PROCEEDINGS OF THE 11TH SYMPOSIUM ON RING THEORY

HELD AT YAMAGUCHI UNIVERSITY, YAMAGUCHI

JULY 28, 1978

EDITED BY

Takasi NAGAHARA
Okayama University

WITH THE COOPERATION OF

Shizuo ENDO
Tokyo Metropolitan University

Manabu HARADA
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OKAYAMA, JAPAN

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CONTENTS

PREFACE

This volume contains the articles presented at the 11th Symposium on Ring Theory held at Yamaguchi University, July 28, 1978.

The main aims of the Symposium are to provide a means for the dissemination of recent theories on rings and modules which are not yet widely known and to give algebraists an opportunity to report on recent progress in the ring theory.

The 11th Symposium itself and this proceedings were partially supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture.

Finally we would like to thank Prof. Y. Kurata for unending patient and kind hospitality to the participants of the Symposium.

T. Nagahara

ON REGULAR RINGS AND π-REGULAR RINGS

Yasuyuki Hirano

1. Intoruction. Firstly, the notions of right p.p. rings, right CPP-rings and right CPF-rings, introduced primarily for rings with identity, will be defined for s-unital rings. Using these notions, we shall characterize (von Neumann) regular rings (possibly without identity). Furthermore, we shall present a characterization of an s-unital right CPP-rings, which will deduce the main theorem in [5]. Next, we shall consider π-regular rings. In his paper [9], H. Tominaga proved that if A is a π-regular ring of bounded index then $(A)_n$ is strongly π-regular for any positive integer n. We shall show that the same is true for N-rings, and give several equivalent conditions for an N-ring and for a CI-ring to be strongly π-regular.

Throughout A will represent a ring (possibly without identity), N_0 the set of all nilpotent elements of A, N the prime radical of A, and J the Jacobson radical of A.

If M is a right (resp. left) A-module and S is a subset of A, then we set $\ell_M(S) = \{u \in M \mid uS = 0\}$ (resp. $r_M(S) = \{u \in M \mid Su = 0\}$). As usual, we write $\ell(S) = \ell_A(S)$ and $r(S) = r_A(S)$.

1. Regular rings. Following [10], a non-zero right (resp. left) A-module M is said to be s-unital if $u \in uA$ (resp. $u \in Au$) for each $u \in M$. If A_A (resp. ${}_AA$) is s-unital, A is called a right (resp. left) s-unital ring. In case A is right and left s-unital, we merely say s-unital. If F is a finite subset of a right s-unital ring (resp. an s-unital ring) A, then there exists an element $e \in A$ such that $ae = a$ (resp. $ea = ae = a$) for all $a \in F$.

A right A-module M is said to be p-injective if for any principal right ideal $|a)$ of A and $f: |a)_A \to M_A$ there exists an element $u \in M$ such that $f(x) = ux$ for all $x \in |a)$. Let A be a right s-unital ring, and M_A an s-unital module. If M_A is p-injective then, for each $a \in A$ there holds $\ell_M(r(a)) = Ma$, and conversely. In particular, for a domain A with 1, a unital module M_A is p-injective if and only if M_A is divisible. As is well known, A is a regular ring if and only if every right A-module is p-injective.

A right s-unital ring A is called a right p.p.ring if every r(a) is a direct summand of A_A. A right s-unital ring A is called a right CPP-ring (resp. CPF-ring) if for each non-zero right ideal R of A either R is a direct summand of A_A (resp. R is a s-unital ring) or A/R_A is p-injective (see [11]). As was noted in [10, Proposition 1], it is well known that a non-zero right ideal R of A with 1 is a left s-unital ring if and only if A/R_A is flat. It is easy to see that every regular ring is a right CPP-ring and every s-unital right CPP-ring is a right CPF-ring. Moreover, every homomorphic image of right CPP-ring (resp. CPF-ring) is also a right CPP-ring (resp. CPF-ring). If A is an s-unital, right CPP-ring then A is a fully right idempotent, right p.p. ring (see [7]). Now, we shall present some characterizations of regular rings.

Theorem 1. ([7]). The following are equivalent:

1) A is a regular ring.

2) A is a right CPP-ring and A_A is p-injective.

3) A is a right p.p.ring and A_A is p-injective.

4) A is a right s-unital ring such that A_A and every singular homomorphic image of A_A are p-injective.

5) Every essential right ideal of A is a left s-unital ring.

6) A is an s-unital, right CPF-ring such that every principal right ideal is either a direct summand of A_A or the right annihilator of an element.

7) A is an s-unital ring such that for each essntial right ideal R either R is a left s-unital ring or A/R_A is p-injective, and that every principal right ideal is either a direct summand of A_A or the right annihilator of an element.

2') - 7') The left-right analogues of 2) - 7).

Let A be a non-regular, right CPP-ring. If $A = R_1 \oplus R_2$ with right ideals R_1 and R_2, then R_1 or R_2 is completely reducible (see [7]). Using this fact, we obtain the following

Theorem 2. ([7]). The following are equivalent:

1) A is an s-unital, right CPP-ring.

2) A is a regular ring or $A = S \oplus T$ where S is a right (and left) completely reducible, semi-prime ring and T is a simple domain (not a division ring) all of whose proper cyclic right modules are divisible.

In [8], B. Osofsky proved that if every cyclic right A-module is injective, then A is Artinian, semi-primitive. Also, C. Faith [4] proved that if each proper cyclic right A-module is injective, A is either Artinian, semi-primitive or a right semi-hereditary, right Ore domain. Using these results, we obtain [5, Theorem] as a Corollary of Theorem 2.

Corollary 1. If A contains 1, then the following are equivalent:

1) Every cyclic right (unital) A-module is injective

or projective.

2) $A = S \oplus T$ where S is an Artinian, semi-primitive ring and T is 0 or a simple, right semi-hereditary, right Ore domain (not a division ring) all of whose proper cyclic right modules are injective.

2. π-regular rings. A is said to be π-regular if for each a in A there exists an x in A and a positive integer n such that $a^n = a^n x a^n$. A is called right (resp. left) π-regular if for each a in A, there exists an x in A and a positive integer n such that $a^n = a^{n+1}x$ (resp. $a^n = xa^{n+1}$). A ring which is both left and right π-regular is called strongly π-regular. Recently, F. Dischinger [2] has proved that every right (or left) π-regular ring is strongly π-regular. He has also announced the following ; For each integer $n \geq 1$, $(A)_n$ is strongly π-regular if and only if A has the property that injective endomorphisms of finitely generated right (left) A-modules are isomorphisms (see [1]).

If N coincides with N_0, or equivalently, if A/N

is a reduced ring, then A is called an N-ring. As was noted in [10], every P_1-ring is an AC-ring, and every AC-ring is an N-ring. Following [3], A is called a CN-ring (resp. CI-ring) if every nilpotent (resp. idempotent) element of A is central. As is easily seen, every CN-ring is a CI-ring (and an N-ring), but not conversely.

M. P. Drazin gave the following sufficient condition for A to be a CN-ring.

Proposition 1. ([3, Theorem 2]). If for each x, y in A there exists some z in A such that $[x - x^2z, y] = 0$, then A is a CN-ring.

Theorem 3. ([6]). If A is an N-ring then the following are equivalent:

1) A is strongly π-regular.

2) A is π-regular.

3) J is nil and A/J is π-regular.

4) A/I is π-regular for some nil ideal I.

5) A/N is strongly regular.

6) Every proper prime ideal of A is a maximal one-

sided ideal.

7) Every proper completely prime ideal of A is a maximal one-sided ideal.

8) $(A)_n$ is strongly π-regular ($n = 1, 2, \ldots$).

Next, as a combination of Theorem 3 and Proposition 1, we obtain the following

Corollary 2. ([6]). The following are equivalent:

1) A is a strongly π-regular CN-ring.

2) A is a CN-ring whose proper prime ideals are maximal one-sided ideals.

3) For each $x \in A$ there exists some y such that $x - x^2y$ is a central nilpotent element.

4) A is a π-regular ring such that for each $x, y \in A$ there exists some z with $[x - x^2z, y] = 0$.

Corresponding to Corollary 2, we have the next

Theorem 4. ([6]). If A is a CI-ring, then the following are equivalent:

1) A is strongly π-regular.

2) A is π-regular.

3) J is nil and A/J is π-regular.

4) A/I is π-regular for some nil ideal I.

5) Every prime factor ring of A is either a nil ring or a local ring with Jacobson radical nil.

6) J is nil and every element of A is either π-regular or quasi-regular.

7) Every non-nil right ideal of A contains a non-zero idempotent and every element annihilated by some non-zero idempotent is π-regular.

Any PI-ring contains a unique maximal π-regular ideal and, more generaly, we have the following result.

Theorem 5. (see [6], [9]). If A is a ring such that each prime factor ring of A is of bounded index, then A contains a unique maximal π-regular ideal M and A/M has no non-zero π-regular ideals.

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NOTES ON DECOMPOSITIONS OF INJECTIVE MODULES

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Let R be a commutative ring with a unit, Spec(R) the set of all prime ideals in R and let F(R) be the set of all elements P in Spec(R) such that the localization R_P of R at P is a noetherian ring. Let P be an element in F(R). We shall denote by G(P) the generalization of P. A non-empty subset X of F(R) is said to be of open type if, for any P in X, $G(P) \subseteq X$. Let X be a subset of F(R) of open type. We shall denote by N'[X] the set of all R-modules such that $M_P = 0$ for all P in X and by N[X] the set of all R-modules L such that $Hom_R(M, L) = 0$ for all M in N'[X].

Now, we state the following conditions.

(I). Every injective R-module in N[X] can be expressed as a direct sum of indecomposable injective R-modules in N[X] of the form $E_R(R/P)$, $P \in X$, where $E_R(R/P)$ is the injective hull of R/P.

(II). A direct sum of any family of injective R-modules in N[X] is injective as an R-module.

If R is a noetherian ring, then, for any subset X of Spec(R) of open type, the conditions (I) and (II) hold in N[X]. In this note, we shall study decompositions of injective R-modules in N[X], and observe the conditions for R and X such that the conditions (I) and (II) hold in N[X]. Among others, if X satisfies the maximal condition, then we shall have such decompositions are characterized by the injectivity of $\sum_{P \in S} \oplus E_R(R/P)$, where S is the set of all maximal elements in X.

We shall say that R satisfies the condition H[X] if, for any proper ideal A of R such that

$$A = \bigcap_{P \in X} (AR_P \cap R),$$

we have that $\mathrm{Ass}(R/A) \cap X \neq \emptyset$.

Let X be a subset of F(R) of open type. Then, in § 1 we shall observe basic properties of injective R-modules in N[X] provided that R satisfies the condition H[X]. In § 2, we shall study properties of ideals A of R such that R/A belongs to N[X], and in § 3 we shall observe indecomposable decompositions of injective R-modules in N[X].

1. Basic properties. Throughout this note, we shall denote by R a commutative ring with a unit, by X a subset of F(R) of open type, and assume that F(R) is not empty for R considered in this note. We shall also denote by $E_R(M)$ the injective hull of an R-module M.

For X, we are easy to see that N[X] contains a non-zero R-module. Furthermore, if an R-module M belongs to N[X], then every submodule of M belongs to N[X].

PROPOSITION 2.1. Let M be an R-module. Then, M belongs to N[X] if and only if $E_R(M)$ belongs to N[X].

PROOF. Assume that M belongs to N[X]. For any submodule $M' \neq 0$ of $E_R(M)$, we have $M' \cap M \neq 0$. Moreover, if an R-module N belongs to N'[X], every submodule of N belongs to N'[X]. Thus any non-zero submodule of $E_R(M)$ does not belong to N'[X], and so $E_R(M)$ belongs to N'[X].

REMARK. Using Proposition 2.1, we have that

N[X]) becomes a hereditary torsion theory.

LEMMA 1.2. Let R satisfies the condition H[X] and M a non-zero R-module in N[X]. Then, $Ass(M) \cap X \neq \emptyset$.

PROOF. Let $0 \neq x$ be an element of M and set $Ann_R(x) = A$. Then, it is easily seen that

$$A = \bigcap_{\substack{P \in X \\ P \supseteq A}} (AR_P \cap R).$$

Since R satisfies the condition H[X], $Ass(R/A) \cap X \neq \emptyset$. As $R/A = Rx \subseteq M$, we have $Ass(R/A) \subseteq Ass(M) \cap X \neq \emptyset$.

THEOREM 1.3. The following conditions are equivalent.

(1). R satisfies the condition H[X].

(2). Any non-zero injective R-module in N[X] contains an indecomposable injective R-module isomorphic to $E_R(R/P)$ for some P in X.

PROOF. (1)$\longrightarrow$(2). Let M be a non-zero injective R-module in N[X]. Then, by Lemma 1.2 there exists an element P in X such that $M \supseteq M' \cong R/P$. Since M is injective, $M \supseteq E_R(M') \cong E_R(R/P)$, which is an indecomposable injective R-module by Theorem 2.4 of [6]. Hence M contains an indecomposable injective R-module isomorphic to $E_R(R/P)$.

(2)$\longrightarrow$(1). Let A be any proper ideal in R such that

$$A = \bigcap_{\substack{P \in X \\ P \supseteq A}} (AR_P \cap R).$$

Then, it is easy to see that R/A is contained in N[X] and by Proposition 1.1 $E_R(R/A) \in N[X]$. Thus, by the assumption $E_R(R/A)$ contains a submodule E' isomorphic to $E_R(R/P)$ for some P in X. Hence, as $Ass(R/A) = Ass(E_R(R/A))$, we have that $Ass(R/A) \supseteq Ass(E') = Ass(E_R(R/P)) \ni P$, and so, Ass(R/A)

$\bigcap X \neq \emptyset$.

PROPOSITION 1.4. Assume that R satisfies the condition H[X]. Then, every injective R-module in N[X] is the injective hull of a direct sum of indecomposable injective R-modules in N[X], each of which is isomorphic to $E_R(R/P)$ for some P in X.

PROOF. Let M be any injective R-module in N[X] and $(E_i)_{i \in I}$ the set of all submodules of M which are indecomposable injective R-modules. Now, set K = $\{J \subseteq I \mid \sum_{i \in J} E_i$ is a direct sum$\}$. Then, by Theorem 1.3, K is not empty if M is not zero. By Zorn's Lemma, there exists a maximal element J_o in K with respect to the canonical inclusion. Then, we infer that $M = E_R(M_o)$, where $M_o = \sum_{i \in J_o} E_i$.

For, if $E_R(M_o) \subsetneqq M$, then $E_R(M_o)$ is a proper diect summand of M, that is, $M = E_R(M_o) \oplus M'$ and M' is a non-zero injective R-module belonging to N[X]. Thus, by Theorem 1.3, M' contains an indecomposable injective R-module which is isomorphic to $E_R(R/P)$ for some P in X and this contradicts to the maximality of J_o. Hence $M = E_R(M_o)$.

COROLLARY. If R satisfies the condition H[X], then any indecomposable injective R-module in N[X] is of the form $E_R(R/P)$, $P \in X$.

LEMMA 1.5. Let P and P' be prime ideals of R such that $P \subseteq P'$. Then, we can regard $E_R(R/P)$ as an $R_{P'}$-module and it is an indecomposable injective $R_{P'}$-module.

PROOF. It is easy to see that the mapping T_r: $E_R(R/P) \longrightarrow E_R(R/P)$ defined by $T_r(x) = rx$ for x $E_R(R/P)$ is an automorphism for every r in R - P. Thus, $E_R(R/P)$ can be regarded as an $R_{P'}$-module and the indecomposablity of $E_R(R/P)$ as an $R_{P'}$-module follows from that of $E_R(R/P)$ as an R-module.

Assume that R satisfies the condition H[X] and let M be an injective R-module in N[X]. Then by Theorem 1.3 Ass(M) $\neq \emptyset$ if M $\neq$ 0 and by Proposition 1.4 M can be expressed as follows;

$$\text{(a)} \qquad M = E_R(\sum_{i \in I} \oplus E_i),$$

where each of $E_i (i \in I)$ is isomorphic to $E_R(R/P_i)$ for some P_i in X; let us call such an expression of M, an expression of M. For each P in Ass(M), let us set

$$M(P) = E_R(\sum_{\substack{i \in I \\ P_i = P}} \oplus E_i),$$

$$M[P] = E_R(\sum_{\substack{P' \in Ass(M) \\ P' \subseteqq P}} \oplus M(P')).$$

We shall then call M(P) the P-component of M and M[P] the local component of M at P, with respect to the expression (a) of M. For convenience, let us set M(P) = 0 and M[P] = 0 when $P \in X$ - Ass(M) and $G(P) \cap Ass(M) = \emptyset$, respectively.

We infer that $M(P) = \sum_{\substack{i \in I \\ P_i = P}} \oplus E_i$ and it is a direct summand of M. By Lemma 1.5,

$$M' = \sum_{\substack{P' \in Ass(M) \\ P' \subsetneqq P}} \oplus M(P')$$

can be regarded as an R_P-module and for each $i \in I$, E_i, which appears in M', is injective as an R_P-module. Thus

M' is injective as an R_P- module because R_P is a noetherian ring. Therefore, M' is injective as an R-module, that is,

$$M[\mathfrak{X}]_{P} = \sum_{\substack{P' \in Ass(M) \\ P' \subseteqq P}} \oplus M(P') = \sum_{\substack{i \in I \\ P_i \subseteqq P}} \oplus E_i.$$

By Corollary 4.2 of [14] ro by Theorem 6 of [2], the P-components of M with respect to any two expressions of M are isomorphic. From these facts, we obtain the following proposition.

PROPOSITION 1.6. Assume that R satisfies the condition H[X] and let M be an injective R-module belonging to N[X]. Then we obtain the followings.

(1). For each element P in Ass(M), the P-component of M and the local component of M at P, with respect to an expression of M, can be written as direct sums of indecomposable injective R-modules belonging to N[X].

(2). For each element P in Ass(M), the P-components of M with respect to any two expressions of M are isomorphic and the local components of M at P with respect to any two expressions of M are isomorphic.

(3). Let M(P')(P'$\in$Ass(M)) be the P'-components of M with respectto an expression of M. We then obtain

$$M = E_R\left(\sum_{P' \in Ass(M)} \oplus M(P')\right).$$

Let M be an injective R-module belonging to N[X] and suppose that R satisfies the condition H[X]. For an element P in X, let us set $(E_i)_{i \in I}$ the set of all submodules of M each of which is isomorphic to $E_R(R/P')$ for some P' in X which is contained in P and set $V = \{L \mid L \subseteq I, \sum_{i \in L} E_i$ is a direct sum$\}$. By Zorn's Lemma, there is a maximal element L_o in V with respect to the order by the canonical inclusion. Then, let us call $M' = \sum_{i \in L_o} \oplus E_i$ a local component of M at P. By Lemma 1.5, we infer that M' is injective as an R-module. Furthermore, it is easy to see that $G(P) \cap Ass(M'') = \emptyset$, where $M = M' \oplus M''$. Since M'' is injective as an R-module and belongs to N[X], by Proposition 1.4 M'' can be expressed as

$$M'' = E_R(\sum_{j \in J} \oplus E'_j),$$

where $E'_j (j \in J)$ are indecomposable injective R-modules belonging to N[X]. Thus, we have that

$$M = (\sum_{i \in L_o} \oplus E_i) \oplus E_R(\sum_{j \in J} \oplus E'_j)$$

is an expression of M and the local component M[P] of M at P is equal to M'. Thus, by Proposition 1.6, any two local components of M at P are isomorphic (we also denote by M[P] a local component of M at P). From these facts, we obtain the following proposition.

PROPOSITION 1.7. Assume that R satisfies the condition H[X] and let M be an injective R-module in N[X]. Then, we have the followings.

(1). For each element P in Ass(M), a local component of M at P is a direct summand of M.

(2). Any two local components of M at P are isomorphic.

2. N[X]-ideals. An ideal A of R is called an N[X]-ideal if R/A belongs to N[X]. We shall study , in this section, properties of N[X]-ideals of R.

PROPOSITION 2.1. (1). An ideal A of R is an N[X]-ideal if and only if $A = \bigcap_{P \in X} (AR_P \cap R)$.
(2). For any family $\{A_j\}_{j \in J}$ of N[X]-ideals of R, $\bigcap_{j \in J} A_j$ is an N[X]-ideal.
(3). If P is an element in X, then every P-primary ideal of R is an N[X]-ideal.

PROOF. (1) is immediate. (2) Since $R/A_j \in N[X]$ for all $j \in J$, we obtain that

$$\prod_{j \in J} (R/A_j) \in N[X] \text{ and } \bigcap_{j \in J} A_j = \mathrm{Ann}_R((\bar{1}_j)_{j \in J}),$$

where $\bar{1}_j$ is the canonical image of 1 in R/A_j. Thus, $\bigcap_{j \in J} A_j$ is an N[X]-ideal of R.

(3) Let Q be an arbitrary P-primary ideal of R. Then since QR_P is PR_P-primary and since R_P is a noetherian ring, QR_P has an irredundant irreducible decomposition;

$$QR_P = Q'_1 \cap Q'_2 \cap \cdots \cap Q'_n,$$

and it is easily seen that Q'_i is irreducible PR_P-primary. Thus, by Theorem 2.3 and Proposition 3.1 of [6], we have that

$$E_{R_P}(R_P/QR_P) = E_{R_P}(R_P/Q'_1) \oplus \cdots \oplus E_{R_P}(R_P/Q'_n)$$

$$= E_{R_P}(R_P/PR_P) \oplus \cdots \oplus E_{R_P}(R_P/PR_P)$$

which is isomorphic to $E_R(R/P) \oplus \cdots \oplus E_R(R/P)$. Moreover, it is easy to check that $E_{R_P}(R_P/QR_P) = E_R(R/Q)$. Hence,

$E_R(R/Q)$ belongs to N[X] because $E_R(R/P) \in N[X]$, and this implies that R/Q belongs to N[X] or Q is an N[X]-ideal.

COROLLARY. For any R-module M in N[X], $Ann_R(M)$ becomes an N[X]-ideal.

PROPOSITION 2.2. If P is in X, then an ideal Q of R is an irreducible P-primary ideal if and only if Q = Ann_R (x) for some $x \neq 0$ in $E_R(R/P)$.

PROOF. It is clear that an ideal A of R contained in P is irreducible P-primary if and only if $A = AR_P \cap R$ and AR_P is irreducible PR_P-primary. Since R_P is a noetherian ring, by Lemm 3.2 of [6] an ideal B of R_P is irreducible PR_P-primary if and only if $B = Ann_{R_P}(y)$ for some $0 \neq y \in E_{R_P}(R_P/PR_P)$. Thus we obtain the result because $E_R(R/P) = E_{R_P}(R_P/PR_P)$ and $Ann_R(z) = Ann_{R_P}(z) \cap R$, for $z \in E_R(R/P)$.

PROPOSITION 2.3. If R satisfies the condition H[X], then every N[X]-ideal of R is the intersection of irreducible primary ideals of R each of which belongs to some prime ideal belonging to X.

PROOF. Let A be an N[X]-ideal of R. Then, R/A belongs to N[X] and by Proposition 1.1 $E_R(R/A)$ belongs to N[X]. Thus, by Proposition 1.4, $E_R(R/A)$ can be expressed as

$$E_R(R/A) = E_R(\sum_{i \in I} \oplus E_i),$$

where for each $i \in I$, $E_i \cong E_R(R/P)$ for some P in X.

Since $E_R(\sum_{i \in I} \oplus E_i)$ can be regarded as a submodule of $\prod_{i \in I} E_i$, for $\bar{1}$(=the canonical image of 1 in R/A) = $(x_i)_{i \in I}$ in $E_R(R/A)$,

$$A = Ann_R(\bar{1}) = \bigcap_{i \in I} Ann_R(x_i),$$

and by Propostion 2.2 $Ann_R(x_i)$ is an irreducible primary ideal belonging to some prime in X whenever $x_i \neq 0$. This is the required result.

PROPOSITION 2.4. If A is any N[X]-ideal of R, then (A : B) is an N[X]-ideal for every ideal B of R.

PROOF. It is easy to see that $(A : B) = Ann_R(A + B/A)$ and $(A + B)/A$ belongs to N[X] because it is s submodule of R/A. Thus (A : B) is an N[X]-ideal of R.

PROPOSITION 2.5. Suppose that R satisfies the ascending chain condition on N[X]-ideals of R. Then, if Q is an irreducible N[X]-ideal of R, Q is a primary ideal and its radical belongs to X.

PROOF. Assume that Q is not primary. Then, there are elements a, b in R such that $ab \in Q$, $a \notin Q$ and $b^n \notin Q$ for all $n \geq 1$. Then, by Proposition 2.4, $(Q : Rb^n)$ is an N[X]-ideal for all n and

$$(Q : Rb) \subseteqq (Q : Rb^2) \subseteqq (Q : Rb^3) \subseteqq \cdots\cdots.$$

As R satisfies the ascending chain condition on N[X]-ideals, there is an integer m such that

$$(Q : Rb^m) = (Q : Rb^{m+1}) = \cdots.$$

Under these situations, we infer that

$$Q = (Q : Rb^m) \cap (Q + Rb^m),$$

$$Q \subsetneqq (Q : Rb^m),\ (Q + Rb^m),$$

and this contradicts to the irreducibility of Q. Hence Q is primary. Since $(R/Q)_{P'} = 0$ if $P' \not\supseteqq Q$ and $R/Q \neq 0$ belongs to N[X], $(R/Q)_P \neq 0$ for some P in X and this implies that $Q \subsetneqq P$ or the radical of Q belongs to X because X is of open type.

Assume that X satisfies the maximal condition with respect to the canonical inclusion. Then, we shall denote by S the set of all maximal elements in X. Now, we have the following theorem.

THEOREM 2.6. If $\sum_{P \in S} \oplus E_R(R/P)$ is injective as an R-module, then R satisfies the ascending chain condition on N[X]-ideals of R.

PROOF. Let $A_1 \subsetneqq A_2 \subsetneqq A_3 \subsetneqq \ldots$ be a strictly ascending chain of N[X]-ideals of R and $A = \bigcup_i A_i$. Then $Rx_i = R/A_i \in N[X]$ for all i and $Ax_i \neq 0$ for all i. Set $M = \sum_i \oplus E_R(Rx_i)$ and let g be an R-homomorphism of A into M defined by

$$g(r) = \sum_i rx_i, \quad r \in A.$$

For each i, since $\bigcap_{P \in S} (A_i R_P \cap R) = A_i$ by Proposition 2.1 and $A_i \subsetneqq A$, there exists an element P_i in X such that $A \nsubseteq (A_i R_{P_i} \cap R)$. Let $s_i \in A - (A_i R_{P_i} \cap R)$. Then, $(A_i : s_i) \subseteqq P_i$. Therefore, there is the canonical homomorphism

$$f_i' : (Rs_i + A_i)/A_i \longrightarrow R/P_i$$

defined by $f_i'(rs_i + A_i) = r\bar{1}_i$, $r \in R$, where $\bar{1}_i$ is the canonical image of 1 in R/P_i and f_i' can be extended to an R-homomorphism f_i of $E_R(R/A_i)$ into $E_R(R/P_i)$. Let h_i be the canonical projection of $\sum_j \oplus E_R(R/A_j)$ onto $E_R(R/A_i)$. Then, we obtain an R-homomorphism of A into $\sum_i \oplus E_R(R/P_i)$ defined by

$$f = \sum_i f_i h_i g$$

whose image is not contained in a direct sum of any finite number of $E_R(R/P_i)$, $i = 1, 2, 3, \ldots$ by the construction of f. Thus, f can not be extended to an R-homomorphism of R into $\sum_i \oplus E_R(R/P_i)$ and this contradicts to the injectivity of $\sum_i \oplus E_R(R/P_i)$ obtained by the following Lemma 2.7. Thus we have the result.

LEMMA 1.7. If $\sum_{P \in S} \oplus E_R(R/P)$ is injective as an R-module, then for any subset S' of S, $\sum_{P \in S'} \oplus J(P)$ is injective, where J(P) is a direct sum of arbitrary copies of $E_R(R/P)$.

PROOF. It is easy to see that for each $P \in S'$, J(P) is injective as an R-module. Now, if $\sum_{P \in S'} \oplus J(P)$ is not injective, there esist an ideal B of R and an R-homomorphism f of B into $\sum_{P \in S'} \oplus J(P)$ such that the image of f is not contained in a direct sum of any finite number of J(P), $P \in S'$. Moreover, if the canonical projection h_P of f(B) to J(P) is not zero, then the canonical projection of J(P) to $E_R(R/P)$ for some component is not zero and let us denote this map by g_P. Now, $\sum_{P \in S'} g_P h_P f$ is an R-homomorphism of B into $\sum_{P \in S'} \oplus E_R(R/P)$ whose image is not contained in a direct sum of any finite number of $E_R(R/P)$, $P \in S' \subseteqq S$. This is impossible and we obtain the result.

3. Decompositions of injective modules in N[X]. Assume that X satisfies the maximal condition and let S be the set of all maximal elements of X. Then, we obtain the following theorem.

THOREM 3.1 (Proposition 3.5 of [4]). For N[X], the following conditions are equivalent.

(1). $\sum_{P \in S} \oplus E_R(R/P)$ is injective as an R-module.

(2). For any coutable subset S' of S, $\sum_{P \in S'} \oplus E_R(R/P)$ is injective as an R-module.

(3). R satisfies the ascending chain condition on N[X]-ideals of R.

(4). Every N[X]-ideal of R can be written as the intersection of a finite number of irreducible primary ideals of R each of which belongs to some prime in X.

(5). Every injective R-module in N[X] can be expressed as a direct sum of indecomposable injective R-modules in N[X] each of which is isomorphic to $E_R(R/P)$ for aome P in X.

(6). A direct sum of any family of injective R-modules belonging to N[X] is injective as an R-module.

(7). A direct sum of any countably many injective R-modules belonging to N[X] is injective as an R-module.

REMARK. The injectivity of $\sum_{P \in S} \oplus E_R(R/P)$ characterrizes that the conditions (I) and (II) hold in N[X].

Let Y be a non-empty subset of F(R) and let us set $\overline{Y} = \bigcup_{P \in Y} G(P)$. Then, we shall say that $\overline{Y}$ is the open closure of Y. A subset Y of F(R) is said to be of finite type if there is a finite subset of F(R) such that its open closur contains Y. For any non-empty subset Y of X, it is easy to see that $N[\overline{Y}] \subsetneqq N[X]$. Thus, we have the following corollary.

COROLLARY. If X is of finite type, then the conditions (I) and (II) hold in N[X].

Let A be an ideal of R and f an R-homomorphism of A into $E_R(R/P)$ ($P \in Spec(R)$), and assume that $A \subseteq u^{-1}(0)$, where $u : R \longrightarrow R_P$ is the canonical homomorphism. Then, it is easily seen that f is trivial. Using this fact, we infer the following theorem.

THEOREM 3.2 (Theorem 2.3 of [5]). Assume that R satisfies the condition H[X] and X satisfies the maximal condition. Let S be the set of all maximal elements of X. Then, if the set $\{P \in S \mid AR_P \neq 0\}$ is a finite set for any proper ideal A of R, the conditions (I) and (II) hold in N[X].

Let A be an ideal of R and f an R-homomorphism of A into $E_R(R/P)$ ($P \in Spec(R)$). Then, it is easy to see that f is trivial whenever $Ker(f) \nsubseteq P$. From this fact, we infer the following theorem.

THEOREM 3.3 (Theorem 2.4 of [5]). Assume that X satisfies the maximal condition and every non-zero ideal of R is contained in only a finite number of maximal elements in X. Then, we have the following facts.

(1). R satisfies the condition H[X].

(2). The conditions (I) and (II) hold in N[X].

Let R be a Krull domain and set $X' = \{P \in Spec(R) \mid \text{the height of } P \leqq 1\}$. Then, X' is of open type. I. Beck investigated in [1] the structures and the decompositions of injective R-modules in N[X'], and completely determined these. We can obtain his results as the corollary of Theorem 3.3 because R and X' satisfy the conditions of Theorem 3.3.

COROLLARY. Let R and X' be as above. Then, we have the following facts.

(1). R satisfies the condition H[X'].

(2). Every indecomposable injective R-modules in N[X'] is of the form $E_R(R/P)$, $p \in X'$.

(3). The conditions (I) and (II) hold in N[X'].

Let r be an element of R and set $V(r) = \{P \in Spec(R) \mid P \not\ni r\}$. Then, Spec(R) becomes a topological space with $\{V(r) \mid r \in R\}$ as the system of basic open sets. This topology is known as the Zariski topology. Let Y a subset of Spec(R). Then, a point P in Y is called an isolated point if there is an element r in R such that $V(r) \cap Y = \{P\}$, and Y is said to be a discrete subset if every point of Y is isolated. For commutative regular rings (von Neumann), we have the following propositions.

PROPOSITION 3.4 (Proposition 2.1 of [3]). Let R be a commutative regular ring. Thne, the following conditions are equivalent.

(1). R satisfies the condition H[X].

(2). Every non-empty subset of X contains an isolated point.

PROPOSITION 3.5. Let R be a regular ring and assume that every non-empty subset of X contains an isolated point. Then, we have the followings.

(1). Every injective R-module in N[X] is a injective hull of semi-simple module in N[X].

(2). Let M be an injective R-module in N[X] and (M(P) ((P∈ Ass(P)) the P-components of M. Then,

$$M = \prod_{P \in Ass(M)} M(P).$$

Let R be a ring which is the direct product of a family $(K_i)_{i \in I}$ of fields K_i. Then, F(R) = Spec(R) since R is a regular ring. Let P be a prime ideal in R. Then, we shall call P "of first kind" when P is generated by an idempotent and P "of second kind" when P contains $\sum_{i \in I} \oplus K_i$. We shall denote by X_1 the set of all prime ideals of R of first kind and by X_2 the set of all prime ideals in R of second kind. Since X_1 is a discrete subset of Spec(R), any non-empty subset of X_1 is discrete. Thus we infer the followings for any non-empty subset X of X_1.

(1). R satisfies the condition H[X].

(2). Every injective R-module in N[X] is a injective hull of a semi-simple R-module in N[X].

(3). Let M be any injective R-module in N[X] and M(P) ($P \in Ass(M)$) the P-components of M. Then, $M = \prod_{P \in Ass(M)} M(P)$.

We can show that X_2 contains no isolated points if X_2 is not empty. From this we obtain the following proposition.

PROPOSITION 3.6. If X_2 is not empty, then we have the following facts.

(1). R does not satisfy the condition $H[X_2]$. In particular, R does not satisfy the condition H[Spec(R)].

(2). There exists an injective R-module in $N[X_2]$ such that it can not be expressed as the injective hull of any semi-simple R-module in $N[X_2]$.

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ON MAXIMAL QUOTIENT RINGS OF QF-3, 1-GORENSTEIN RINGS WITH ZERO SOCLE

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This note is an abstract of the author's papers [12] and [13], and includes historical notes of study of 1-Gorenstein rings. A ring is said to be right 1-Gorenstein if it is left and right noetherian and its right self-injective dimension at most one. A left and right 1-Gorenstein ring is called 1-Gorenstein in short. As for such rings, Jans [5] gave a characerization. T. Sumioka posed a problem whether an artinian QF-3, 1-Gorenstein ring has a QF (Quasi-Frobenius) maximal quotient ring or not, at Azumaya Symposium held at Tokyo University of Education, September 1-2, 1975. (See his paper [15, Theorem 1].) The problem is unsettled even now. So we convert its artinian condition, the definition of QF-3 ring, into a noetherian one defined below. A ring R is said to be left QF-3 if every finitely generated submodule of $E({}_R R)$ is torsionless.

Remark 1. If R is a noetherian ring, then it is left QF-3 if and only if it is left QF-3 in the sense of Morita [8], that is, $E({}_R R)$ is flat. So R is right QF-3 in this case (see [11, Theorem 1.1]). On the other hand, if R is left and right perfect, then it is left QF-3 if and only if it has a minimal faithful left module. The latter result was obtained by T. Sumioka, who has shown moreover that a right 1-Gorenstein ring is QF-3 if and only if its maximal right quotient ring is its left quotient ring [17].

So we have

Generalized Sumioka's Problem. Has any QF-3, 1-Gorenstein ring a QF maximal (two-sided) quotient ring ?

We showed in [11],

Theorem. Let R be a 1-Gorenstein ring and Q its left maximal quotient ring. Then the following statements are equivalent.

(1) The canonical inclusion $R \to Q$ is a two-sided flat epimorphism.

(2) R is either left or right QF-3 and Q is a QF ring.

When the above statements hold, R is (left and right) QF-3 and Q is at the same time a right maximal quotient ring of R.

However it is difficult to verify whether a 1-Gorenstein ring satisfies the criterion (1) or not. So it is of much interest to find a sufficient condition for a QF-3, 1-Gorenstein ring to have a QF (two-sided) quotient ring.

On the other hand, the following theorem holds.

Theorem ([4], [11]) Let R be a 1-Gorenstein ring and E the injective hull of ${}_RR$. Then the following statements hold.

(1) $E \oplus E/R$ is an injective cogenerator.

(2) E is a cogenerator if and only if R is its own maximal left quotient ring if and only if R is a QF ring.

(3) E/R is a cogenerator if and only if Soc R = 0.

Remark 2. A (left and right) noetherian ring R has the largest artinian left ideal which is at the same time the largest artinian right ideal. We denote it by A(R) or by A, and call it the artinian radical of R. So for a noetherian ring R, $\mathrm{Soc}({}_RR) = 0$ if and only if $\mathrm{Soc}(R_R) = 0$. (See [6] and [12].)

In view of the above theorem, it is of much interest to study maximal quotient rings of QF-3, 1-Gorenstein rings with zero socle. Noetherian rings with zero socle are, so to say, purely noetherian (see [2]).

Now we can state our first theorem as follows.

Therem A. Any QF-3 right 1-Gorenstein ring with zero socle is a two-sided order in a QF ring.

But we can deduce the above theorem from a more general theorem. For this purpose we shall give some definitions and auxiliary results. A family of left ideals is said to be a topology if it is a Gabriel topology in the sense of Stenström's book [14]. Thus a perfect topology in this note is corresponding to a perfect Gabriel topology in [14]. As for Krull dimension of modules, refer [3].

Let $\underline{G}$ be a left topology on a ring R. Then we have the notion of $\underline{G}$-dimension of R-modules and elementary properties for it, Proposition 1, Corollary 2 and Corollary 3. (All of them are in the author's paper [13].) A chain of submodules of a left R-module M ;

$$M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_r$$

is called a $\underline{G}$-chain if each M_{i-1}/M_i is not a $\underline{G}$-torsion module. A $\underline{G}$-chain is said to be maximal if it has no proper refinement of $\underline{G}$-chain.

Proposition 1. If ${}_RM$ has a finite maximal $\underline{G}$-chain of length r, then any $\underline{G}$-chain of M has a finite length s and $s \leq r$.

Hence we can give a definition of $\underline{G}$-dimension of M which is denoted by $\underline{G}$-dim M, as follows. Define $\underline{G}$-dim M = r if M has a finite maximal $\underline{G}$-chain of length r, and define $\underline{G}$-dim M = ∞ otherwise.

Corollary 2. For any exact sequence of R-modules ;

$0 \to M' \to M \to M'' \to 0$,

we have $\underline{G}$-dim M = $\underline{G}$-dim M' + $\underline{G}$-dim M' .

Corollary 3. Let $\underline{G} \subseteq \underline{G}'$ be topologies on R, and M a left R-module. Then $\underline{G}$-dim M $\geq$ $\underline{G}'$-dim M .

In the remainder of this note, we assume that R is a left noetherian ring with left Krull dimension α.

Proposition 4. ([12, Theorem 3.1]) For any $\beta < \alpha$, the family $\underline{F}_\beta = \{ {}_R I \subseteq R \mid \text{K-dim } R/I \leq \beta \}$ is a topology.

Let $\underline{D}$ be the topology of dense left ideals of R. Our interest lies in the connection between $\underline{F}_\beta$ and $\underline{D}$. We say that a ring R satisfies the β-restricted minimum condition for left ideals if $\underline{D} \subseteq \underline{F}_\beta$. If $\beta = 0$, we omit the letter β.

Remark 3. If R is non-singular, our notion of restricted minimum condition coincies with Chatters' one in [1]. We should remark that Chatters deals with this notion only in the theory of non-singular rings.

Denote by $\text{rad}^\beta({}_R R)$ the largest left ideal of R whose Krull dimension is not more than β. Then we have

Proposition 5. ([12, Proposition 3.2]) For any $\beta < \alpha$, $\text{rad}^\beta({}_R R) = 0$ if and only if $\underline{D} \supseteq \underline{F}_\beta$.

By [16, Proposition 1], a QF-3 noetherian ring has finite left $\underline{D}$-dimension. Any right 1-Gorenstein ring satisfies the restricted minimum condition for left ideals (see [10, Theorem 2.4] and [12, Theorem 5.2]). Thus we have

Proposition 6. ([12, Theorem 5.3]) Any QF-3 1-Gorenstein ring has left Krull dimension at most one.

So we generalize Theorem A as follows.

Theorem B. ([13, Theorem 9]) Let R be a QF-3 noetherian ring satisfying the restricted minimum condition for left ideals. Denote its artinian radical by A. Then R/A has a QF classical two-sided quotient ring which is isomorphic to $R_{\underline{F}}$, the quotient ring with respect to the topology $\underline{F}$, where $\underline{F} = \underline{F}_0$.

In order to prove the above theorem, we shall apply the following theorem of Lenagan.

Lenagan's Theorem ([6, Theorem 3.6], [7, Theorem 3.2]) Let R be a noetherian ring with left Krull dimension one, and A its artinian radical. Let $\overline{R}$ = R/A and denote by $\overline{x}$ the canonical image of x ε R. Let S = { s ε R | $\overline{s}$ is a regular element in $\overline{R}$ } and $\Sigma(S)$ = { Rs | s ε S }. Then (1) $\Sigma(S)$ is cofinal in $\underline{F}$ (= $\underline{F}_0$) and (2) $\overline{R}$ has a classical two-sided quotient ring $Q(\overline{R})$.

Remark 4. In fact, Lenagan has established this theorem by showing that R satisfies the regularity condition of Small. Therefore $Q(\overline{R})$ is artinian. But we can show in our case that the existence of $Q(\overline{R})$ implies its artinianness, by using the notion of $\underline{G}$-dimension.

Lemma 7. ([13, Lemma 7, Lemma 8]) $R_{\underline{F}}$ is isomorphic to $Q(\overline{R})$ and $\underline{F}$ is a perfect topology.

From the above lemma, we can show that $R_{\underline{F}}$ satisfies Rutter's condition [9, Corollary 6]. So it is a QF ring.

Finally we have a characerization of noetherian two-sided orders with left Krull dimension one in QF rings, from Therem B and [2, Theorem 10].

Theorem C. ([13, Theorem 10]) For a noetherian ring R, the following statements are eqivalent.

(1) R is a two-sided order in a QF ring and K-dim $_R R = 1$.

(2) R is a ring direct sum, say $R = A \oplus B$, where A is a QF ring and B is a QF-3 ring with zero socle satisfying the restricted minimum condition for left ideals.

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BICOMMUTATORS OF LOCALLY PROJECTIVE MODULES

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The bicommutator of a (finitely generated) projective module was studied by many authors [1][2][5][7]. In this paper, we extend some results to the bicommutator of a locally projective module. Here, following Zimmermann[10], a module U is called locally projective, if, for all diagrams;

$$\begin{array}{ccccc} 0 & \longrightarrow & V & \longrightarrow & U & & \\ & & & & \big\downarrow g & & \\ & & X & \xrightarrow{f} & Y & \longrightarrow & 0 \end{array}$$

with exact rows and V a finitely generated modules, there exists $g' \in \mathrm{Hom}(U,X)$ such that $g|_V = fg'|_V$. She characterized locally projective modules by many ways. Locally projective modules are flat by [10]. We shall collect the facts which are necessary for this paper from [10].

Theorem 1. The following conditions are equivalent for a module U_R.

(1) U is locally projective.

(2) For each element $u \in U$, there exist $x_1, \ldots, x_n \in U$ and

$f_1,\ldots,f_n \in U^*=Hom(U,R)$ such that $m=\Sigma_j[x_j,f_j]m$, where $[x,f]\in S=End(U)$ is defined by $[x,f]y=x(fy)$ for $x,y\in U$ and $f\in U^*$.

(3) For every S-submodule V of U, there exists a left ideal A of R such that TA=A and V=UA, where T is a trace ideal of U_R.

(4) U=UT and $(S/\Delta)_S$ is flat, where $\Delta=\{\Sigma[x_i,f_i]\ ;\ x_i\in U, f_i\in U^*\}$ is an ideal of S.

Proof. This follows from [10, Theorem 2.1 and 3.1].

Now, we state some notation and definitions. Throughout this paper, let U be a locally projective right R-module over a ring R with 1, $S=End(U_R)$ an endomorphism ring of U, $C=End({}_SU)$ a bicommutator of U, T a trace ideal of U, $\mathcal{T}=\{{}_RX\ ;\ TX=0\}$, $\mathcal{F}=\{{}_RY\ ;\ Ty=0,\ y\in Y \Longrightarrow y=0\}$, $\mathcal{f}=\{{}_RI\subset R\ ;\ T\subset I\}$. Then $(\mathcal{T},\mathcal{F})$ forms a hereditary torsion theory over the category of all left R-modules with its filter $\mathcal{f}$ and the quotient ring of R with respect to $(\mathcal{T},\mathcal{F})$ is C[6]. For the torsion theory and related topics, the reader is referred to [8]. Firstly, we study the case when a ring homomorphism $\rho:R \to C$ is a finite left localization.

Theorem 2. The following conditions are equivalent.

(1) $(\mathcal{T},\mathcal{F})$ is a perfect torsion theory.

(2) A ring homomorphism $\rho:R \to C$ is a finite left

localization.

(3) ${}_RT$ is finitely generated and projective relative to the class of epimorphisms $\{X \to Y \to 0; X,Y \in F\}$.

(4) ${}_SU$ is finitely generated projective.

(5) C=CT.

Proof. For the proof, see [6], however it is noted that the equivalence (4)$\Longleftrightarrow$(5) is well-known, since S= End(U_C) and CT is the trace ideal of U_C.

The following lemma is fundamental in this paper and this is also obtained in [4] under the assumption of U to be projective.

Lemma 3. It holds that all simple factor modules of ${}_SU$ can be embedded into ${}_SU$ if and only if CT is a minimal dense left ideal of C.

Proof. We can conclude that $E({}_C\mathrm{Hom}_S(U,X)) \simeq \mathrm{Hom}_S(U, E({}_SX))$ as left C-modules for a left S-module X by Theorem 1, (4) and [9, Lemma 2.3 and Theorem 2.4]. However, we state the direct proof for the later application. Since U_R is flat and $U \otimes_R C \simeq U$, U_C is also flat. By the adjoint relation $\mathrm{Hom}_C(-, \mathrm{Hom}_S(U, E({}_SX))) \simeq \mathrm{Hom}_S(U \otimes_C -, E({}_SX))$, we conclude that $\mathrm{Hom}_S(U, E({}_SX))$ is injective as a left C-module. Take any nonzero $\phi \in \mathrm{Hom}_S(U, E(X))$. Then there exists $u \in U$

such that $0\neq u\phi\in X$. Let $u=\Sigma[x_i,f_i]u$ for $x_i\in U$, $f_i\in U^*$ by Theorem 1. It holds that $0\neq[u_i,f_i](u\phi)=u_i((f_iu)\phi)$ for some i. For any $u'\in U$, $u'((f_iu)\phi)=[u',f_i](u\phi)\in X$ implies $0\neq(f_iu)\phi$ $\in Hom_S(U,X)$, that is, ${}_CHom_S(U,E(X))$ is an essential extension of ${}_CHom_S(U,X)$. Hence $E(Hom_S(U,X))\simeq Hom_S(U,E(X))$. Putting X=U we have $E({}_CC)\simeq Hom_S(U,E(U))$. Now, assume that all simple factor modules of ${}_SU$ are embedded into ${}_SU$. Then $E({}_SU)$ cogenerates all the modules of the form ${}_SU\otimes_CX$ for any left C-module X by a routin computation. Therefore, A is a dense left ideal of $C\Leftrightarrow Hom_C(C/A,Hom_S(U,E(U)))=0\Leftrightarrow Hom_S(U\otimes C/A,E(U))=0\Leftrightarrow U\otimes C/A=0\Leftrightarrow U=UA\Leftrightarrow A\supset CT$ which implies that CT is a minimal dense left ideal of C. Conversely, let U' be a simple factor module of ${}_SU$. Then $U'\simeq U/V$ for an S-submodule V of U, where V=UA for a left ideal A of R with TA=A by Theorem 1, and then $U'\simeq U\otimes_CC/CA$. Thus $Hom_S(U',E({}_SU))\simeq Hom_S(U\otimes C/CA,E(U))\simeq Hom_C(C/CA,Hom_S(U,E(U)))\simeq Hom_C(C/CA,E({}_CC))$. If $Hom_S(U',E(U))=0$, then CA is a dense left ideal of C which implies $CA\supset CT$ by hypothesis. We conclude that $V=UA=UCA\supset UCT=U$ which is a contradiction. Hence $Hom_S(U',E(U))\neq 0$, that is, U' can be embedded into ${}_SU$.

A ring D is called a left S-ring, if ${}_DD$ is lower distinguished. It is well-known that D is a left S-ring if and only if D has no proper dense left ideal [8, Ch. XI, Lemma 5.1].

Combining Theorem 2 with Lemma 3 we get the following.

Theorem 4. C is a left S-ring if and only if all simple factor modules of ${}_SU$ can be embedded into ${}_SU$ and ${}_SU$ is finitely generated projective.

For the injectivity of ${}_CC$, we have the following.

Theorem 5. The following conditions are equivalent.

(1) C is a left self-injective ring.

(2) CT is quasi-injective as a left C-module.

(3) U is a quasi-injective left S-module.

Proof. (1)$\Longrightarrow$(2); Since CT is an idempotent ideal, we have $\mathrm{Hom}_C(CT,C)=\mathrm{Hom}_C(CT,CT)$. Hence (1)$\Longrightarrow$(2) holds. (2)$\Longrightarrow$(3); Let $d:{}_SUA \to {}_SU$ where A is a left ideal of R such that TA=A. Consider the following diagram:

$$\begin{array}{ccc} U^*\otimes_S UA & \xrightarrow{1\otimes d} & U^*\otimes_S U \\ \downarrow & & \downarrow \\ CA & \xrightarrow{\bar{d}} & CT, \end{array}$$

where $U^*=\mathrm{Hom}_C(U,C)$, the vertical maps are canonical ones, and $\bar{d}$ is defined by $(\phi u)\bar{d}=\phi(ud)$ for $\phi\otimes u\in U^*\otimes UA$. We can extend $\bar{d}$ to $c:{}_CCT \to {}_CCT$ by hypothesis, that is, c satisfies $(\phi u)c=\phi(ud)$ for any $\phi\otimes u\in U^*\otimes UA$. Tensoring with U_C we get $1\otimes c:U\otimes_C CT \to U\otimes_C CT$, however the flatness of U_C implies ${}_SU\otimes CT$

$\simeq {}_SU$, and then we get a homomorphism $c' : {}_SU \to {}_SU$. For any $u(fu') \in UA = UTA (u \in U, f \in \mathrm{Hom}_R(U,R), u' \in UA)$, we have $(u(fu'))c' = u((fu')c) = u(f(u'd)) = [u,f]u'd = ([u,f]u')d = (u(fu'))d$. Hence c' is an extension of d. (3)$\Longrightarrow$(1) follows from [11].

Corollary 6. C is a left injective cogenerator ring if and only if all simple factor modules of ${}_SU$ can be embedded into ${}_SU$ and ${}_SU$ is quasi-injective finitely generated projective.

Proof. This follows from Theorem 4 and 5.

Suppose that U is faithful locally projective. Then U is torsionless by Theorem 1 and we have $E({}_RC) = E({}_RR)$ by [3, Theorem 6]. Thus A is a dense left ideal of R if and only if $\mathrm{Hom}_R(R/A, \mathrm{Hom}_S(U, E(U))) = 0$ by the proof of Lemma 2. Therefore, we conclude that all simple factor modules of ${}_SU$ can be embedded into ${}_SU$ if and only if T is a minimal dense left ideal by the same way as Lemma 2. Moreover, in this case, the torsion theory $(\mathcal{T}, \mathcal{F})$ is the Lambek torsion theory, and then C is the maximal left quotient ring of R. Then we get the followings.

Corollary 7. The maximal left quotient ring of R is a left S-ring and a finite left localization of R if and only if there exists a faithful locally projective module U_R such that ${}_SU$ is finitely generated projective and all simple factor modules of ${}_SU$ can be embedded into ${}_SU$, where $S=End(U_R)$. Moreover, the bicommutator of U_R equals the maximal left quotient ring of R.

Proof. This follows from Theorem 2, 4 and [7, Theorem 2.3].

Corollary 8. The maximal left quotient ring of R is a left injective cogenerator ring and a finite left localization of R if and only if there exists a faithful locally projective module U_R such that ${}_SU$ is finitely generated projective quasi-injective and all simple factor modules of ${}_SU$ can be embedded into ${}_SU$. Moreover, the bicommutator of U_R equals the maximal left quotient ring of R.

Proof. This follows from Theorem 2, 5, Corollary 6, and [7, Theorem 2.3 and Corollary 2.8].

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REMARKS ON ADJOINTS AND TORSION THEORIES

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J. Lambek and B. A. Rattray viewed localization functors as equalizers of suitable norphisms. In fact, they showes [2] that the ordinary localization functor with respect to a hereditary torsion theory can be seen as a kind of their functor. Such a view point is recobnized in K. Morita [4] or [5]. Since Lambek-Rattray's localization is categorical, it yields automatically the concept of colocalization as it dual. On the other hand, torsion theoretical colocalization is treated in K. Ohtake [6], and he proved that the corresponding torsion theory is co-hereditary, i.e. a torsion theory whose torsion-free class is closed under taking factor modules. And, also in this case, the torsion theoretical colocalization is one of Lambek-Rattray's colocalizations. So it is natural to search for the case when Lambek-Rattray's localization or colocalization coincides with the torsion theoretical one, and this is the main purpose in our present paper.

Let R, S be rings and ${}_RU_S$ a unitary bimodule.

Then we can consider the following functors and canonical natural transformations.

$$\text{R-Mod} \underset{H=\mathrm{Hom}_R(U,-)}{\overset{T=(U\otimes_S -)}{\rightleftarrows}} \text{S-Mod} ,$$

$$\text{R-Mod} \underset{D=\mathrm{Hom}_R(-,U)}{\overset{D=\mathrm{Hom}_S(-,U)}{\rightleftarrows}} \text{Mod-S} , \text{ and}$$

$TH \overset{\varepsilon}{\to} 1_{R\text{-Mod}}$, $1_{S\text{-Mod}} \overset{\eta}{\to} HT$ and $1_{R\text{-Mod}} \overset{\pi}{\to} D^2$. And let us denote by $TH \overset{p}{\to} C$, $L \overset{t}{\to} HT$ and $\Gamma \overset{w}{\to} D^2$, the cokernel of $(TH)^2 \xrightarrow{\varepsilon TH - TH\varepsilon} TH$, the kernel of $HT \xrightarrow{\eta HT - HT\eta} (HT)^2$ and the kernel of $D^2 \xrightarrow{\pi D^2 - D^2 \pi} D^4$. Then ε,η and π factor through p,t and w respectively:

$$TH \overset{\varepsilon}{\to} 1 = TH \overset{p}{\to} C \overset{\varepsilon}{\to} 1 ,$$

$$1 \overset{\eta}{\to} HT = 1 \overset{\ell}{\to} L \overset{t}{\to} HT , \text{ and}$$

$$1 \overset{\pi}{\to} D^2 = 1 \overset{\gamma}{\to} \Gamma \overset{w}{\to} D^2 .$$

In this paper, we shall consider the condition for U under which $C \overset{\varepsilon}{\to} 1$, $1 \overset{\ell}{\to} L$ and $1 \overset{\gamma}{\to} \Gamma$ are the torsion theoretical colocalization functor or localization functor.

We begin with the consideration of the torsion theories determined by ε,η and π .

A preradical is a subfunctor of an identity functor.

As its dual, we call a factor functor of an identity functor a precoradical. It is obvious that any preradical $t \rightarrowtail 1$ determines a precoradical $1 \twoheadrightarrow t^{\circ}$, and conversely, any precoradical $1 \twoheadrightarrow s$ determines a preradical $s^{\circ} \rightarrowtail 1$. For each pair of preradicals t_1, t_2 , a preradical $(t_1:t_2)$ is defined as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & t_1 & \longrightarrow & (t_1:t_2) & \longrightarrow & t_2t_1^{\circ} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & t_1 & \longrightarrow & 1 & \longrightarrow & t_1^{\circ} & \longrightarrow & 0 \end{array} ,$$

and a preradical t is a radical if $(t:t)=t$. Dually, a precoradical $(s_1:s_2)$ is defined for a pair of precoradicals s_1, s_2:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & s_1^{\circ} & \longrightarrow & 1 & \longrightarrow & s_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & s_2s_1^{\circ} & \longrightarrow & (s_1:s_2) & \longrightarrow & s_1 & \longrightarrow & 0 \end{array} .$$

And we call a preradical s a coradical if $(s:s)=s$. For a preradical t and a precoradical s, we associate the classes of modules as follows:

$\underline{T}(t)=\{ X \mid t(X)=X \}$,

$\underline{F}(t)=\{ X \mid t(X)=0 \}$,

$\underline{T}(s)=\{ X \mid s(X)=0 \}$, and

$\underline{F}(s)=\{ X \mid s(X)=X \}$.

<u>Lemma 1.</u> For a preradical t and a precoradical s, the following hold.

(a) $(\underline{T}(t),\underline{F}(t))$ is a torsion theory if and only if t

is an idempotent radical.

(b) $(\underline{T}(t),\underline{F}(t))$ is a hereditary torsion theory if and only if t is a left exact radical.

(c) $(\underline{T}(s),\underline{F}(s))$ is a torsion theory if and only if s is an idempotent coradical.

(d) $(\underline{T}(s),\underline{F}(s))$ is a cohereditary torsion theory if and only if s is a right exact coradical.

Lemma 2. The following hold.

(a) $\mathrm{Ker}\pi$ is a radical and $\underline{T}(\mathrm{ker}\pi)=\mathrm{Ker}(D)$ and $F(\mathrm{Ker}\pi)=\{ X \mid \pi_X$ is a monomorphism $\}$.

(b) $\mathrm{Cok}\varepsilon$ is a coradical and $\underline{T}(\mathrm{Cok}\varepsilon)=\{ X \mid \varepsilon_X$ is an epimorphism $\}$ and $\underline{F}(\mathrm{Cok}\varepsilon)=\mathrm{Ker}(H)$.

(c) $\mathrm{Ker}\eta$ is a radical and $\underline{T}(\mathrm{Ker}\eta)=\mathrm{Ker}(T)$ and $\underline{F}(\mathrm{Ker}\eta)=\{ X \mid \eta_X$ is a monomorphism $\}$.

Definition.

(1) ${}_RU$ is pseudo-injective(self-pseudo-injective) if D satisfies the following condition: For any monomorphism $0 \to {}_RA \overset{u}{\to} {}_RB$ (with π_{Coku} is a monomorphism), if $D(u)=0$ then $D(A)=0$.

(2) ${}_RU$ is pseudo-projective(self-pseudo-projective) if H satisfies the following condition: For any epimorphism $_RB \overset{p}{\to} {}_RA \to 0$ (with $\varepsilon_{\mathrm{Cokp}}$ is an epimorphism), if $H(p)=0$

then $H(A)=0$.

(3) U_S is pseudo-flat(self-pseudo-flat) if T satisfies the following condition: For any monomorphism $0 \to {}_SA \overset{u}{\to} {}_SB$ (with η_{Coku} is a monomorphism), if $T(u)=0$ then $T(A)=0$.

Proposition 3. The following are equivalent.

(a) $(\mathrm{Ker}(D),\{X|\ \pi_X$ is mono $\})$ is a torsion theory.

(b) $\mathrm{Ker}\pi$ is an idempotent radical.

(c) ${}_RU$ is self-pseudo-injective.

Proposition 4. The following are equivalent.

(a) $(\{X|\ \varepsilon_X$ is epi $\},\mathrm{Ker}(H))$ is a torsion theory.

(b) $\mathrm{Coke}\varepsilon$ is an idempotent coradical.

(c) ${}_RU$ is self-pseudo-projective.

Proposition 5. The following are equivalent.

(a) $(\mathrm{Ker}(T),\{X|\ \eta_X$ is mono $\})$ is a torsion theory.

(b) $\mathrm{Ker}\eta$ is an idempotent radical.

(c) U_S is self-pseudo-flat.

Theorem 6. The following are equivalent.

(a) $(\mathrm{Ker}(D),\{X|\ \pi_X$ is mono $\})$ is a hereditary torsion theory.

(b) $\mathrm{Ker}\,\pi$ is a left exact radical.

(c) For any diagram $0 \to {}_RA \overset{u}{\to} {}_RB$,

$$\begin{array}{ccc} 0 \to & {}_RA & \overset{u}{\to} {}_RB \\ & f \downarrow & \\ & {}_RU & \end{array}$$

it holds that $\mathrm{Ker} f \supseteq A \cap \bigcap_{h \in D(B)} \mathrm{Ker} h$.

(d) For any monomorphism $0 \to {}_RU \overset{u}{\to} {}_RE$, there exists a morphism ${}_RE \overset{\gamma}{\to} \Pi_R U$ such that the composition ${}_RU \overset{u}{\to} {}_RE \overset{\gamma}{\to} \Pi_R U$ is a monomorphism.

(e) For any diagram

$$\begin{array}{ccc} 0 \to & {}_RA & \overset{u}{\to} {}_RB \\ & 0 \neq f \downarrow & \\ & {}_RU & \end{array}$$

there exist ${}_RU \overset{\alpha}{\to} {}_RU$ and ${}_RB \overset{\beta}{\to} {}_RU$ such that $\alpha \cdot f = \beta \cdot u \neq 0$.

(f) ${}_RU$ is pseudo-injective.

(g) $E({}_RU) \hookrightarrow \Pi_R U$, where $E({}_RU)$ is the injective envelope of ${}_RU$.

Remark. This theorem is essentially proved in G. M. Tsukerman [8].

<u>Theorem 7.</u> The following are equivalent.

(a) $(\{X \mid \varepsilon_X \text{ is epi}\}, \mathrm{Ker}(H))$ is a cohereditary torsion theory.

(b) $\mathrm{Coke}\varepsilon$ is a right exact coradical.

(c) For any diagram

$$\begin{array}{ccc} & {}_RU & \\ & \downarrow f & \\ {}_RB \underset{p}{\to} & {}_RA & \to 0 \end{array},$$

it holds that $\mathrm{Im} f \subseteq \sum_{g \in H(B)} \mathrm{Im}(p \cdot g)$.

(d) For any epimorphism ${}_RP \xrightarrow{p} {}_RU \to 0$, there exists a morphism $\oplus_R U \xrightarrow{\gamma} {}_RP$ such that the composition $\oplus_R U \xrightarrow{\gamma} {}_RP \xrightarrow{p} {}_RU$ is an epimorphism.

(e) For any diagram

$$\begin{array}{ccc} & & {}_RU \\ & & \downarrow f \neq 0 \\ {}_RB & \xrightarrow[p]{} & {}_RA \to 0 \end{array}$$

, there exist ${}_RU \xrightarrow{\alpha} {}_RU$ and ${}_RU \xrightarrow{\beta} {}_RA$ such that $f \cdot \alpha = p \cdot \beta \neq 0$.

(f) ${}_RU$ is pseudo-projective.

(g) $I \cdot U = U$, where $I = \sum_{f \in H(R)} \mathrm{Im} f$ is the trace ideal.

<u>Theorem 8.</u> The following are equivalent.

(a) $(\mathrm{Ker}(T), \{X \mid \eta_X \text{ is mono}\})$ is a hereditary torsion theory.

(b) $\mathrm{Ker}\,\eta$ is a left exact radical.

(c) U_S is pseudo-flat.

(d) ${}_S\mathrm{Hom}_R({}_RU_S, {}_RW)$ is pseudo-injective for an injective cogenerator ${}_RW$.

(e) For any left ideal ${}_SJ \subseteq {}_SS$, if $U \otimes (S/J)$ is zero in $U \otimes_S E({}_SS/J)$ then $0 = U \otimes_S (S/J)$, where $E({}_SS/J)$ is the injective envelope of ${}_SS/J$.

For a torsion theory $(\underline{T}, \underline{F})$ in a category of left modules over some ring, a module X in this category is said to be $\underline{T}$-injective if $\mathrm{Hom}(-,X)$ is exact on all exact sequences $0 \to Y' \to Y \to Y'' \to 0$ with $Y'' \in \underline{T}$, and the F-projectivity is defined as its dual. Similarly, a right module Z over the same ring is said to be $\underline{T}$-flat if $(Z \otimes -)$ is exact on all exact sequences $0 \to Y' \to Y \to Y'' \to 0$ with $Y'' \in \underline{T}$.

Theorem 9.

(a) If ${}_RU$ is self-pseudo-injective and Ker(D)-injective, then ${}_RX \xrightarrow{\gamma_X} {}_R\Gamma(X)$ is the localization of any R-module ${}_RX$, with respect to $(\mathrm{Ker}(D),\{X|\pi_X \text{ is mono}\})$.

(b) If ${}_RU$ is self-pseudo-projective and Ker(H)-projective, then ${}_RC(X) \xrightarrow{c_X} {}_RX$ is the colocalization of any R-module ${}_RX$, with respect to $(\{X|\varepsilon_X \text{ is epi}\},\mathrm{Ker}(H))$.

(c) If U_S is self-pseudo-flat and Ker(T)-flat, then ${}_SX \xrightarrow{\ell_X} {}_SL(X)$ is the localization of any S-module ${}_SX$, with respect to $(\mathrm{Ker}(T),\{X|\eta_X \text{ is mono}\})$.

K.Ohtake [6] proved that a torsion theory $(\underline{T},\underline{F})$ is cohereditary (resp. hereditary) if and only if each module has its $(\underline{T},\underline{F})$-colocalization (resp. $(\underline{T},\underline{F})$-localization). Thus, in the above theorem, the terms "self-pseudo-injective","self-psseudo-projective" and "self-pseudo-flat" may be replaced by "pseudo-injective","pseudo-projective" and "pseudo-flat".

Next we consider the conditions under which $\Gamma(X)= D^2(X)$, $C(X)=TH(X)$ and $L(X)=HT(X)$. From the definitions of γ, c and ℓ, it is obvious that these are repectively equivalent to $\pi D^2{}_X=D^2\pi_X$, $\varepsilon TH_X=TH\varepsilon_X$ and $\eta HT_X=HT\eta_X$.

<u>Proposition 10.</u> The following are equivalent.

(a) $\pi D^2_X=D^2\pi_X$.

(b) πD_X is an isomorphism.

(c) πD^2_X is an isomorphism.

<u>Proposition 11.</u> The following are equivalent.

(a) $\varepsilon TH_X=TH\varepsilon_X$.

(b) ηH_X is an isomorphism.

(c) εTH_X is an isomorphism.

<u>Proposition 12.</u> The following are equivalent.

(a) $\eta HT_X=HT\eta_X$.

(b) εT_X is an isomorphism.

(c) ηHT_X is an isomorphism.

Proposition 11 and 12 show that if $C(X)=TH(X)$ and $L(X)=HT(X)$ hold for any R- or S-module X, then T and H induce an equivalence between Im(T) and Im(H). So in order to investigate the conditions under which C=TH and

L=HT, we have to study the condition $Im(T) \simeq Im(H)$.

Let $End({}_RU)=\overline{S}$, $S \overset{\phi}{\to} \overline{S}$ be the canonical ring morphism, and $\overline{T}=(U\otimes_{\overline{S}}-)$: $\overline{S}$-Mod $\to$ R-Mod, $\overline{H}=Hom_R(U,-)$: R-Mod $\to$ $\overline{S}$-Mod, $\overline{T}\overline{H} \overset{\xi}{\to} 1_{R\text{-Mod}}$, $1_{\overline{S}} \overset{\eta}{\to} \overline{H}\overline{T}$.

<u>Lemma 13.</u> The following statements (a) and (b) are equivalent.

(a) T and H induce an equivalence $Im(T) \simeq Im(H)$.

(b) (i) $\overline{T}$ and $\overline{H}$ induce an equivalence $Im(\overline{T}) \simeq Im(\overline{H})$,

and(ii) ${}_RU\otimes_S\overline{S}_{\overline{S}} \simeq {}_RU_{\overline{S}}$.

Moreover, if (a) and (b) are satisfied, then the following (c) and (d) also hold.

(c) $Im(T)=Im(\overline{T})$.

(d) $({}_S\overline{S}\otimes_{\overline{S}}-)$ and $Hom_S({}_S\overline{S}_{\overline{S}},-)$ induce an equivalence $Im(H) \simeq Im(\overline{H})$.

<u>Definition.</u>

${}_RU$ is pseudo-small if for any morphism ${}_RU \overset{f}{\to} \underset{\alpha}{\oplus}\, {}_RU$, there is an epimorphism $\underset{\beta}{\oplus}\, {}_RU \overset{p}{\to} {}_RU \to 0$ such that $Im(f\cdot p\cdot i_\beta)$ is contained in a finite direct sum $\underset{\alpha_i}{\oplus}\, {}_RU \subseteq \underset{\alpha}{\oplus}\, {}_RU$ for each β-th injection ${}_RU \overset{i_\beta}{\to} \underset{\beta}{\oplus}\, {}_RU$ (we call such a morphism to be finitary).

Theorem 14. The following are equivalent if ${}_RU$ is pseudo-small.

(a) $\overline{T}$ and $\overline{H}$ induce an equivalence $Im(\overline{T}) \simeq Im(\overline{H})$.

(b) For any ${}_RU$-presented R-module ${}_RX$, there exists an epimorphism $\oplus {}_RU \xrightarrow{p} {}_RX \to 0$ such that $\overline{H}(p)$ is an epimorphism and ${}_R\mathrm{Ker}p$ is ${}_RU$-presented.

(c) For any ${}_RU$-presented R-module ${}_RX$, ${}_R\mathrm{Ker}\psi_X$ is also ${}_RU$-presented, where $\underset{\overline{H}(X)}{\oplus} {}_RU \xrightarrow{\psi_X} {}_RX$ is the canonical (epi) morphism.

The following proposition gives the examples of pseudo-small modules.

Proposition 15.

(a) A self-generator is pseudo-small.

(b) A pseudo-projective module is pseudo-small.

(c) If $U_{\overline{S}}$ is pseudo-flat and $\overline{T}$ and $\overline{H}$ induce an equivalence $Im(\overline{T}) \simeq Im(\overline{H})$, then ${}_RU$ is pseudo-small.

Proposition 16. If ${}_RU$ is pseudo-projective, then $Im(\overline{T})$ and $Im(\overline{H})$ are equivalent by $\overline{T}$ and $\overline{H}$.

Combine Theorem 9 with Lemma 13 and Proposition 16, then we have the following.

Proposition 17.

$TH \overset{\varepsilon}{\to} 1_{R\text{-Mod}}$ is the colocalization functor with respect to some cohereditary torsion theory if and only if (i) ${}_RU\otimes_S\bar{S}_{\bar{S}} \simeq {}_RU_{\bar{S}}$ (this is equivalent to $Im(T) \simeq Im(H)$ under the assumption (ii)) and (ii) ${}_RU$ is pseudo-projective and Ker(H)-projective.

Proposition 18.

$1_{S\text{-Mod}} \overset{\eta}{\to} HT$ is the localization functor with respect to some hereditary torsion theory if and only if (i) $Im(T) \simeq Im(H)$ and (ii) U_S is pseudo-flat and Ker(T)-flat.

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THE THIRD SEMINAR ON RINGS

At the time when Prof. F. Kasch was visiting Japan the third seminar was held at Tokyo University of Education.

This note consists of abstracts of lectures presented at this seminar, September 28 - 30, 1970.

On Categories of Indecomposable Modules

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It is natural to consider that the idea of categories is a generalization of idea of modules. From this point of view, there are many results which are generalizations of well known results in the case of modules. However, unfortunately, the author does not know any method to obtain some results in modules which is not contained in the ordinal one in modules, but contained in ideas of categories.

In this lecture, the author would like to show one method to obtain results in modules via method of categories.

Let R be a ring with identity. An R-module M is called completely indecomposable if $End_R(M)$ is a (non commutative) local ring.

We define an additive category A as follows:

The objects in A consist of all directsums of completely indecomposable modules.

The morphisms in A consist of all R-homomorphisms of objects in A.

We can define ideals in an additive category and its factor category.

We shall denote I a subfamily of morphisms in A such that for any object $M = \sum + M_\alpha$, $N = \sum + N_\beta$ in A; M_α, N_β are completely indecomposable, $I \cap [M, N]_R = \{f \mid \epsilon[M, N], p_\beta f i_\alpha : M_\alpha \longrightarrow N_\beta$ is not isomorphic for every α, β where i_α and p_β are injection and projection, respectively$\}$.

We can easily show from the assumption that I is an ideal in A. Then we have

Theorem. Let A and I be as above. Then A/I is a completely reducible C_3-abelian category.

As corollaries of that theorem, we can prove easily Krull-Remak-Schmidt-Azumaya's theorem. Furthermore, we can generalize above theorem (see [1]).

If we restrict ourselves to cases of projective or injective modules, we can find easily proof for many well known theorems (see [1]).

[1] M. Harada and Y. Sai, On categories of indecomposable modules I, II, to appear in Osaka J. Math.

Retracts and Corestracts of Categories of Modules

By Yutaka Kawada, Kyoto Technical University

Let A and B be rings with identity, ${}_A M$ amd ${}_B M$ categories of all (unitary) left A-modules and all left B-modules respectively. Suppose that there exist covariant functors $S: M_A \longrightarrow {}_B M$ and $T: {}_B M \longrightarrow {}_A M$ such that T is a right adjoint of S. If $ST \approx 1$ (resp. $1 \approx TS$), then ${}_B M$ will be called a retract (resp. coretract) of ${}_A M$. Recently retracts of ${}_A M$ have been characterized by K. Morita [2]. The purpose of the present paper is to discuss the case of coretracts.

At first let U be a left B-module. For a left B-module Y, we shall say that U-dimension of Y is $\geq n$ if there is an exact sequence

$$Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y \longrightarrow 0$$

such that each Y_i ($i = 1, \ldots, n$) is a direct sum of copies of U. Next a right A-module U will be called a weak generator in $\mathcal{M}_A$ if it holds that $X = 0$ whenever $U \otimes_A X = 0$. Then the main theorem is stated as follows: ${}_B\mathcal{M}$ is a coretract of ${}_A\mathcal{M}$ if and only if there exists a B-A-module U such that U is a weak generator in $\mathcal{M}_A$, $A = \mathrm{End}({}_BU)$ and that ${}_A\mathcal{M} \approx \mathcal{P}({}_BU)$, where $\mathcal{P}({}_BU)$ denotes the full subcategory of ${}_B\mathcal{M}$ consisting of all left B-modules of U-dimension ≥ 2, and $\mathrm{End}({}_BU)$ denotes the opposite ring of endomorphism ring of ${}_BU$.

As for the Grothendieck category $\mathcal{P}({}_BU)$ obtained in the main theorem, we get further (1) $B \in \mathcal{P}({}_BU)$ if and only if ${}_BU$ is a finitely generated, projective generator, and (2) $\mathcal{P}({}_BU)$ is an exact subcategory of ${}_B\mathcal{M}$ in the sense of P. Freyd [1] if and only if U-dim $Y = \infty$ for every $Y \in \mathcal{P}({}_BU)$.

Finally it is shown that if there is a left B-module M of type FP (in the sense of [2]) with $A = \mathrm{End}({}_BM)$, then ${}_B\mathcal{M}$ is a coretract of ${}_A\mathcal{M}$, while $\mathcal{M}_A$ is a retract of $\mathcal{M}_B$ as was proved by Morita [2].

[1] P. Freyd, Abelian Categories, Harper and Row, New York, 1964.

[2] K. Morita, Localization in categories of modules, I, Math. Z. 114 (1970), 121-144.

Localizations in Category of Modules

By Kiiti Morita, Tokyo University of Education

Let A be a ring with an identity and ${}_A\mathcal{M}$ the category of all unitary left A-modules. For a finitely cogenerating, injective left A-module V, let $\mathcal{D}(V)$ (resp. $\mathcal{T}(V)$) be the full subcategory of ${}_A\mathcal{M}$ consisting of all $X \in {}_A\mathcal{M}$ such that V-domi.dim. $X \geqq 2$ (resp. $\mathrm{Hom}_A(X,V) = 0$). In a previous paper (Localizations in categories of modules. I, Math. Zeitschr. 114 (1970), 121-144), the author proved that ${}_A\mathcal{M}/\mathcal{T}(V) \cong \mathcal{D}(V)$ and that the quotient ring of A obtained by Gabriel's process of localization with respect to $\mathcal{T}(V)$ coincides with the double centralizer of V. Thus the subcategory $\mathcal{D}(V)$ plays an important role in the theory of localization. Corresponding to a characterization of $\mathcal{T}(V)$ as a localizing subcategory, the author gives a necessary and sufficient condition for a full subcategory of ${}_A\mathcal{M}$ to be expressed as $\mathcal{D}(V)$ with a suitable V. As an application of this characterization of $\mathcal{D}(V)$ he proved that for any ring A there exists a maximal left flat bimorphism $A \to M(A)$ in the category of rings; for the case of A being commutative this result has been obtained recently by D. Lazard.

On Cofinitely Generated Modules

By Takeshi Onodera, Hokkaido University

A left R-module ${}_RM$ is called "cofinitely generated", when for every non-empty set $\{N_\alpha\}_{\alpha\in\Lambda}$ of submodules of M such that $\cap_{\alpha\in\Lambda} N_\alpha = 0$ there exist finitely many $\alpha_1, \alpha_2, \dots, \alpha_n$ such that $\cap_{i=1}^{n} N_{\alpha_i} = 0$.

This notion, which is considered as natural dual of a finitely generated module, was first introduced by F. Kasch with the terminology "durchschnittendlicher Modul".

Theorem 1. A left R-module ${}_RM$ is cofinitely generated if and only if the socle of ${}_RM$ is finitely generated and large (= essential) in ${}_RM$.

Let R and S be rings with identies. Regarding to the duality between the category of finitely generated R-modules and that of cofinitely generated S-modules by the use of two-sided cogenerator ${}_RQ_S$ we have the following

Theorem 2. For a two-sided R-S-module ${}_RQ_S$ following conditions are equivalent:

(1) ${}_RQ$, Q_S are cogeneators, $S = \mathrm{End}({}_RQ)$ (= the endomorphism ring of ${}_RQ$) and $R = \mathrm{End}(Q_S)$.

(2) ${}_RQ$ and Q_S are injective cogenerators, $S = \mathrm{End}({}_RQ)$ and $R = \mathrm{End}(Q_S)$.

(3) (i) For every finitely generated left R-module ${}_RM$ $M^*_S = \mathrm{Hom}_R(M,Q)_S$ is cofinitely generated and ${}_RM$ is Q-reflexive, that is, ${}_RM \overset{\text{nat.}}{\cong} {}_R\mathrm{Hom}_S(\mathrm{Hom}_R(M,Q),Q)$.

(ii) For every cofinitely generated left R-module ${}_RM'$, M'^*_S is finitely generated and ${}_RM'$ is Q-reflexive.

(iii) For every finitely generated right S-module N_S, ${}_RN^*$ is cofinitely generated and N_S is Q-reflexive.

(iv) For every cofinitely generated right S-module N'_S, ${}_RN'^*$ is finitely generated and N'_S is Q-reflexive.

Corollary. For a ring R following conditions are equivalent:

(1) R is two-sided cogenerator (= both ${}_RR$ and R_R are cogenerators).

(2) R is two-sided injective cogenerator.

(3) For every cofinitely generated left R-module ${}_RM$, $M_R^* = \mathrm{Hom}_R(M,R)_R$ is finitely generated and ${}_RM$ is (R-) reflexive, and conversely, for every finitely generated right R-module N_R, ${}_RN^*$ is cofinitely generated and N_R is reflexive.

(4) For every finitely generated ${}_RM'$, M'^*_R is cofinitely generated and ${}_RM'$ is reflexive, and conversely, for every cofinitely generated N'_R, RN'^* is finitely generated and N'_R is reflexive.

Artinian Classical Quotient Rings

By Hiroyuki Tachikawa, Tokyo University of Education

Let R be a ring having a regular element r_o and W a right R-module. We shall call that W is r_o-torsion free, if $xr_o = 0$, $x \in W$ implies $x = 0$. If W is r_o-torsion free, injective, then a family $\mathcal{F}$, consisting of right ideals D of R such that $\mathrm{Hom}_R(R/D,W) = 0$, becomes an idempotent topologizing filter and we can define the localizing functor H in the following way: $H(X) = \varinjlim_{D\in\mathcal{F}} \mathrm{Hom}_R(D,X/X_{\mathcal{F}})$, where X is a right R-module and $X_{\mathcal{F}} = \{x \in X \mid \mathrm{Ann}_R\, x \in \mathcal{F}\}$. H(R) has a ring-structure and H(X) becomes a right H(R)-module. With these notations we can prove that the following statements I to III are equivalent:

I. R has a right Artinian right classical quotient ring.

II. There exists a faithful, r_o-torsion free, injective right R-module W such that the following conditions are satisfied:

(a) For every right ideal J of H(R) there is a right ideal I of R such that H(I) = J.

(b) The descending chain condition holds for the right annihilators of subset of W in R.

(c) The prime radical of R coincides with the set consisting of all elements r of R such that r annihilates H(V), where V is a large R-submodule of W.

III. There exists a faithful, r_o-torsion free, injective right R-module W such that the following conditions are satisfied:

(a) The double centralizer Q of W is right Artinian.

(b) W is a cogenerator in the category of right Q-modules.

(c) The prime radical of R coincides with the intersection of R and the radical of Q.

Brauer Groups of Algebraic Function Fields and Their Adèle Rings

By Yutaka Watanabe and Kenji Yokogawa, Osaka University

In the 1968 Symposium at Norikura, G. Azumaya showed that the Brauer group of an adèle ring A_K of an algebraic number field K is isomorphic to the direct sum $\oplus_{\mathfrak{p}} B(K_{\mathfrak{p}})$ of all local Brauer groups. Here, we will consider the Brauer group of an adèle ring of an algebraic function field.

Let F be an algebraic function field of one variable over a perfect field k and $A_F = A_{F/k}$ be its adèle ring. Tensoring over A_F an A_F-central separable algebra Λ with $F_{\mathfrak{p}}$, we get a group homomorphism $\phi_o : B(A_F) \to \prod_{\mathfrak{p}} B(F_{\mathfrak{p}})$ [$\Lambda \to (\dots , \Lambda_{\mathfrak{p}}, \dots)$], where $\mathfrak{p}$ runs over all prime divisors of F over k. But we can prove that $\Lambda_{\mathfrak{p}}$ has an unramified (i.e. separable) maximal order over the valuation ring $\mathcal{O}_{\mathfrak{p}}$ of $\mathfrak{p}$ for almost all $\mathfrak{p}$. Therefore, combining the following exact sequence of Witt and Auslander-Brumer

$$0 \to B(\mathcal{O}_{\mathfrak{P}}) \to B(F_{\mathfrak{P}}) \to \chi(G_{\mathfrak{P}}) \to 0$$
$$B(\mathcal{O}_{\mathfrak{P}}) \cong B(k_{\mathfrak{P}})$$

we obtain a homomorphism ϕ: $B(A_F) \to \oplus_{\mathfrak{P}} \chi(G_{\mathfrak{P}})$. ($\chi(G_{\mathfrak{P}})$ denotes the character group of the decomposition group of $\mathfrak{P}$ and $k_{\mathfrak{P}}$ denotes the residue class field $\mathcal{O}/\mathfrak{P}$.) The homomorphism is indeed an epimorphism.

To compute the kernel of ϕ, we must prove a few fact.

One of these is the injectivity of ϕ_0. The proof of this fact is slightly complicative. By much the same argument we can also prove that the canonical mapping $\alpha : B(\prod K_\iota) \to \prod B(K_\iota)$ [K_ι's are fields and the cardinal number of the index set $I = \{\iota\}$ is utterly arbitrary] is monomorphic. And we get the following commutative diagram:

$$\begin{array}{ccccccc} 0 \to \prod_{\mathfrak{P}} B(k_{\mathfrak{P}}) & \to & \prod_{\mathfrak{P}} B(F_{\mathfrak{P}}) & \to & \prod \chi(G_{\mathfrak{P}}) \to 0 \\ \uparrow \alpha\text{:mono} & & \uparrow \phi_0\text{:mono} & & \uparrow \text{mono (of course!)} \\ B(\prod_{\mathfrak{P}} k_{\mathfrak{P}}) & \to & B(A_F) & \underset{\phi}{\to} & \oplus_{\mathfrak{P}} \chi(G_{\mathfrak{P}}) \to 0 . \end{array}$$

Using this diagram, we can show that the kernel of ϕ is just $B(\prod_{\mathfrak{P}} k_{\mathfrak{P}})$. Accordingly, if the constant field k is finite, ϕ gives an isomorphism $B(A_F) \cong \oplus_{\mathfrak{P}} \chi(G_{\mathfrak{P}}) \cong \oplus_{\mathfrak{P}} B(F_{\mathfrak{P}})$.

Secondly we will mention about the canonical mapping ψ : $B(F) \to B(A_F)$.

A consequence is that the kernel of ψ is isomorphic to $H^1(\overline{CJ})$ [the first Galois cohomology of the ideal class group of $\bar{F} = F\cdot\bar{k}$]. So, if k is a $\mathfrak{p}$-adic number field, ψ is injective by Theorem of Tate. Furthermore, the exact sequence

$$0 \to H^1(\overline{CJ}) \to B(F) \to B(A_F)$$

can be imbedded in the following commutative diagram with exact raws and columns

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & H^1(\overline{CU}) & \to & B(k) & \to & B(\prod_{\xi} k_{\xi}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & H^1(\overline{CJ}) & \to & B(F) & \to & B(A_F) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & H^1(\overline{CD}) & \to & H^2(\bar{H}) & \to & \oplus_{\xi} \chi(G_{\xi}) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & 0 \end{array} .$$

The Center of (*)-Rings

By Friedrich Kasch, München University

A ring R with identity is called a (*)-ring if for every c in the center of R the chain

$$cR \supseteq c^2R \supseteq c^3R \supseteq \ldots$$

is ultimately constant. Then the main theorem states that the center of a (*)-ring is a π-regular ring (in the sense of N. H. McCoy). As special cases we conclude the following results.

1) The center of a semi-primary ring is semi-primary;

2) The center of a perfect ring is perfect;

3) The center of an Artinian ring is perfect and has nilpotent radical.

The last result can not be improved since by an example of U. Oberst there exist Artinian rings for which the center is not Artinian (even not Noetherian).

Note on Endomorphism Ring of Projective Modules

Kazuhiko Hirata, Chiba University

1. Let Γ be a ring with a unit, M a left unitary Γ-module which is finitely generated and projective over Γ. It

is well known that $\mathrm{Hom}_\Gamma(M,\Gamma)\otimes_\Gamma M$ is isomorphic to $\Omega = \mathrm{End}({}_\Gamma M)$ by the map $f\otimes x \to (z \to f(z)x)$ where $f\otimes x \in \mathrm{Hom}_\Gamma(M,\Gamma)\otimes_\Gamma M$ and $z \in M$. Let $\sum_{i=1}^n f_i \otimes x_i$ be the element in $\mathrm{Hom}_\Gamma(M,\Gamma)\otimes_\Gamma M$ which corresponds to the identity 1_M in Ω. Then x_i's generate M over Γ: $x = \sum_{i=1}^n f_i(x)x_i$, $x \in M$, and f_i's generate $\mathrm{Hom}_\Gamma(M,\Gamma)$ over Γ: $f = \sum_{i=1}^n f_i f(x_i)$, $f \in \mathrm{Hom}_\Gamma(M,\Gamma)$. Ω acts on the left side of M by $\omega x = \omega(x)$ and M becomes a left Ω-module, and then $\mathrm{Hom}_\Gamma(M,\Gamma)$ becomes a right Ω-module. With these notations we have

Lemma 1. For any $\omega \in \Omega$ we have $x_i = \sum_{j=1}^n f_i(\omega x)x_j$ and $f_j\omega = \sum_{i=1}^n f_i f_j(\omega x_i)$.

Assume further that M is a Γ-generator. Then there are elements $g_j \in \mathrm{Hom}_\Gamma(M,\Gamma)$ and $y_j \in M$, $j = 1, 2, \ldots, m$, such that $\sum_{j=1}^m g_j(y_j) = 1$.

Lemma 2. For any $\gamma \in \Gamma$, $g_j \otimes \gamma x_i = \sum_{k=1}^m (g_k \otimes x_i) \otimes (g_j \otimes \gamma y_k)$ and $\sum_{j=1}^m (g_j \otimes \gamma y_k)(f_i \otimes y_j) = f_i\gamma \otimes y_k$.

Let Γ' be the subset of Γ consisting of $f_j(\omega x_i)$, $1 \leq i, j \leq n, \omega \in \Omega$, and let Δ be the subring of Ω generated by $g_j \otimes \gamma y_k$, $1 \leq j, k \leq m$, $\gamma \in \Gamma'$ and 1_M. Then we have

Theorem 1. Ω is a separable extension of Δ.

Proof. With above notations the element $\sum_{i,j} (g_j \otimes x_i) \otimes (f_i \otimes y_j)$ in $\Omega \otimes_\Delta \Omega$ provides the separability of Ω over Δ.

2. The endomorphism ring of a finitely generated projective module over a commutative ring C is a Frobenius algebra over C. Even if C is not commutative, the total matrix ring $M_n(C)$ (the endomorphism ring of a free module of rank n) is a Frobenius extension of C. These facts are unified in the following theorem.

Theorem 2. Let Γ be a ring, M a left Γ-module which is finitely generated projective and a generator over Γ. Let Δ

be a subring of $\Omega = \mathrm{End}({}_{\Gamma}M)$. Then Ω is a Frobenius extension of Δ if and only if $\mathrm{Hom}({}_{\Gamma}M, {}_{\Gamma}\Gamma)_{\Gamma,\Delta} \cong \mathrm{Hom}({}_{\Delta}M, {}_{\Delta}\Delta)_{\Gamma,\Delta}$, and ${}_{\Delta}M$ is finitely generated projective.

On Cyclic Extension of Rings

By Kazuo Kishimoto, Shinshu University

Let B be an algebra over $GF(p)$ (p a prime) without central idempotents except 0 and 1, G a cyclic group of order p with a generator .

The purpose of this lecture is to give a necessary and sufficient condition for B to have a G-cyclic extension A without central idempotents except 0 and 1 such that B is a B-direct summand of A. The result depends on [1], [3] and [4], and the details of the proof will appear in forthcomming paper [2].

The theorem can be stated as follows:

Theorem. In order that B have a G-Galois extension A, it is necessary and sufficient that there exist an element b_o in B and a derivation D in B satisfying

1) $D^p - D = (b_o)_r - (b_o)_\ell$ and $D(b_o) = 0$,

2) $X^p - X - b_o$ is indecomposable in $B[X;D]$.

More precisely, if there exist b_o, D satisfying 1) and 2), then $M = (X^p - X - b_o)B[X;D]$ is a two-sided ideal of $B[X;D]$ and $A^* = B[X;D]/M$ is a G^*-Galois extension of B, where G^* is a cyclic group of order p with a generator σ^* defined by $\sigma^*(y) = y + 1$ and y is the residue class of X modulo M. Conversely, if A is a G-cyclic extension of B, then we can find such b_o, D satisfying 1) and 2) that there holds a B-isomorphism $\phi^* : A^* \cong A$ with the following commutative diagram

$$\begin{array}{ccc} A^* & \xrightarrow{\phi^*} & A \\ {\scriptstyle \sigma^*}\downarrow & & \downarrow{\scriptstyle \sigma} \\ A^* & \xrightarrow{\phi^*} & A \end{array}$$

Corollary. Let B be a local ring (domain). If A is a G-Galois extension of B such that A is a local ring (domain), then, for each positive integer e, there exists a local ring (domain) T such that T is an H-Galois extension of B satisfying $T \supseteq A$ and $\tau|B = \sigma$, where H is a cyclic group of order p^e with a generator τ.

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[2] K. Kishimoto, On abelian extension of rings I (to appear in Math. J. Okayama Univ.)

[3] G. J. Janusz, Separable algebras over commutative rings, Trans. Amer. Math. Soc., 122 (1966), 461-479.

[4] Y. Miyashita, Finite outer Galois theory of non-commutative rings, J. Fac. Sci. Hokkaido Univ., 19 (1966), 114-134.

On Galois Theory of Primitive Rings

By Takasi Nagahara, Okayama University

Ln 1952, T. Nakayama presented a Galois theory of simple (artinian) rings and in 1955, A. Rosenberg and D. Zelinsky succeeded in generalizing it to primitive rings with non-zero socles (cf. [1]). They are about the Galois extensions of finite dimension. Since, in 1955-1956, N. Nobusawa and N. Jacobson found a key of the treaty of Galois theory of infinite dimension for division rings, a number of important developments have taken place in this direction for division rings and simple rings (cf. [2]). Recently, we found the fact that Galois theory of infinite dimension for simple rings is lifted to primitive rings with non-zero socle.

Let A be a closed, right primitive ring with a non-zero socle S. A subring T of A will be called regular if T is a right primitive ring with non-zero socle such that A/T has a right height and a right index, and the centralizer of T in A is a simple ring. Moreover, a group H of automorphisms in A will be called regular if the fixring U of H in A is regular and H contains all the inner U(-ring) automorphisms of A. Now, let B be a subring of A, V the centralizer of B in A, and G the group of all B (-ring) automorphisms of A. The extension A/B will be called Galois if B is a regular subring of A and it is the fixring of G in A. Moreover, the notion of right locally finiteness of A/B is defined in some way, and on A and G we may place a topology induced by the finite topology of S^S. Then, our lifting, for example, contains the following theorem:

Let A/B be Galois and right locally finite. Then, G is locally compact if and only if V is finite over the center of A. In this case, there exists a 1-1 dual correspondence between closed regular subgroups of G and closed regular intermediate rings of A/B.

In general, the following theories are lifted to primitive rings with non-zero socle: h-Galois theory, q-Galois theory, and a theory of generating elements of Galois extensions which were considered for simple rings (cf. [2]).

[1] N. Jacobson, Structure of Rings, Providence (1956).

[2] H. Tominaga and T. Nagahara, Galois Theory of Simple Rings, Okayama Math. Lectures (1970).