\mathsf{CM}^{\mathbb{Z}}A$, and the natural inclusion induces a fully faithful triangle functor $\underline{\mathsf{CM}}_0^{\mathbb{Z}}A \to \underline{\mathsf{CM}}^{\mathbb{Z}}A$. As in classical Auslander-Reiten theory for orders, $\mathsf{CM}_0^{\mathbb{Z}}A$ behaves much nicer than $\mathsf{CM}^{\mathbb{Z}}A$. In fact, it enjoys Auslander-Reiten-Serre duality, and hence it has almost split sequences as follows.

Theorem 3. *Let A be an AS-Gorenstein algebra of dimension 1. Then there exists an invertible A-bimodule ω such that $(-)\otimes_A \omega$ induces a Serre functor*

$$(-)\otimes_A \omega : \underline{\mathsf{CM}}_0^{\mathbb{Z}}A \longrightarrow \underline{\mathsf{CM}}_0^{\mathbb{Z}}A.$$

We recall the definition of tilting objects. Let $\mathcal{T}$ be a Hom-finite and Krull-Schmidt triangulated k-category (e.g. $\underline{\mathsf{CM}}_0^{\mathbb{Z}}A$ for an AS-Gorenstein algebra A of dimension 1). An object $T \in \mathcal{T}$ is *tilting* if it satisfies (i) $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for all $i \neq 0$, and (ii) T generates $\mathcal{T}$, that is, the smallest triangulated subcategory of $\mathcal{T}$ containing T and closed under direct summands is $\mathcal{T}$. We call T *silting* if T satisfies (ii) and (i′) $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for all $i > 0$.

If $\mathcal{T}$ has a tilting object T, then we have a triangle equivalence $\mathcal{T} \simeq \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\mathrm{End}_{\mathcal{T}}(T))$. So the existence of tilting objets and calculating their endomorphism algebras are very important to study triangulated categories.

We state our main result.

Theorem 4. *Let A be an AS-Gorenstein algebra of dimension 1 with Gorenstein parameters $(p_i)_{i\in\mathbb{I}}$. Assume that A is ring-indecomposable and* $\mathrm{gldim}\, A_0 < \infty$*.*

- (a) *For sufficiently large $N > 0$, $V = \bigoplus_{i=1}^{N} A(i)_{\geq 0}$ is a silting object of $\underline{\mathsf{CM}}_0^{\mathbb{Z}}A$.*
- (b) *If $p_i \leq 0$ for any $i \in \mathbb{I}$, then V is tilting.*
- (c) *In the case (b), we have a description of* $\mathrm{End}(V)$ *by using A, Q and $(p_i)_{i\in\mathbb{I}}$.*
- (d) *The following statements are equivalent.*
  - (i) *$\underline{\mathsf{CM}}_0^{\mathbb{Z}}A$ has a tilting object.*
  - (ii) *$\sum_{i\in\mathbb{I}} p_i \leq 0$ or* $\mathrm{gldim}\, A < \infty$*.*

Note that this theorem is a natural generalization of the result of [3], which is one aspect of the motivation for our study.

4. Tiled orders and endomorphism algebras

In this section, we give a definition of tiled orders, and calculate endomorphism algebras of tilting objects of them. We refer [7] for basic properties of Gorenstein tiled orders.

Throughout this section, let $R = k[x]$ be the ring of polynomials in one variable with $\deg x = 1$. We denote by $\mathfrak{m} = (x) = Rx$ a maximal ideal of R. Let $\mathbb{I} := \{1, 2, \ldots, n\}$. A *Gorenstein tiled order* A is an R-subalgebra of $\mathrm{M}_n(R)$ of the form

$$A = \begin{bmatrix} R & \mathfrak{m}^{m(1,2)} & \cdots & \mathfrak{m}^{m(1,n)} \\ \mathfrak{m}^{m(2,1)} & R & \cdots & \mathfrak{m}^{m(2,n)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{m}^{m(n,1)} & \mathfrak{m}^{m(n,2)} & \cdots & R \end{bmatrix} \tag{4.1}$$

for some $m(i,j) \in \mathbb{Z}_{\geq 0}$ satisfying $\mathrm{Hom}_R(A_A, R) \in \mathsf{proj}({}_A A)$.

Gorenstein tiled orders over R are clearly Gorenstein R-orders. So by Example 2(1), Gorenstein tiled orders over R are AS-Gorenstein algebras of dimension 1. Let A be a Gorenstein tiled order of the form (4.1). We can see that the Nakayama permutation ν of A is determined by an equation

$$m(\nu(i), j) + m(j, i) = m(\nu(i), i) \quad \text{for each } i, j \in \mathbb{I}.$$

The Gorenstein parameters $(p_i)_{i \in \mathbb{I}}$ can be calculated as $p_i = 1 - m(\nu(i), i)$ for each i.

Example 5. Let $p, q, r \in \mathbb{Z}_{\geq 0}$ with $p + q + r > 0$. Then

$$\begin{bmatrix} R & \mathfrak{m}^p & \mathfrak{m}^{p+q} \\ \mathfrak{m}^{q+r} & R & \mathfrak{m}^q \\ \mathfrak{m}^r & \mathfrak{m}^{r+p} & R \end{bmatrix}$$

is a Gorenstein tiled order such that $\nu = (1\ 2\ 3)$ and $p_1 = -q - r,\ p_2 = 1 - r - p,\ p_3 = 1 - p - q$.

Let A be a Gorenstein tiled order of the form (4.1) with the Nakayama permutation ν and Gorenstein parameters $(p_i)_{i \in \mathbb{I}}$. To calculate the endomorphism algebras of tilting objects, we need the following notation.

For an A-module M of the form $M = [\mathfrak{m}^{\ell_1}\ \mathfrak{m}^{\ell_2}\ \cdots\ \mathfrak{m}^{\ell_n}]$, let

$$\mathrm{v}(M) := (\ell_1, \ell_2, \ldots, \ell_n) \in \mathbb{Z}^n.$$

For $v, w \in \mathbb{Z}^n$, we write $v \leq w$ if $v_i \leq w_i$ for each $i \in \mathbb{I}$. This defines a partial order on $\mathbb{Z}^n$. We have a finite poset $(\mathbb{V}_A, \leq)$, where

$$\mathbb{V}_A := \{\mathrm{v}(e_i A(j)_{\geq 0}) \mid i \in \mathbb{I},\ 1 \leq j \leq -p_{\nu^{-1}(i)}\} \cup \{0\} \subset \mathbb{Z}^n.$$

Then we sate our theorem.

Theorem 6. *Assume that $p_i \leq 0$ for each $i \in \mathbb{I}$. Let $V = \bigoplus_{i=1}^N A(i)_{\geq 0}$ be the tilting object of $\underline{\mathsf{CM}}_0^{\mathbb{Z}} A$ as in Theorem 4. Then* $\mathrm{End}(V)$ *is Morita equivalent to the incidence algebra $k(\mathbb{V}_A^{\mathrm{op}})$ of the opposite poset of $(\mathbb{V}_A, \leq)$.*

We end this article by giving two examples.

Example 7. For $a, b \in \mathbb{Z}$ with $a + b > 0$, let A be the Gorenstein tiled order in Example 2(2). We have that $p_1, p_2 \leq 0$ if and only if $a, b \geq 1$. Then

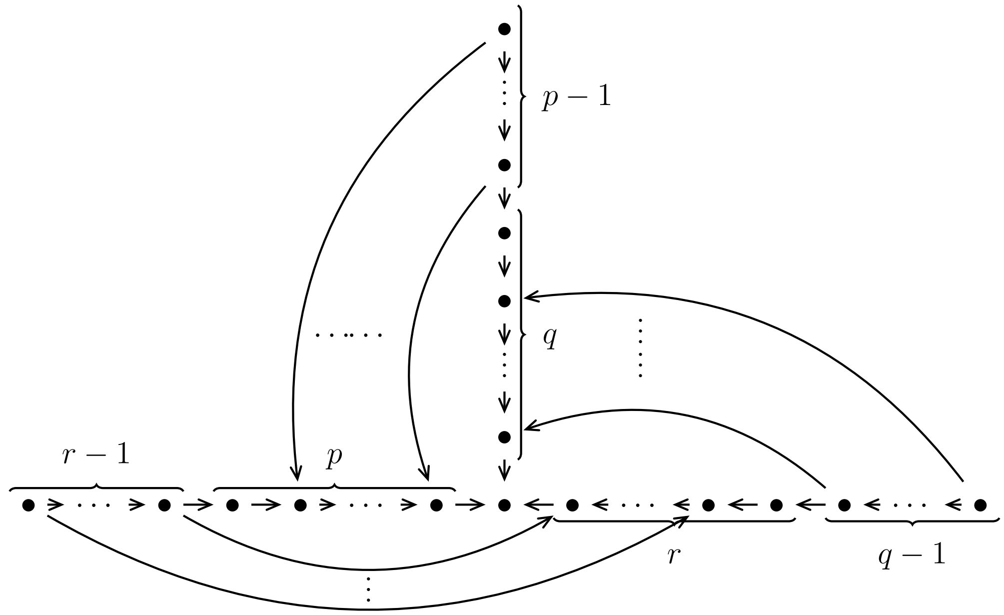
$$\mathbb{V}_A = \{(0\ i), (0\ 0), (j\ 0) \mid 1 \leq i \leq a - 1,\ 1 \leq j \leq b - 1\}$$

and $\mathrm{End}(V)$ is Morita equivalent to $k(\mathbb{V}_A^{\mathrm{op}}) \simeq kQ$ for a quiver Q as follows

$$\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow \bullet_0 \longleftarrow \bullet \longleftarrow \cdots \longleftarrow \bullet \longleftarrow \bullet$$

where $\bullet_0$ corresponds to $(0, 0) \in \mathbb{V}_A$. There are $a - 1$ vertices arranged to the left of $\bullet_0$, and there are $b - 1$ vertices arranged to the right of $\bullet_0$.

Example 8. For $p, q, r \in \mathbb{Z}_{\geq 0}$ with $p + q + r > 0$, let A be the Gorenstein tiled order in Example 5. Assume that $p_1, p_2, p_3 \leq 0$ ($\Leftrightarrow q+r,\ r+p,\ p+q \geq 1$). Then $\mathrm{End}(V)$ is Morita equivalent to $k(\mathbb{V}_A^{\mathrm{op}}) \simeq kQ/I$, where Q is as follows and I is generated by commutative relations:



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