ON THE EXISTENCE OF COUNTEREXAMPLES FOR VANISHING PROBLEMS OF EXT AND TOR

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Abstract. In this article, we investigate the relationships among various homological properties for commutative noetherian local rings. As an application, we prove that there exist a Gorenstein local UFD A having an isolated singularity such that $\operatorname{Tor}^A_{>0}(M,N)=0$ does not imply $\operatorname{depth}(M\otimes_A N)=\operatorname{depth}M+\operatorname{depth}N-\operatorname{depth}A$, and a Cohen–Macaulay local UFD B having an isolated singularity such that $\operatorname{Ext}^{>0}_B(M,B)=0$ does not imply that M is totally reflexive.

Key Words: vanishing of Ext/Tor, AB ring, totally reflexive module, depth formula.

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1. Main result

Throughout this article, let R be a commutative noetherian local ring and denote by $\mathfrak{m}$ the unique maximal ideal of R. Huneke and Jorgensen [6] introduce the notion of an *AB ring*: the local ring R is said to be an *AB ring* if R is Gorenstein and there exists an integer n such that whenever one has $\operatorname{Ext}^{\gg 0}_R(M,N)=0$ for finitely generated R-modules M,N it holds that $\operatorname{Ext}^{>n}_R(M,N)=0$. Auslander conjectured that any finite dimensional algebra over a field is AB. However, Jorgensen and Şega [9] constructed, for a field k that is not algebraic over a finite field, an artinian Gorenstein equicharacteristic local ring $(A,\mathfrak{m}_A,k)$ which is not AB. We say that R satisfies (**dep**) if all finitely generated R-modules M and N with $\operatorname{Tor}^R_{>0}(M,N)=0$ satisfy *Auslander's depth formula* [1], i.e.,

$$\operatorname{depth}(M\otimes_R N)=\operatorname{depth}M+\operatorname{depth}N-\operatorname{depth}R.$$

Huneke and Wiegand [7] proved that every local complete intersection satisfies (**dep**), and it was extended to AB rings by Christensen and Jorgensen [4]. On the other hand, it has been an open question for several decades now whether every local ring, or even every Gorenstein local ring, satisfies (**dep**), and much work has been put towards providing sufficient conditions for (**dep**) to hold; see the introduction of [3] for an overview on the history of this problem. In this work, we provide a negative answer to this question. Our result in this direction comes as a consequence of the following theorem.

Theorem 1. *A Gorenstein local ring of positive dimension satisfies* (**dep**) *if and only if it is an AB ring.*

We say that R satisfies (**tr**) if a finitely generated R-module M is totally reflexive whenever $\operatorname{Ext}^{>0}_R(M,R)=0$; recall that M is called *totally reflexive* if the natural map $M\to M^{**}$ is an isomorphism and $\operatorname{Ext}^{>0}_R(M,R)=\operatorname{Ext}^{>0}_R(M^*,R)=0$, where $(-)^*$ is the

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The detailed version [11] of this paper will be submitted for publication elsewhere.

R-dual functor. The property (**tr**) is the same as the *weakly Gorenstein* property in the sense of Ringel and Zhang [12]. A (chain) complex of projective R-modules is called *totally acyclic* if it and its R-dual are both acyclic. We say that R satisfies (**tac**) if every acyclic complex of finitely generated projective R-modules is totally acyclic. This is a finitely generated module version of the property studied by Iyengar and Krause [8]. The following describes the relationships between (**tr**), (**tac**) and (**dep**).

Theorem 2. *A local ring satisfying* (**tac**) *satisfies* (**tr**). *A Cohen–Macaulay local ring of positive dimension satisfying* (**dep**) *satisfies* (**tr**).

Jorgensen and Şega [10] construct, for a field k that is not algebraic over a finite field, an artinian equicharacteristic local ring $(B, \mathfrak{m}_B, k)$ which does not satisfy (**tr**). (This shows that the second assertion of Theorem 2 does not necessarily hold without the assumption of positive dimension.) By considering the lifting of properties of local rings to higher Krull dimension, we obtain the following theorem. Here, we note that Heitmann's existence theorem [5] plays a crucial role in the proof.

Theorem 3. *Let k be a field which is not algebraic over a finite field.*

1. *For every $d \geq 2$, there is a d-dimensional Gorenstein equicharacteristic local unique factorization domain $(R, \mathfrak{m}_R, k)$ with an isolated singularity which does not satisfy* (**dep**).
2. *There exists a 1-dimensional Gorenstein equicharacteristic local domain $(S, \mathfrak{m}_S, k)$ that does not satisfy* (**dep**).
3. *For every integer $d \geq 2$, there exists a d-dimensional Cohen–Macaulay equicharacteristic local unique factorization domain $(R, \mathfrak{m}_R, k)$ with an isolated singularity which does not satisfy* (**tr**). *Therefore, R does not satisfy* (**tac**). *Moreover, R is a non-Gorenstein ring that does not satisfy* (**dep**).
4. *There is a 1-dimensional Cohen–Macaulay equicharacteristic local domain $(S, \mathfrak{m}_S, k)$ which does not satisfy* (**tr**). *Hence, S does not satisfy* (**tac**). *Also, S is a non-Gorenstein ring that does not satisfy* (**dep**).

It is claimed in [13, Theorem 1.1] that every generically Gorenstein ring satisfies (**tac**), and it is claimed in [13, Corollary 1.3] that every generically Gorenstein ring satisfies (**tr**). Here, a *generically Gorenstein* ring is defined as a ring which is locally Gorenstein on the associated prime ideals. Since every domain is a generically Gorenstein ring, these claims turn out to be incorrect in any positive dimension.

2. Comments on proofs of the main results

We give an outline of the proof of Theorem 1, that is, we want to prove that any Gorenstein local ring satisfying (**dep**) is AB. Actually, the following more general assertion holds true.

Theorem 4. *Let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring of dimension $d > 0$ with a canonical module ω. If R satisfies* (**dep**), *then for all two finitely generated R-modules M and N such that $\operatorname{Ext}_R^{\gg 0}(M, N) = 0$ one has $\operatorname{Ext}_R^{>d}(M, N) = 0$.*

Let M be a finitely generated R-module. Take a minimal free resolution $\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to M \to 0$ of M. The image of the ith differential map ∂_i is called the *ith syzygy* of M and denoted by $\Omega^i M$ (or $\Omega^i_R M$ to specify the base ring R). We set $\Omega^0 M = M$ and $\Omega M = \Omega^1 M$.

Sketch of Proof of Theorem 4 Assume that R satisfies (**dep**). Let $n \geq 0$ be an integer, and let M and N be finitely generated R-modules such that $\mathrm{Ext}_R^{>n}(M,N) = 0$ and $\mathrm{Ext}_R^n(M,N) \neq 0$. We claim that if M and N are maximal Cohen–Macaulay R-modules, then $n = 0$. If this claim is established, we can replace the two R-modules with their *maximal Cohen–Macaulay approximations* to obtain the conclusion of the theorem. Assume $n > 0$. Let $\boldsymbol{x} = x_1, \dots, x_d$ be a system of parameters of R. This is a regular sequence on $K := \Omega^{n-1} M$. Since $\mathrm{Ext}_R^{\geq 1}(K,N) = 0$ and $\mathrm{Ext}_R^1(K,N) \neq 0$, it can be proved that $\mathrm{Ext}_R^{\geq d+1}(K/\boldsymbol{x}K, N) = 0$ and $\mathrm{Ext}_R^{d+1}(K/\boldsymbol{x}K,N) \neq 0$. The dth syzygy $L = \Omega^d_R(K/\boldsymbol{x}K)$ is a maximal Cohen–Macaulay R-module such that $\mathrm{Ext}_R^{\geq 1}(L,N) = 0$ and $\mathrm{Ext}_R^1(L,N) \neq 0$. As L is locally free on the punctured spectrum of R and $d > 0$, we get $\mathrm{Tor}_i^R(L, \mathrm{Hom}(N,\omega)) \cong \mathrm{Hom}_R(\mathrm{Ext}_R^{d+i}(L,N), \mathrm{E}_R(k)) = 0$ for all $i > 0$, where $\mathrm{E}_R(k)$ is the injective hull of k. Since R satisfies (**dep**), we have $\operatorname{depth}(L \otimes_R \mathrm{Hom}_R(N,\omega)) = \operatorname{depth} L + \operatorname{depth} \mathrm{Hom}_R(N,\omega) - \operatorname{depth} R = d$, and hence $L \otimes_R \mathrm{Hom}_R(N,\omega)$ is maximal Cohen–Macaulay. We can see that $\mathrm{Ext}_R^i(L,N) = 0$ for all integers $1 \leq i \leq d$. This contradicts the fact that $\mathrm{Ext}_R^1(L,N) \neq 0$; recall that $d > 0$. Thus we must have $n = 0$. The claim follows. □

We state some comments on the second part of Theorem 2. The key point is the following characterization of local rings of positive depth that satisfy (**tr**).

Theorem 5. *Let R be a local ring with* $\operatorname{depth} R > 0$. *Then the following are equivalent.*

(1) *Every finitely generated R-module M with* $\mathrm{Ext}_R^{>0}(M,R) = 0$ *is totally reflexive, that is, R satisfies* (**tr**).
(2) *Every finitely generated R-module M with* $\mathrm{Ext}_R^{>0}(M,R) = 0$ *satisfies* $\operatorname{depth} M > 0$.

To show the theorem above, we need the following lemma. For a finitely generated module M over a ring R we denote by $\operatorname{Gdim}_R M$ the G-dimension of M. Note that M is totally reflexive if and only if $\operatorname{Gdim}_R M \leq 0$. We denote by $\operatorname{mod} R$ the category of finitely generated R-modules. A *resolving subcategory* of $\operatorname{mod} R$ is by definition a full subcategory of $\operatorname{mod} R$ containing R and closed under direct summands, extensions and syzygies. For the details of G-dimension and resolving subcategories, we refer the reader to [2].

Lemma 6. *Let $(R, \mathfrak{m})$ be a local ring with* $\operatorname{depth} R > 0$. *Let M be a finitely generated R-module such that* $\operatorname{depth} M > 0$ *and* $\operatorname{Gdim} M = \infty$. *Then there exists a finitely generated R-module N such that N is locally free on the punctured spectrum of R and contained in the smallest resolving subcategory of* $\operatorname{mod} R$ *containing the module M,* $\operatorname{depth} N > 0$ *and* $\operatorname{Gdim} N = \infty$.

We denote by $\operatorname{proj} R$ the full subcategory of $\operatorname{mod} R$ consisting of projective modules, and by $(-)^*$ the R-dual functor $\mathrm{Hom}_R(-,R)$. The *first cosyzygy* $\Omega^{-1}M$ of a finitely generated R-module M is defined as the cokernel of a *left* $\operatorname{proj} R$*-approximation* (or $\operatorname{proj} R$*-preenvelope*) $f : M \to F$, that is, f is a morphism in $\operatorname{mod} R$ with F projective such that

$f^* : F^* \to M^*$ is surjective. For an integer $n \geq 2$ the *nth cosyzygy* $\Omega^{-n}M$ is defined inductively by $\Omega^{-n}M = \Omega^{-1}(\Omega^{-(n-1)}M)$.

Proof of Theorem 5 Assume that (1) does not hold but (2) does. Then there is an R-module M which is not totally reflexive but satisfies $\mathrm{Ext}_R^{>0}(M,R) = 0$, so $\mathrm{Gdim}\, M = \infty$. By (2) we have $\operatorname{depth} M > 0$. Also, M belongs to the full subcategory $\mathcal{X}$ of $\operatorname{mod} R$ consisting of modules X with $\mathrm{Ext}_R^{>0}(X,R) = 0$. As $\mathcal{X}$ is resolving, by Lemma 6 we find an R-module $N \in \mathcal{X}$ which is locally free on the punctured spectrum, $\operatorname{depth} N > 0$ and $\mathrm{Gdim}\, N = \infty$. Then there is an exact sequence $0 \to N \to F^0 \to \Omega^{-1}N \to 0$ with F^0 free. This exact sequence implies $\mathrm{Gdim}\, \Omega^{-1}N = \infty$ and $\mathrm{Ext}^{>0}(\Omega^{-1}N, R) = 0$. By (2) again we have $\operatorname{depth} \Omega^{-1}N > 0$ and get an exact sequence $0 \to \Omega^{-1}N \to F^1 \to \Omega^{-2}N \to 0$ with F^1 free. Iterating this procedure yields an exact sequence

$$0 \to N \to F^0 \xrightarrow{\partial^1} F^1 \xrightarrow{\partial^2} F^2 \xrightarrow{\partial^3} \cdots$$

such that for each $i > 0$ we have that F^i is free, the image of ∂^i is $\Omega^{-i}N$, and $\mathrm{Ext}^{>0}(\Omega^{-i}N, R) = 0$. Applying the functor $(-)^*$ gives rise to an exact sequence $\cdots \to (F^2)^* \to (F^1)^* \to (F^0)^* \to N^* \to 0$, and applying $(-)^*$ again restores the original exact sequence. This shows that N is totally reflexive. However, this contradicts the fact that $\mathrm{Gdim}\, N = \infty$. We now conclude that (2) implies (1). □

No counterexample has been found so far to each of the implication (**tac**) ⇒ (**tr**). Once a counterexample of an artinian equicharacteristic local ring is found, one can lift it to a counterexample of a (unique factorization) domain with an isolated singularity. Finally, we give some comments on [13].

Remark 7. Theorem 3 says that the assertions of [13, Theorem 1.1 and Corollary 1.3] are both incorrect. In their proofs, [13, Theorem 8.5] plays an essential role, and the authors wonder if the proof of [13, Theorem 8.5] contains gaps. On the other hand, the ring R produced by Theorem 3 is not excellent. So, even if the proof of [13, Theorem 8.5] contains gaps, the assertion itself may be true in the case where the base ring is excellent. However, the theory developed in [13] does not seem to be related to the excellence of the base ring, so even if the assertion of [13, Theorem 8.5] is true for excellent rings, we would need another approach to show it.

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