PROCEEDINGS OF THE 12TH SYMPOSIUM ON RING THEORY

HELD AT HOKKAIDO UNIVERSITY, SAPPORO

AUGUST 31—SEPTEMBER 1, 1979

EDITED BY

MANABU HARADA

WITH THE COOPERATION OF

SHIZUO ENDO TAKASI NAGAHARA

HIROYUKI TACHIKAWA HISAO TOMINAGA

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OKAYAMA, JAPAN

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CONTENTS

PREFACE

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PREFACE

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M. Harada

TRIVIAL EXTENSION OF A COMMUTATIVE RING WITH BALANCED CONDITION

Hideaki SEKIYAMA

Throughout this paper, we assume that all rings are commutative rings with units, and further, all modules will be assumed unitary. For a ring B and a B-module E, the direct sum $B \oplus E$ is turned into a ring by the multiplication composition $(b, e)(b', e') = (bb', b'e + be')$. This ring will be called the trivial extension of B by E. An R-module M is called balanced if the canonical ring homomorphism of R into the double centralizer of M is surjective. Moreover, a ring R is said to be QF-1 if every faithful R-module is balanced. As is easily seen, every generator is balanced, and in case R is a QF ring, every faithful R-module is a generator. Hence, any QF ring is QF-1. Further, a ring R is said to be PF if R is an injective cogenerator as R-module. As is well known, a ring R is PF if and only if every faithful R-module is a generator. This implies that any PF ring is QF-1.

Now, in [1] and [3], it has been proved that any commutative QF-1 artinian ring is QF. Moreover, in [10], [13] and [15], this result has been generalized to commutative QF-1 noetherian (cr perfect) rings. On the other hand, B. L. Osofsky [9] gave an example of a local commutative PF ring without chain conditions, which is the trivial extension of the ring of p-adic integers by the Prüfer group for some prime p. As is well known, a PF ring has a non-zero socle. and the ring of p-adic integers is a valuation ring. In this paper, we shall prove that a trivial extension QF-1 ring of a valuation ring has a non-zero socle, and that it is PF under some conditions.

At first, we shall give a general information on commutative QF-1 rings. Any commutative QF-1 ring R has the principal extension property, that is, every homomorphism of an arbitrary ideal of R into R can be extended to that of R ([2], [12]). Moreover, if a ring R has the principal extension property then every principal ideal of R satisfies the annihilator condition, that is, $\mathrm{Ann}_R(\mathrm{Ann}_R(Rf)) = Rr$ for all r in R ([17]). These facts play an important rôle in our study. Now, in [10], C. M. Ringel proved that if a commutative QF-1 ring is local and has a non-zero socle then it is uniform. Our first study is the following

Theorem 1. A commutative QF-1 ring is local if and only if it is uniform.

Proof. Let R be a commutative QF-1 ring. First, we shall prove that if R is local then it is uniform. We assume that R is local. Moreover, by the result of C. M. Ringel as in the above, we may assume that the socle of R is zero. Then, the maximal ideal W (= Rad(R)) of R is faithful. We now suppose that R is not uniform. Then, there exist non-zero elements x, y in R such that $Rx \cap Ry = \{0\}$. Here we suppose that $\mathrm{Ann}_R(x) + \mathrm{Ann}_R(y) \neq R$. Then W contains $\mathrm{Ann}_R(x) + \mathrm{Ann}_R(y)$. Since $Wx \cap Wy \subset Rx \cap Ry = \{0\}$, the R-module $R/Wx \oplus R/Wy$ is faithful and so balanced. Moreover, since $Rx \neq Wx$ and $Ry \neq Wy$, we can define a non-zero map ϕ of $R/Wx \oplus R/Wy$ into itself as follows $(a, b) \to (xa, yb)$. By using the Camillo's criterion [1, Lemma 11], we see that ϕ is an element of the double centralizer of $R/Wx \oplus R/Wy$. Hence ϕ is the multiplication determined by an element r of R. Then, the element $(1 + Wx, 1 + Wy)$ in $R/Wx \oplus R/Wy$ is mapped on $(r + Wx, r + Wy) = (x + Wx, y + Wy)$. Thus, we have $r - x \in Wx$, $r - y \in Wy$, and so, $r \in Rx \cap Ry = \{0\}$, a contradiction. Therefore, it follows that $\mathrm{Ann}_R(x) + \mathrm{Ann}_R(y) = R$. Since R is local, there holds that either $\mathrm{Ann}_R(x) = R$ or $\mathrm{Ann}_R(y) = R$. This implies that either $x = 0$ or $y = 0$, which is also a contradiction. We proved therefore that R is uniform.

Conversely, we assume that R is uniform. Then, every regular element of R is a unit by the principal extension property. Now, let x be an arbitrary non-unit element of R. Then, for any element r of R, we have $\mathrm{Ann}_R(rx) \cap \mathrm{Ann}_R(1 - rx) = \{0\}$. Since R is uniform, $1 - rx$ is regular, and so, it is a unit element. Hence x is contained in the radical of R. This proves that R is local, completing the proof.

Now, for the purpose of reference, it is convenient to introduce the following

Definition. A module E is called to be uniserial if the lattice of submodules of E is linearly ordered by inclusion, and a ring B is called a valuation ring if B is uniserial as B-module. Moreover, a valuation ring B is said to be maximal if every system of pairwise solvable congruences of the form

$$x \equiv x_\alpha \pmod{I_\alpha} \quad (\alpha \in A,\ x_\alpha \in B,\ I_\alpha \text{ an ideal of } B)$$

has a simultaneous solution in B. We say B is almost maximal if the above congruences have a simultaneous solution whenever $\bigcap_{\alpha \in A} I_\alpha \neq \{0\}$ (cf. C. Faith [4]).

Now, we shall consider the socle condition on a trivial extension of a valuation ring. The following theorem is one of our main results.

Theorem 2. Let R be the trivial extension ring of a valuation ring B by a non-zero B-module E. If R is QF-1, then the following hold.

(a) E is faithful and uniserial.

(b) The socle of R is equal to $(0, \mathrm{Soc}_B(E))$ and is not zero.

Proof. (a) By Theorem 1, we see that R is a commutative local ring with the maximal ideal (Rad(B), E), and it is uniform. Since $(\mathrm{Ann}_B(E), 0)$ and $(0, E)$ are ideals with zero intersection, it follows from the uniformness of R that $\mathrm{Ann}_B(E)$ is zero, and whence E is faithful. For the second assertion, it is sufficient to show that for any two elements x, y of E, it holds that either $Bx \subset By$ or $Bx \supset By$. This is easily seen by using the annihilator condition for two principal ideals $R(0, x) = (0, Bx)$ and $R(0, y) = (0, By)$.

(b) The first assertion will be easily seen. Moreover, in case $\mathrm{Soc}_B(B)$ is not zero, one will easily see that $\mathrm{Soc}_R(R)$ is also not zero. Hence, to see the second assertion, we may assume that $\mathrm{Soc}_B(B)$ is zero. Then, we can see that E is not a cyclic B-module and any proper submodule of E is not faithful. Now, let W be the radical of B. Obviously, W is a faithful ideal of B. We suppose here that $Soc_R(R)$ is zero. Then the radical (W, E) of R is faithful and has a zero socle. Hence $(W, E) \neq Rad(W, E) = (W^2, WE)$ by V. P. Camillo [1, Lemma 2]. Since WE is a faithful B-module, it is equal to E. Thus, W is generated by one element w. Therefore, the radical of R is generated by (w, 0), which leads that R has a non-zero socle by applying the proof of C. M. Ringel [10, Lemma 3], a contradiction.

Next, in order to get more informations on the structures of B and R, we shall consider the case that E is injective.

Corollary 3. Let R be the trivial extension ring of a valuation ring B of an injective non-zero B-module E. If R is QF-1, then the following hold.

(a) E is the miniaml injective cogenerator, so B is an almost maximal valuation ring.

(b) If E is cyclic, then B and R are PF.

(c) If B is not an integral domain, then R is PF.

Proof. (a) The assertion follows from Theorems 1, 2 and the result of C. Faith [4, Theorem 20. 49].

(b) If E is cyclic, E is isomorphic to B, and hence, by (a) and B. J. Müller [8, Theorem 10], R is PF.

(c) We shall show that the endomorphism ring of E is canonically isomorphic to B. This implies that R is injective by R. M. Fossum et al. [6, Corollary 4. 37]. Now let f be any element of the endomorphism ring of E, and $\{e_\alpha\}_{\alpha \in A}$ a set of generators of E. For every e_α, a map of $R(0, e_\alpha)$ into R, as follows $r(0, e_\alpha) \to (0, bfe_\alpha)$ for $r = (b, e) \in R$, is a well-defined R-homomorphism, so there exists an element b_α of B such that $fe_\alpha = b_\alpha e_\alpha$ by the principal extension property. Then we consider the system of congruences as follows:

$$x \equiv b_\alpha \pmod{I_\alpha} \quad (\alpha \in A,\ b_\alpha \in B,\ I_\alpha = \mathrm{Ann}_B(e_\alpha)).$$

This system is pairwise solvable. There exists a solution of it, since B is maximal ([4, Proposition 20. 46]). This solution induces f.

As other corollary to Theorem 2, we shall give the necessary and sufficient condition in order that a trivial extension QF-1 ring is a valuation ring.

Corollary 4. Let R be the trivial extension QF-1 ring of a ring B by a non-zero B-module E. Then the following are equivalent:

(a) R is a valuation ring.

(b) B is an integral domain and is a valuation ring.

Now, we shall conclude the study with the following corollary which is obtained by combining our result with R. M. Fossum et al. [6, Corollary 4. 37].

Corollary 5 (C. Faith [5, Theorem 6A]). Let R be the trivial extension of a ring B by a non-zero B-module E. Then the following are equivalent:

(a) R is a PF valuation ring.

(b) B is an integral domain and is an almost maximal valuation ring, E is the injective hull of $B/\mathrm{Rad}(B)$ and $B \simeq \mathrm{End}_B(E)$.

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TRIVIAL EXTENSIONS OF ARTIN ALGEBRAS

Takayoshi WAKAMATSU

1. Introduction. Let A be an artin algebra with a center C, I the injective envelope of $C/\mathrm{Rad}C$ and Q the bi-A-module $\mathrm{Hom}_C(A, I)$. Then we can construct a new artin algebra $R = A\ltimes Q$, i.e. $R = A\oplus Q$ as an additive group and the multiplication is given by the following:

$$(a,q)\cdot(a',q') = (aa',aq'+qa') \quad \text{for } (a,q),(a',q')\in R.$$

This ring R is called the trivial extension of A by Q. Our purpose is to study the relationship between representation types of A and R. This probrem was already considered by some authers [4],[6]. In this note we shall give a construction of indecomposable R-modules and, as its result, it will be shown that $n(R)\geq 2\cdot n(A)$ and $n(R) = 2\cdot n(A)$ if and only if A is hereditary, where $n(R)$ and $n(A)$ denote the number of indecomposable R-modules and A-modules respectively.

2. There are isomorphisms of C-modules

$$\begin{aligned} R = A\oplus Q &\simeq \mathrm{Hom}_C(\mathrm{Hom}_C(A\oplus Q,I),I) \\ &\simeq \mathrm{Hom}_C(\mathrm{Hom}_C(A,I),I)\oplus\mathrm{Hom}_C(\mathrm{Hom}_C(Q,I),I) \\ &\simeq \mathrm{Hom}_C(Q,I)\oplus\mathrm{Hom}_C(A,I) \\ &\simeq \mathrm{Hom}_C(Q\oplus A,I) \simeq \mathrm{Hom}_C(R,I), \end{aligned}$$

and it is easily veryfied that the composition of the above isomorphisms $R \simeq \mathrm{Hom}_C(R,I)$ is bi-R-module morphism. So R is symmetric and hence R is quasi-Frobenius. For a primitive idempotent $e \in A$, we have a primitive idempotent

$(e,0) \in R$. Identifying e with $(e,0)$, we have a projective (= injective) indecomposable R-module eR. And it is well known that every projective indecomposable R-module has such a form.

Since A is a subring of R, for a given R-module M we have the following short exact sequence of A-modules

$$0 \to M\Omega \to M \to M/M\Omega \to 0,$$

and the operation of Ω to M is considered as the epimorphism $M/M\Omega \otimes_A \Omega \to M\Omega \to 0$ because $(M\Omega)\Omega = M\Omega^2 = 0$. Conversely, for a couple of a short exact sequence of A-modules $0 \to X \to Y \overset{\rho}{\to} Z \to 0$ and an epimorphism $Z \otimes_A \Omega \overset{\phi}{\to} X \to 0$ Y is considered as an R-module by the following:

$y(a,q) = ya + \phi(\rho(y) \otimes q)$ for $y \in Y$, $(a,q) \in R$.

Dually, any R-module M is identified with the couple of the short exact sequence $0 \to \mathrm{ann}_M \Omega \to M \to M/\mathrm{ann}_M \Omega \to 0$ and the monomorphism $0 \to M/\mathrm{ann}_M \Omega \to \mathrm{Hom}_A(\Omega, \mathrm{ann}_M \Omega)$.

At the first we have

Theorem 1. For a projective A-module P and an epimorphism $P \otimes_A \Omega \overset{\phi}{\to} L \to 0$, the R-module

$$0 \to L \to L \oplus P \to P \to 0$$
$$P \otimes \Omega \overset{\phi}{\to} L \to 0$$

is indecomposable if and only if

(i) $\mathrm{Ker}\phi$ is indecomposable as A-module and

(ii) $\mathrm{Ker}\phi$ is large in $P \otimes \Omega$.

This result means that we can construct an indecomposable R-module from a given non-injective indecomposable A-module. And it is easy to see that any indecomposable projective (= injective) R-module eR has a form

$$0 \to e\Omega \to e\Omega \oplus eA \to eA \to 0$$

$$eA \otimes \Omega \overset{\simeq}{\to} e\Omega \to 0.$$

As the dual of the above result we have

Theorem 2. For an injective A-module E and a monomorphism $0 \to K \xrightarrow{\psi \neq 0} \mathrm{Hom}_A(\Omega, A)$, the R-module

$$0 \to E \to E \oplus K \to K \to 0$$

$$0 \to K \overset{\psi}{\to} \mathrm{Hom}(\Omega, E)$$

is indecomposable if and only if

(i) $\mathrm{Cok}\psi$ is indecomposable as A-module and

(ii) K is small in $\mathrm{Hom}_A(\Omega, E)$.

By Theorem 1 or Theorem 2, we have the following.

Proposition 3. $n(R) \geq 2\, n(A)$.

If A is hereditary then $M\Omega$ is an injective A-module for any R-module M since $M\Omega$ is a factor of $\oplus\Omega$ and Ω is an injective A-module. So every R-module has the form

$$0 \to E \to E \oplus \Omega \to P \to 0$$

$$0 \to P \to \mathrm{Hom}_A(\Omega, E),$$

where E is an injective A-module and P is a projective A-module. Thus we have

Proposition 4. If A is hereditary, then for a given indecomposable R-module M

(i) $M\Omega = 0$ $(\mathrm{ann}_M \Omega = 0)$, i.e. M is an indecomposable A-module, or

(ii) M is projective (= injective), or

(iii) $M = E \oplus P$, E is an injective A-module, P is a projective small A-submodule of $\mathrm{Hom}_A(\Omega, E)$ and $\mathrm{Hom}_A(\Omega, E)/P$ is an indecomposable A-module. Especially, R is of finite representation type if and only if so is A, and in this case, $n(R) = 2\cdot n(A)$ holds.

In the above we see that $n(R) = 2\cdot n(A)$ if A is hereditary. To verify the converse, we give

Theorem 5. For non-projective indecomposable R-modules

$$(\phi):\ 0 \to L \to L\oplus P \to P \to 0,\quad P\otimes_A Q \xrightarrow{\phi} L \to 0 \quad \text{and}$$

(P is a projective A-module)

$$(\psi):\ 0 \to E \to E\oplus K \to K \to 0,\quad 0 \xrightarrow{\psi} \mathrm{Hom}_A(Q,E),$$

(E is an injective A-module)

(ϕ) is isomorphic to (ψ) if and only if $\mathrm{inj.dim.Ker}\phi = 1$, $\mathrm{proj.dim.Cok}\psi = 1$, $\mathrm{Hom}_A(Q,\mathrm{Ker}\phi) = 0$, $\mathrm{Cok}\psi\otimes Q = 0$, $\mathrm{Cok}\psi \simeq \mathrm{TrD}(\mathrm{Ker}\phi)$ and $\mathrm{Ker}\phi \simeq \mathrm{DTr}(\mathrm{Cok}\psi)$, where $D = \mathrm{Hom}_C(-,I)$ and Tr is Auslander's functor "transpose".

Using the above result, we can count the number of projective R-modules and indecomposable A-modules and indecomposable R-modules constructed by the methods in Theorem 1 or 2. And we have the following.

Proposition 6. $n(R) = 2\cdot n(A)$ if and only if A is hereditary.

Lastly we remark that the algebras constructed from Brauer-trees (see [5]) without exceptional vertex can be seen as the trivial extensions of suitable algebras. So the class of the symmetric algebras which are the trivial extensions is more general than one thought.

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WEAKLY REGULAR MODULES

Tsuguo MABUCHI

This note is an abstract of the author's paper [5].

Recent years, several authors have been investigating weakly regular rings, where a ring R is called a left weakly regular ring if $a \in RaRa$ for every $a \in R$ (see [2]). In this note we shall define a weakly regular (right) module: A right R-module M is called a weakly regular module if

$$m \in \mathrm{Hom}_R(M,M)(m)\mathrm{Hom}_R(M,R)(m)$$
$$= \{\textstyle\sum_i s_i(m)f_i(m) \mid s_i \in \mathrm{Hom}_R(M,M),\ f_i \in \mathrm{Hom}_R(M,R)\}$$

for every $m \in M$. It is easy to see that R is a left weakly regular ring if and only if R_R is a weakly regular module.

1. Preliminaries

Throughout this note, R will represent an associative ring with 1, and M a unitary right R-module. Every (right or left) module is unitary and unadorned $\times$ means $\times_R$, unless otherwise stated. We set $M^* = \mathrm{Hom}_R(M,R)$ and $S = \mathrm{Hom}_R(M,M)$. For any S-R-submodule N of M, we set $T_N = \sum_{f \in M^*} f(N) = \mathrm{Hom}_R(M,R)(N)$. Obviously, $T = T_M$ is the trace ideal of M_R. Given ${}_RA$, $U_S({}_SN \times A)$ will denote the set of all S-submodules of $N \times A$. Further, $U_{T_N}({}_RA)$ will denote the set of all R-submodules A' of A with $T_NA' = A'$. Especially, $U_T({}_RR)$ is the set of all left ideals I of R such that $TI = I$. Finally, let $\Gamma_R(M,A) : M \times A \to \mathrm{Hom}_R({}_RM^*, {}_RA)$ be the unique map such that $\Gamma_R(M,A)(m \times a)(U)$

$= U(m)a$ for $m \in M$, $a \in A$ and $U \in M^*$ (see [1]).

A right R-module M is called a weakly regular module (abbr. w.regular module) if $m \in S(m)M^*(m)$ for every $m \in M$. A submodule N_R of M_R is said to be ideal pure if $N \cap MI = NI$ for every left ideal I of R, or equivalently, $i \otimes 1 : N \otimes R/I \to M \otimes R/I$ is monic for every left ideal I of R, where $i : N \to M$ is the inclusion (see [1]).

2. Weakly regular modules

Theorem 1. The following conditions are equivalent:

1) M_R is a w.regular module.

2) M_R is l.projective and every S-R-submodule of M is ideal pure.

3) M_R is l.projective and SmR_R is ideal pure for each $m \in M$.

4) For any S-R-submodule N of M, N_R is flat and for each left R-module A the lattices $U_{T_N}({}_RA)$ and $U_S({}_SN \otimes A)$ are isomorphic via the inverse assignments

$\Psi : U_{T_N}({}_RA) \to U_S({}_SN \otimes A)$; $A' \mapsto N \times A'$

$\Phi : U_S({}_SN \otimes A) \to U_{T_N}({}_RA)$; ${}_SB \mapsto \{ \sum_i f_i(n_i)a_i \mid f_i \in M^*, n_i \times a_i \in B\}$.

5) For any S-R-submodule N of M, the lattice isomorphism $U_{T_N}({}_RR) \to U_S({}_SN)$; $I \mapsto NI$, is surjective.

6) M_R is l.projective and $J = IJ$ for each pair $I, J \in U_T({}_RR)$ such that $I \supseteq J$ and I is a two-sided ideal of R.

7) M_R is l.projective and $TI = TI^2$ for each left ideal I of R.

Proof. See [5, Theorem 7].

The next corresponds to a theorem of Ware concerning regular modules (see [3, Corollary 4.2].

Theorem 2. If M_R is w.regular, then S is a left w.regular ring.

Proof. See [5, Theorem 8].

Corollary 3. Let N be an S-R-submodule of M. If M_R is w.regular and M/N_R is f.g., then $\mathrm{Hom}_R(M/N,M/N)$ is a left w.regular ring.

Proof. See [5, Corollary 9].

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STRONGLY SEMIPRIME RINGS AND NONSINGULAR QUASI-INJECTIVE MODULES

Mamoru KUTAMI

In this paper, we first give several characterizations of a right strongly semiprime ring. For example, it is shown that a ring R is such a ring if and only if

(1) Q(R) is a direct sum of simple rings, and

(2) eQ(R)eR = eQ(R) for all idempotents e in Q(R)

where Q(R) denotes the maximal ring of right quotients of R. Using these conditions (1) and (2), we shall investigate the following conditions:

(a) Every nonsingular quasi-injective right R-module is injective.

(b) Any finite direct sum of nonsingular quasi-injective right R-modules is quasi-injective.

(c) Any direct sum of nonsingular quasi-injective right R-modules is quasi-injective.

(d) Any direct product of nonsingular quasi-injective right R-modules is quasi-injective.

It is shown that the conditions (a), (b) and (d) are equivalent; indeed, the rings satisfying one of these conditions are determined as rings R such that R/G(R) is a right strongly semiprime ring, where G(R) denotes the right Goldie torsion submodule of R. A ring R satisfying the condition (c) is also characterized as a ring R such that R/G(R) is a semiprime right Goldie ring.

1. Preliminaries and Notations.

Throughout this paper all rings considered have identity and all modules are unitary.

Let R be a ring. Q(R) denotes its maximal ring of right quotients. Let M be a right R-module. By $E_R(M)$, nM, Z(M) and G(M) we denotes its injective hull, the direct product of n-copies, its singular submodule and its Goldie torsion submodule, respectively. (Note that Z(M/Z(M)) = G(M)/Z(M).) For a given two right R-modules N and M, we adopt the symbol $N \subsetneqq M$ to denote the fact that N is isomorphic to a submodule of M, and use the symbol $N \subseteq_e M$ to indicate N to be an essential submodule of M.

Now, for a nonsingular right R-module M, the following statements hold:

(1) MG(R) = 0; so M become a right R/G(R)-module by usual way,

(2) M is also nonsingular as a right R/G(R)-module, and

(3) M is R-injective (R-quasi-injective) if and only if M is R/G(R)-injective (R/G(R)-quasi-injective).

Noting that R/G(R) is a right nonsingular ring, we conclude from [2, Theorem 2.2] that any nonsingular injective right R-module has a unique right Q(R/G(R))-module structure compatible with the R-module structure. So, for a nonsingular right R-module M, we have $M \subseteq_e MQ(R/G(R)) \subseteq_e E_R(M)$.

It is well known (e.g. [2, Theorem 3.2]) that every finitely generated nonsingular right module over a right self-injective regular ring is both projective and injective. Therefore, if M is a finitely generated nonsingular injective right R-module, then M is both Q(R/G(R))-

projective and Q(R/G(R))-injective.

Lemma 1. Let R be a ring and set $\bar{R}$ = R/G(R) and Q = Q($\bar{R}$). If M is a nonsingular right Q-module, then the following statements hold:

(a) M is nonsingular as a right R-module. (Of course, M becomes a right R-module by a natural way.)

(b) M is Q-quasi-injective if and only if M is R-quasi-injective.

Lemma 2. If M is a quasi-injective right R-module such that R $\lesssim$ nM for some positive integer n, then M is injective.

2. Strongly semiprime rings.

We recall some definitions introduced by Handelman and Lawrence [4] and Handelman [5]. An right ideal I of a ring R is insulated if there exists a finite set of I whose right annihilator in R is zero. A ring R is said to be a right strongly semiprime ring if every ideal I of R with I $\subseteq_e$ R as a right ideal is insulated as a right ideal.

Definition. For an element a in a ring R, we call a finite set $\{r_1,\ldots,r_n;\ b\} \subseteq R$ is a right semi-insulator of a when RaR $\cap$ RbR = 0 and the right annihilator of $\{ar_1,\ldots,ar_n\} \cup bR$ is zero.

Proposition. For a given ring, the following conditions are equivalent:

(a) R is a right strongly semiprime ring.

(b) (1) Q(R) is a direct sum of simple rings, and

(2) Q(R)eR = Q(R)eQ(R), or equivalently, eQ(R)eR = eQ(R) for all idempotents e in Q(R).

(c) (1) R contains no infinite direct sums of ideals,

(2) every element of R has a right semi-insulator.

(d) $Q(R)I = Q(R)$ for any essential right ideal I of R.

(e) There exists a ring extention S of R with the same identity satisfying $SI = S$ for any essential right ideal I of R.

3. Nonsingular quasi-injective modules.

Lemma 3. If R is a simple ring, then every nonsingular quasi-injective right R-module is injective.

Theorem. For a given ring R, the following conditions are equivalent:

(a) R/G(R) is a right strongly semiprime ring.

(b) Every nonsingular quasi-injective right R-module is injective.

(c) Any finite direct sum of nonsingular quasi-injective right R-module is also quasi-injective.

(d) Any direct product of nonsingular quasi-injective right R-module is quasi-injective.

Proof. Set $\bar{R} = R/G(R)$ and $Q = Q(R/G(R))$.

(b)$\Longrightarrow$(d)$\Longrightarrow$(c): Obvious.

(a)$\Longrightarrow$(b). Since $\bar{R}$ is a right strongly semiprime ring, Proposition says that Q is a direct sum of simple rings and $eQe\bar{R} = eQ$ for all idempotents e in Q. Now, let M ($\neq 0$) be a nonsingular quasi-injective right R-module. In order to show M is injective, we show $M = MQ$. Let $0 \neq x \in M$. Since xQ is Q-projective, there exists an idempotent e in Q and an isomorphism ψ: $xQ \approx eQ$ with $\psi(x) = e$. Inasmuch as xQ is Q-injective, $E_R(M) = xQ \oplus Y$ for some submodule Y. Since M is quasi-injective, this yields $M = (xQ \cap M) \oplus (Y \cap M)$. As a result, $xQ \cap M$ is quasi-injective.

Put $Z = \Psi(xQ \cap M)$. Inasmuch as $x\overline{R} \subseteq_e xQ \cap M \subseteq_e xQ$, we infer that $E_R(xQ \cap M) = xQ$; whence $E_R(Z) = eQ$. Observing $eQ = eQe\overline{R} = End_Q(eQ)e\overline{R} = End_R(eQ)e\overline{R} \subseteq End_R(eQ)Z = Z$, we see $eQ = Z = \Psi(xQ \cap M)$. Consequently $xQ = xQ \cap M$ and it follows $xQ \subseteq M$. Therefore $MQ = M$. Since M is a nonsingular quasi-injective right R-module, then MQ is nonsingular Q-quasi-injective. Hence, by Lemma 3, MQ is Q-injective; whence MQ = M is R-injective.

(c) $\Rightarrow$(a). In view of Proposition, it is enough to show that $eQe\overline{R} = eQ$ for all idempotents e in Q and Q is a direct sum of simple rings. Let $e = e^2 \in Q$ and set $T = eQe\overline{R} \oplus (1-e)Q(1-e)\overline{R}$. Then T is a nonsingular quasi-injective right R-module because both $eQe\overline{R}$ and $(1-e)Q(1-e)\overline{R}$ are so. Since $R \lesssim T$, it follows that T is injective; whence so is $eQe\overline{R}$. Thus we get $eQe\overline{R} = eQeQ = eQ$. Now, assume that Q can not be expressed as a direct sum of prime rings. Then Q itself is not prime. Hence there exist non-zero two-sided ideals A,B such that AB = 0. Let A',B' be the injective hull of A,B in Q, then they are also two-sided ideals and generated by central idempotents by [3, Corollary 1.10]. Since Q is semiprime, $A \cap B = 0$. Then $A' \cap B' = 0$. Hence there exist orthogonal central idempotents $\{e_i\}_1^3$ such that $\sum_1^3 e_i = 1$. By assumption, at least one of e_iQ, say e_jQ, is not prime. Use the same aregument for the ring e_jQ, then there exists another set $\{e_i'\}_1^5$ of orthogonal central idempotents of Q such that $\sum_1^5 e_i' = 1$. Repeating these procedures, we see that there exist infinite orthogonal non-zero central idempotents $\{e_i \mid i = 1,2,\ldots\}$ in Q. Since $\sum_{i=1}^{\infty} e_iQ$ is nonsingular Q-quasi-injective, it is also non-singular R-quasi-injective (Lemma 1).

Putting $T = (1-e_1)Q \times (\sum_{i=1}^{\infty} e_i Q)$, T is then a nonsingular quasi-injective right R-module, since both $(1-e_1)Q$ and $\sum_{i=1}^{\infty} e_i Q$ are so. As a result, it follows from $R \subsetneqq T$ that T is injective and $\sum_{i=1}^{\infty} e_i Q \lessdot\!\!\oplus Q$, a contradiction.
Hence Q must be written as a direct sum of prime rings, say $Q = Q_1 \oplus \cdots \oplus Q_n$. Let X be a non-zero ideal of Q_i. Then X is a nonsingular quasi-injective right Q-module and hence it is nonsingular R-quasi-injective by Lemma 1. Take a non-zero idempotent e in X and consider $X \times (1-e)Q$. Since both X and $(1-e)Q$ are nonsingular quasi-injective right R-module, so is $X \times (1-e)Q$. Inasmuch as $R \subsetneqq X \times (1-e)Q$, it follows that $X \times (1-e)Q$ is injective; whence $X \lessdot\!\!\oplus Q_i$. Since Q_i is a prime ring, this shows $X = Q_i$. Accordingly each Q_i is simple.

Boyle and Goodearl [1] showed that every nonsingular quasi-injective right R-module over a semiprime right Goldie ring is injective. However, as is easily seen, a semiprime right Goldie ring is a right and left strongly semiprime ring, since every essential ideal of the ring has a regular element. Thus, by Theorem, we have

Corollary 1. If R is a semiprime right Goldie ring, then every nonsingular quasi-injective right R-module is injective and every nonsingular quasi-injective left R-module is also injective.

Corollary 2. For a given ring R, the following conditions are equivalent:

(a) R/G(R) is a semiprime right Goldie ring.

(b) Any direct sum of nonsingular quasi-injective right R-module is quasi-injective.

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COMMUTATIVE RINGS ALL OF WHOSE MODULES ARE QF-3'

Koichiro OHTAKE

Let R be a ring with identity and Mod-R the category of unital right R-modules. A subfunctor of the identity functor of Mod-R is called a preradical. Then a preradical t is called a radical if $t(M/t(M)) = 0$ for all $M \in$ Mod-R, and is called idempotent if $t(t(M)) = 0$ for all $M \in$ Mod-R. Also we can define precoradicals, coradicals and idempotent precoradicals. For example, an endofunctor r of Mod-R is called a coradical if there exists an idempotent preradical t of Mod-R such that $r = 1/t$.

Proposition 1. ([5]). Let $\mathcal{C}$ be a class of right R-modules. Let r be a functor such that

$$r_{\mathcal{C}}(M) = \bigcap\{\text{Ker } f \mid f \in \text{Hom}_R(M,C),\ C\}$$

for $M \in$ Mod-R. Then $r_{\mathcal{C}}$ is a radical. Conversely every radical in Mod-R is obtained like this.

Remark. In general $\mathcal{C}$ is not a set. So we have to make clear the meaning of $r_{\mathcal{C}}(M)$. Put $\mathfrak{S} = \{\mathcal{C}' \subset \mathcal{C} \mid \mathcal{C}'$ is a set$\}$. Let $\mathcal{T} = \{r_{\mathcal{C}'}(M) \mid \mathcal{C}' \in \mathfrak{S}\}$. Then $\mathcal{T}$ is a set since $r_{\mathcal{C}'}(M)$ is a submodule of M. So $r_{\mathcal{C}}(M)$ is defined via $\bigcap\{r_{\mathcal{C}'}(M) \mid r_{\mathcal{C}'}(M) \in \mathcal{T}\}$.

Proposition 1'. ([5]). Let $\mathcal{C}$ be a class of right R-modules. Let $t_{\mathcal{C}}$ be a functor such that

$$t_{\mathcal{C}}(M) = \Sigma\{\text{Im } f \mid f \in \text{Hom}_R(C,M),\ C \in \mathcal{C}\}$$

for $M \in$ Mod-R. Then $t_{\mathcal{C}}$ is an idempotent preradical. Conversely every idempotent preradical is obtained like this.

A torsion theory $(\mathcal{T}, \mathcal{F})$ in Mod-R is a couple of classes of right R-modules with the following properties:

(1) $\mathcal{T} \cap \mathcal{F} = \{0\}$.

(2) $\mathcal{T}$ is closed under direct sums, factor modules and group extensions.

(3) $\mathcal{F}$ is closed under direct products, submodules and group extensions.

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory, then $t_{\mathcal{T}} = r_{\mathcal{F}}$ (by Propositions 1 and 1'). Hence $t_{\mathcal{T}}$ is an idempotent radical. Conversely let t be an idempotent radical and put $\mathcal{T} = \{M \in \text{Mod-}R \mid t(M) = M\}$ and $\mathcal{F} = \{M \in \text{Mod-}R \mid t(M) = 0\}$. Then $(\mathcal{T}, \mathcal{F})$ is a torsion theory and $t = t_{\mathcal{T}} = r_{\mathcal{F}}$ holds. Hence it is an easy consequence that there is a bijective correspondence between torsion theories in Mod-R and idempotent radicals in Mod-R. A torsion theory $(\mathcal{T}, \mathcal{F})$ is called hereditary if $\mathcal{T}$ is closed under submodules, and this is equivalent to say that the corresponding idempotent radical is left exact. This is also equivalent to say that $\mathcal{F}$ is closed under injective envelopes. In fact a left exact radical is also idempotent.

In [3] Bronowitz and Teply offered a problem to characterize rings R such that:

(*) Every torsion theory in Mod-R is hereditary.

To approach this problem there appeared another problem to characterize rings R such that:

(**) Every radical in Mod-R is left exact.

A torsion theory $(\mathcal{T}, \mathcal{F})$ is said to be cohereditary if $\mathcal{F}$ is closed under factor modules (in this case $\mathcal{F}$ is called a TTF class ([4])).

Since (*) and (**) are categorical properties, the dual problems exist, namely:

(*)' Every torsion theory in Mod-R is cohereditary.

(**)' Every idempotent preradical is epi-preserving.

The property (**)' has come from the equivalent property:

every coradical (in the sense [11]) is right exact.

In [3, Theorem 3] rings with (*)' were determined, while it is easy to determine rings with (**)'. In fact R is a ring with (**)' if and only if R is semisimple artinian.

To characterize rings with either (*) or (**) are still open. On the other hand, two implications hold: (**) $\Rightarrow$ (*) and (*)' $\Rightarrow$ (*). But these properties are not equivalent.

Example 1. Every torsion theory over a proper homomorphic image of Z is hereditary. But there exists a radical which is not left exact.

The other half of examples will be given at the end of this report. A module Q_R is said to be QF-3' if Q_R cogenerates its injective envelope $E(Q_R)$. This definition is equivalent to say that r_Q (a particular case defined in Proposition 1) is left exact. Thus it is clear that (**) implies that every right R-module is QF-3'. In fact the converse holds ([5]).

We know that a simple module is QF-3' if and only if it is injective. Hence (**) implies that R is a right V-ring (i.e. every simple right R-module is injective).

Recently H. Katayama has proved the following:

Proposition 2.([5]). Suppose that R is either left or right semi-artinian. Then the following assertions are equivalent.

(1) Every right R-module is QF-3'.

(2) R is a right V-ring.

(3) Every left R-module is QF-3'.

(4) R is a left V-ring.

Here R is said to be semi-artinian if every nonzero right R-module has a nonzero socle. Also we get the following.

Proposition 3. The implications (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) hold.

(1) R is right semi-artinian and a right V-ring.

(2) Every right R-module is an essential extension of a direct sum of injective modules.

(3) Every nonzero right R-module has a nonzero injective submodule.

(4) Every right R-module is QF-3'.

If R is commutative we have the implication (4) $\Rightarrow$ (3). This fact follows from the so called Kaplansky's result: if R is commutative, R is a V-ring if and only if R is (von Neumann) regular. In fact if R is commutative, the statements in the preceding proposition are equivalent. In order to obtain it we need some lemmas.

Lemma 4. Let M_R be a nonzero QF-3' module with $S = \mathrm{End}(M_R)$. Suppose S is regular. Then M_R has a nonzero injective submodule.

Lemma 5. Let I be an ideal of R such that ${}_R R/I$ is flat. Let M be a right R/I-module. Then M_R is injective if and only if $M_{R/I}$ is injective.

Lemma 6. (Osofsky [9]). Let R be a right self-injective regular ring with an infinite set of orthogonal idempotents $\{e_\lambda \mid \lambda \in \Lambda\}$. Put $I = \sum_{\lambda \in \Lambda} e_\lambda R$. Then R/I is not an injective right R-module.

Finally we state a useful lemma.

Lemma 7. ([6]). The following conditions are equivalent.

(1) Every right R-module is QF-3'.

(2) Every cyclic right R-module is QF-3'.

A ring R is called a right QF-3' ring if R_R is a QF-3' module. Left QF-3' rings are defined similarly. R is called a QF-3' ring if it is both left and right QF-3'. Now we are ready to state our main result.

Theorem 8. Let R be a commutative ring. Then the following assertions are equivalent.

(1) Every R-module is QF-3'.

(2) Every nonzero R-module contains a nonzero injective submodule.

(3) R is regular and every factor ring of R is a QF-3' ring.

(4) R is regular and every factor ring of R has a nonzero injective ideal.

(5) R is regular and semi-artinian.

Finally we give an example indecated by Katayama.

Example 2. Let K be a field and K^I a direct product of copies of K with an index set I. Let R be a subring of K^I generated by $K^{(I)}$ and the identity of K^I. Then R is regular and semi-artinian.

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SEPARABLE POLYNOMIAL AND FROBENIUS POLYNOMIAL

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Throughout, K will represent a ring with 1, ρ an automorphism of K, and D a ρ-derivation of K (i.e. an additive map of K into itself such that $D(ab) = D(a)\rho(b) + aD(b)$ for all $a, b \in K$). Let $R = K[X; \rho, D]$ be the skew polynomial ring, in which the multiplication is given by $aX = X\rho(a) + D(a)$ $(a \in K)$. A monic polynomial f in R is called a separable (resp. Frobenius) polynomial, if $Rf = fR$ and R/Rf is a separable (resp. Frobenius) extension of K.

In case R is a usual polynomial ring over a commutative ring K, separable polynomials were studied by G. J. Janusz [2] and T. Nagahara [7, 8 and 9]. Concerning Frobenius polynomials, Y. Miyashita [5] proved that any monic polynomial in R is Frobenius.

In case $R = K[X; \rho]$ or $R = K[X; D]$, separable polynomials of some special type have been studied by K. Kishimoto [3, 4], and T. Nagahara [10, 11] has made a thorough investigation of polynomials of degree 2.

In his paper [6], Miyashita posed the following question: Is any separable polynomial Frobenius? Some arguments concerning the question have been done in [6, § 3]. Our present intention is to give some sufficient conditions for a separable polynomial to be Frobenius, and sharpen the results of Miyashita [6, Theorems 3.4 and 3.5].

1. In what follows, we use the following convention: Let $f = X^m - X^{m-1}a_1 - \dots - Xa_{m-1} - a_m$ $(m > 1)$ be a monic

polynomial in R with $Rf = fR$, and let

$$Y_0 = X^{m-1} - X^{m-2}a_1 - \dots - Xa_{m-2} - a_{m-1}$$

$$Y_1 = X^{m-2} - X^{m-3}a_1 - \dots - a_{m-2}$$

$$\dots\dots\dots\dots$$

$$Y_{m-2} = X - a_1$$

$$Y_{m-1} = 1 \quad .$$

First, we state the following important results of Miyashita:

Theorem 1 ([6, Theorem 1.8]). If f is separable, then there exists $y \in R$ with $\deg y < m$ such that $\sum_{j=0}^{m-1} Y_j y X^j \equiv 1 \pmod{Rf}$ and $\rho^{m-1}(a)y = ya$ for all $a \in K$, and conversely.

Theorem 2 ([6, Proposition 1.13]). If f is Frobenius, then there exists $r \in R$ with $\deg r < m$ such that $r + Rf$ is invertible in R/Rf and $\rho^{m-1}(a)r = ra$ (or $r\rho^{m-1}(a) = ar$) for all $a \in K$, and conversely.

If $R = K[X; D]$, then any monic polynomial f in R with $Rf = fR$ is Frobenius. (Take 1 as r in Theorem 2.) More generally, if ρ is an inner automorphism effected by an invertible element u in K, then any monic polynomial f in $R = K[X; \rho, D]$ with $Rf = fR$ is Frobenius. (Take u^{m-1} as r in Theorem 2.) However, for general R, the same need not be true.

Example. Let K be a field with an automorphism ρ of order 2, and $R = K[X; \rho]$. Then $f = X^2$ is not a

Frobenius polynomial. In fact, assume that $r = Xc_1 + c_0$ is an element of R such that $\rho(a)r = ra$ $(a \in K)$. Since $\rho \neq 1$, we have $c_0 = 0$ and $r^2 = (Xc_1)^2 = X^2\rho(c_1)c_1 \in fR$. Hence, $r + Rf$ cannot be invertible in R/Rf. According to Theorem 2, this implies that f is not Frobenius.

Now, we shall prove the following

Theorem 3. Let $R = K[X; \rho]$. Assume that there exists an invertible element u in K and a positive integer n such that $\rho(u) = u$ and $\rho^n(a) = uau^{-1}$ $(a \in K)$. If n is invertible in K, then any separable polynomial f in R commuting X is a Frobenius polynomial.

Proof. By Theorem 1, there exists $y \in R$ with $\deg y < m$ such that $\rho^{m-1}(a)y = ya$ $(a \in K)$ and $\sum_{j=0}^{m-1} Y_j y X^j \equiv 1$ (mod Rf). We consider the mapping $\tilde{\rho} : R \to R$ defined by $\tilde{\rho}(\sum_k X^k d_k) = \sum_k X^k \rho(d_k)$, which is easily seen to be a ring automorphism of R such that $\rho^n(h) = uhu^{-1}$ $(h \in R)$. Since $Xf = fX$, we have $\rho(a_i) = a_i$ $(1 \leq i \leq m)$, and therefore $\tilde{\rho}(f) = f$ and $\tilde{\rho}(Y_j) = Y_j$ $(0 \leq j \leq m-1)$. We put $r = n^{-1}\sum_{\nu=0}^{n-1} \tilde{\rho}^{\nu}(y)$. Since $\tilde{\rho}^n(y) = uyu^{-1} = \rho^{m-1}(u)yu^{-1} = yuu^{-1} = y$, we have $\tilde{\rho}(r) = r$, and therefore $rX = Xr$. Moreover, recalling that $\rho(a_i) = a_i$, we obtain

$$a_i r = n^{-1}\sum_{\nu=0}^{n-1} \tilde{\rho}^{\nu}(\rho^{m-1}(a_i)y) = n^{-1}\sum_{\nu=0}^{n-1} \tilde{\rho}^{\nu}(ya_i) = ra_i,$$

and hence $Y_j r = rY_j$. Now in view of $\tilde{\rho}(f) = f$ and $\tilde{\rho}(Y_j) = Y_j$, it follows that

$$r(\sum_{j=0}^{m-1} Y_j X^j) = (\sum_{j=0}^{m-1} Y_j X^j)r = \sum_{j=0}^{m-1} Y_j r X^j$$
$$= n^{-1}(\sum_{\nu=0}^{n-1} \tilde{\rho}^{\nu}(\sum_{\nu=0}^{m-1} Y_j y X^j)) \equiv 1 \ (\text{mod } Rf).$$

Thus, f is Frobenius by Theorem 2.

2. Throughout this section, we assume that $D = 0$, namely $R = K[X; \rho]$. Then the condition $Rf = fR$ implies that $af = f\rho^m(a)$ $(a \in K)$. Hence, we have $aa_i = a_i\rho^i(a)$, whence it follows that $Ka_i = a_iK$. This fact will be used freely in the subsequent study.

First, we prove the following key lemma of this section.

Lemma 1. Let f be separable, and $y = X^{m-1}c_{m-1} + \dots + Xc_1 + c_0$ be as in Theorem 1. Then there exists $d \in K$ such that $a_m d - a_{m-1}c_0 = 1$.

Proof. There exists a polynomial g in R such that $\sum_{j=0}^{m-1} Y_j y X^j = 1 - fg$. Comparing the constant terms of the both sides, we readily obtain $-a_{m-1}c_0 = 1 - a_m d$ with some $d \in K$.

Remark. In Lemma 1, we can easily see that $a_{m-1}c_0$ is in the center of K, since $\rho^{m-1}(a)y = ya$ $(a \in K)$.

Theorem 4 ([1, Theorem 1]). (a) If a_{m-1} or a_m is invertible, then f is Frobenius.

(b) If f is separable and if a_{m-1} or a_m is in the Jacobson radical $\mathrm{rad}(K)$ of K, then f is Frobenius.

Proof. (a) If a_{m-1} is invertible, we can take a_{m-1} as r in Theorem 2. If a_m is invertible, then $a_m \equiv X(X^{m-1} - X^{m-2}a_1 - \dots - a_{m-1}) \pmod{Rf}$ implies that $X + Rf$ is invertible in R/Rf, and therefore we can take X^{m-1} as r in Theorem 2.

(b) By Lemma 1, $\mathrm{rad}(K) + a_{m-1}K = K$ or $\mathrm{rad}(K) + a_mK$

$= K$. Then $a_{m-1}K = K$ or $a_m K = K$, and therefore a_{m-1} or a_m is invertible. Hence, f is Frobenius by (a).

Corollary 1 (cf. [6, Theorem 3.4 (1)]). If $\mathrm{rad}(K)$ is a maximal ideal of K, then every separable polynomial f is Frobenius.

Proof. By Lemma 1, we have $a_m K + a_{m-1}K = K$. Since $\mathrm{rad}(K)$ is the unique maximal ideal of K, it follows that $a_m K = K$ or $a_{m-1}K = K$. Now, the conclusion is immediate by Theorem 4.

Taking the above remark into mind, we can easily see

Corollary 2. If the center C of K is a local ring, then every separable polynomial f is Frobenius.

In the rest of this section, we assume that K is the direct sum of (directly) indecomposable rings K_i $(i = 1, 2, \ldots, r)$. Obviously, the center C of K is the direct sum of the center C_i of K_i. Let e_i be the identity of K_i. Then ρ induces a permutation τ of $\{1, 2, \ldots, r\}$ such that $\rho(e_i) = e_{\tau(i)}$. Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be the orbits of τ, and set $A_j = \oplus_{i\in\gamma_j} K_j$ $(j = 1, 2, \ldots, k)$. Then there holds $R = \oplus_{j=1}^{k} A_j[X; \rho_j]$, where ρ_j is the restriction of ρ onto A_j.

Under the above hypothesis and notations, there holds the following

Lemma 2. If τ is a cycle of length $r > 1$, then every separable polynomial f of R is Frobenius.

Proof. Without loss of generality, we may assume that $\tau = (1, 2, \ldots, r)$. If r does not divide $m - 1$, then $\rho^{m-1}(e_1) \neq e_1$. Hence, $e_1 a_{m-1} = a_{m-1}\rho^{m-1}(e_1) = \rho^{m-1}(e_1)a_{m-1}$, whence it follows $e_1 a_{m-1} = 0$. Similarly, we can prove that $e_i a_{m-1} = 0$, and therefore $a_{m-1} = 0$. Hence f is Frobenius by Theorem 4. On the other hand, if r divides $m-1$, then $\rho^m(e_i) = \rho(e_i) = e_{i+1}$ with the convention $e_{r+1} = e_1$. Hence, $e_i a_m = a_m e_{i+1} = e_{i+1}a_m$, whence it follows $a_m = 0$. Then f is Frobenius by Theorem 4.

Now, let e_j^* be the identity of A_j. According to Theorem 1 (resp. Theorem 2), one can easily see that h is a separable (resp. Frobenius) polynomial in R if and only if each he_j^* is a separable (resp. Frobenius) polynomial in $A_j[X; \rho_j]$. Hence, as a combination of Lemma 2 and Corollary 1, we readily obtain the following which includes [6, Theorem 3.5]:

Theorem 5 ([1, Theorem 2]). Assume that K is the direct sum of indecomposable rings K_i, and that each $\mathrm{rad}(K_i)$ is a maximal ideal of K_i. Then every separable polynomial in R is Frobenius.

As is well known, every commutative Artinian ring is a direct sum of local rings. This together with Lemma 2 and Corollary 2 yields the following

Theorem 6 ([1, Theorem 3]). If the center of K is an Artinian ring, then every separable polynomial in R is Frobenius.

Corollary 3. If K is a commutative Artinian ring, then every separable polynomial in R is Frobenius.

3. Now, we return to the general case, and prove the following slight generalization of [6, Theorem 3.4 (2)].

Theorem 7. Assume that K is a (two-sided) simple ring and f is separable. Let $y = X^n c_n + X^{n-1} c_{n-1} + \dots + c_0$ $(c_n \neq 0)$ be as in Theorem 1. If either $n = 0$ or $(m, n) = 1$, then f is Frobenius.

Proof. If $n = 0$, then $\rho^{m-1}(a)c_0 = c_0 a$ $(a \in K)$. Since K is simple, c_0 is invertible, and therefore f is Frobenius by Theorem 2. Henceforth, we assume that $(m, n) = 1$, and choose positive integers r, s such that $mr - ns = 1$. As is easily verified, $\delta_\nu = \sum_{i=0}^{\nu-1} \rho^i D \rho^{-i}$ $(\nu \geq 1)$ is a ρ-derivation (see, e.g. [12]). The condition $Rf = fR$ implies $af = f\rho^m(a)$ $(a \in K)$. Comparing the coefficients of X^{m-1} in the both sides, we obtain

$$\delta_{m-1}(\rho^{m-1}(a)) - \rho^{m-1}(a)a_1 = - a_1\rho^m(a).$$

Hence, putting $c = \rho^{m-1}(-a)$, we have $\delta_{m-1}(c) = a_1\rho(c) - ca_1$, which means that δ_{m-1} is an inner ρ-derivation. Next, since $\rho^{m-1}(a)y = ya$ $(a \in K)$, there holds that $\rho^{m+n-1}(a)c_n = c_n a$ and $\delta_{n-1}(\rho^{m+n-2}(a))c_n + \rho^{m+n-2}(a)c_{n-1} = c_{n-1}a$. Recalling that $c_n \neq 0$ and K is sinple, we see that c_n is invertible. Hence it follows that

$$\delta_{n-1}(\rho^{m+n-2}(a)) = c_{n-1}ac_n^{-1} - \rho^{m+n-2}(a)c_{n-1}c_n^{-1}$$
$$= c_{n-1}c_n^{-1}\rho^{m+n-1}(a) - \rho^{m+n-2}(a)c_{n-1}c_n^{-1}.$$

Now, putting $d = \rho^{m+n-2}(a)$, we have $\delta_{n-1}(d) = c_{n-1}c_n^{-1}\rho(d) - dc_{n-1}c_n^{-1}$, and therefore δ_{n-1} is an inner ρ-derivation. Then, it is easy to see that

$$\sum_{k=0}^{r-1} \rho^{km}\delta_{m-1}\rho^{-km} - \rho(\sum_{\ell=0}^{s-1} \rho^{\ell n}\delta_{n-1}\rho^{-\ell n})\rho^{-1} = D.$$

Since $\rho^{i}\delta_{m-1}\rho^{-i}$ and $\rho^{i}\delta_{n-1}\rho^{-i}$ are inner ρ-derivations, we see that D is also an inner ρ-derivation. Hence, there exists u K such that $D(a) = u\rho(a) - au$ (a K). Now, we have $K[X; \rho, D] = K[Y; \rho]$ for $Y = X + u$, and therefore f is Frobenius by Corollary 1.

Corollary 4. If K is a simple ring, every separable polynomial of prime degree in R is Frobenius.

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UNITS IN INTEGRAL GROUP RINGS

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Introduction. Let G be a finite group and let ZG be its integral group ring. Let $U(ZG)$ denote the unit group of ZG and put $V(ZG) = \{ u \in U(ZG) \mid \varepsilon(u) = 1 \}$ where $\varepsilon : ZG \longrightarrow Z$ is the augmentation map of ZG. Then $U(ZG) = V(ZG) \times \{\pm 1\}$. A unit of ZG is called trivial if it is of the form $\pm g$, $g \in G$. In [3], Higman showed that, if G is a finite abelian group, every unit of finite order in ZG is trivial. He also showed that every unit in ZG is trivial if and only if G is

(1) abelian and its exponent is 1, 2, 3, 4 or 6, or

(2) the direct product of the quaternion group of order 8 and an elementary abelian 2-group.

However, if G is not an abelian group, there exists very few results concerning $U(ZG)$.

In this paper, we will study the following problems :

Problem 1. Is there a torsion free normal subgroup F of $V(ZG)$ such that $V(ZG) = F \cdot G$?

Problem 2. How many conjugate classes are there in $V(ZG)$ of subgroups of $V(ZG)$ isomorphic to G ?

Problem 3. Construct nontrivial units in ZG.

Let S_n (resp. A_n) denote the symmetric group (resp. alternating group) on n symbols, D_n the dihedral group of order 2n and C_n the cyclic group of order n.

To begin with, we shall state the results concerning Problems 1 and 2 which are obtained until now.

(1) Hughes and Pearson [4] : There is 1 conjugate class in $V(ZS_3)$ of subgroups of $V(ZS_3)$ isomorphic to S_3.

(2) Polcino [7] : There are 2 conjugate classes in $V(ZD_4)$ of subgroups of $V(ZD_4)$ isomorphic to D_4.

(3) Dennis [1] : There is a torsion free normal subgroup F of $V(ZS_3)$ such that $V(ZS_3) = F \cdot S_3$.

(4) Miyata [6] : If n is odd, there is a torsion free normal subgroup F of $V(ZD_n)$ such that $V(ZD_n) = F \cdot D_n$.

If n and the order of the class group of ZD_n are odd, then there are $\phi(n)/2$ conjugate classes in $V(ZD_n)$ of subgroups of $V(ZD_n)$ isomorphic to D_n, where $\phi(\cdot)$ denotes Euler's totient function.

On the other hand, concerning Problem 3, nontrivial units were constructed in the following cases :

$G = S_3$ ([11]), $G = D_4$([7]) etc.

In § 1, we shall consider Problem 1 in the case where G is a metabelian group, and in § 2, the number of conjugate classes will be determined in each of the cases : $G = A_4$ and S_4. Finally, in §3, we will give a method of construction of nontrivial units in ZG of an arbitrary finite group G. For the proofs of theorems in §1 and §2, see [9].

§1. For $N \lhd G$, denote by $\varepsilon_{G,N}$ the natural map from ZG to $Z(G/N)$ and let $I(G,N) = \mathrm{Ker}\ \varepsilon_{G,N}$. For an ideal J of ZG, we write $U(1+J) = U(ZG) \cap (1+J)$, where $1+J$ is the set of all elements of the form $1+j$, $j \in J$. Note that $\varepsilon_{G,G}$ is the augmentation map of ZG and $V(ZG) = U(1+ I(G,G))$. Write $\varepsilon = \varepsilon_{G,G}$ and $I(G) = I(G,G)$. The following results are useful.

Proposition 1.1 ([5]). Let G be a finite group and let $g \in G$. Then $g - 1 \in I(G)^2$ if and only if $g \in G'$.

Proposition 1.2 ([13]). Let G be a finite group and let $N \lhd G$. Then

$$N/N' \cong I(G,N)/I(G)I(G,N)$$

under the map $nN' \longrightarrow n-1 + I(G)I(G,N)$, $n \in N$, where N' denotes the commutator subgroup of N.

Define the map $U(1+I(G,N)) \longrightarrow I(G,N)$ by $1+k \longrightarrow k$, $k \in I(G,N)$. This map induces a group isomorphism

$$U(1+I(G,N))/U(1+I(G)I(G,N)) \longrightarrow I(G,N)/I(G)I(G,N).$$

Thus we get

Corollary 1.3. Let G be a finite group and let $N \lhd G$. Then

$$N/N' \cong U(1+I(G,N))/U(1+I(G)I(G,N)).$$

By (1.1),(1.2) and the theorem of Higman, we get

Proposition 1.4. Let G be a finite abelian group. Then $U(1+I(G)^2)$ is a torsion free subgroup and $V(ZG) = G \times U(1+I(G)^2)$.

The purpose of this section is to state the following

Theorem 1.5. Let G be a finite metabelian group such that the exponent of G/G' is 1, 2, 3, 4 or 6. Then there is a torsion free normal subgroup F of $V(ZG)$ such that $V(ZG) = F\cdot G$.

Remark. Let G be a finite group of order n. By the regular representation of G, there is a ring monomorphism $T : ZG \longrightarrow M_n(Z)$, where $M_n(Z)$ denotes the ring of $n \times n$ matrices over Z. Then the restriction of T to $U(ZG)$ yields a group monomorphism $U(ZG) \longrightarrow GL(n,Z)$, which is denoted by the same symbol T. For an odd prime p, define the natural map $\phi_p : GL(n,Z) \longrightarrow GL(n,Z/pZ)$. Then it is easy to see that $\mathrm{Ker}\,\phi_p$ is a torsion free normal subgroup of $GL(n,Z)$ of finite index in $GL(n,Z)$. Set $F = T^{-1}(\mathrm{Ker}\,\phi_p)$, then F is a torsion free normal subgroup of $U(ZG)$ of finite index in $U(ZG)$. Thus, for any finite group G, $U(ZG)$ has a torsion free normal subgroup of finite index.

§2. Let A_4 be the alternating group on 4 symbols 1, 2, 3 and 4. Set $N = \{1, (12)(34), (13)(24), (14)(23)\} \triangleleft A_4$ and define $\bar{N} = 1 + (12)(34) + (13)(24) + (14)(23)$ in ZA_4.

Hereafter, the unit group of a ring R will be denoted by U(R).

Consider the pullback diagram

$$\begin{array}{ccc} ZA_4 & \longrightarrow & Z(A_4/N) \\ \downarrow & & \downarrow \\ ZA_4/(\bar{N}) & \longrightarrow & (Z/4Z)[(123)]. \end{array}$$

From this diagram we get the exact sequence (e.g. [8]).

$$1 \longrightarrow U(ZA_4) \longrightarrow U(ZA_4/(\bar{N})) \longrightarrow U((Z/4Z)[(123)])/\langle -1, (123)\rangle \longrightarrow 1.$$

Define the representation of $ZA_4/(\bar{N})$ to $M_3(Z)$ by

$$\overline{(12)(34)} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \overline{(123)} \longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Here, for $x \in ZA_4$, $\bar{x}$ denotes the image of x under the natural map $ZA_4 \longrightarrow ZA_4/(\bar{N})$. By this representation $U(ZA_4/(\bar{N}))$ is embedded in GL(3,Z). Making use of the classification of finite subgroups of GL(3,Z) ([10]), we obtain

Theorem 2.1. There are 4 conjugate classes in $V(ZA_4)$ of subgroups of $V(ZA_4)$ isomorphic to A_4.

By more complicated computations, we further obtain

Theorem 2.2. There are 16 conjugate classes in $V(ZS_4)$ of subgroups of $V(ZS_4)$ isomorphic to S_4.

§3. In this section, we will give a method of construction of nontrivial units. Our main result in this section is as follows :

Theorem 3.1. Let G be a finite group. Take a unit u in ZG of finite order n and $f \in ZG$ such that $f + f^{(1)} + \ldots\ldots + f^{(n-1)} = 0$, where $f^{(i)} = u^i f u^{-i}$. Let c be a unit in ZG such that c is commutative with f and u, and set $v = v(u,f,c) = f + (f^{(1)} + c)u + f^{(2)}u^2 + \ldots\ldots. + f^{(n-1)}u^{n-1}$. Then $v^n = c^n$. In particular, v is a unit in ZG.

Proof. For f_i, $g_j \in ZG$, put $x = \sum_{i=0}^{n-1} f_i u^i$ and $y = \sum_{j=0}^{n-1} g_j u^j$. Then

$$
\begin{aligned}
xy &= (f_0 + f_1 u + \ldots\ldots + f_{n-1}u^{n-1})(g_0 + g_1 u + \ldots. + g_{n-1}u^{n-1}) \\
&= f_0 g_0 + f_1 g_{n-1}^{(1)} + \ldots\ldots\ldots + f_{n-1} g_1^{(n-1)} \\
&\quad + (f_0 g_1 + f_1 g_0^{(1)} + \ldots\ldots. + f_{n-1} g_2^{(n-1)})u \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad + (f_0 g_{n-1} + f_1 g_{n-2}^{(1)} + \ldots\ldots + f_{n-1} g_0^{(n-1)})u^{n-1}
\end{aligned}
$$

$$= (f_0, f_1, \ldots\ldots, f_{n-1})\, W \begin{pmatrix} 1 \\ u \\ \vdots \\ u^{n-1} \end{pmatrix}, \quad \text{where}$$

$$W = \begin{pmatrix} g_0 & g_1 & g_2 & \cdots\cdots & g_{n-1} \\ g_{n-1}^{(1)} & g_0^{(1)} & g_1^{(1)} & \cdots & g_{n-2}^{(1)} \\ & & \cdots\cdots\cdots\cdots & & \\ g_1^{(n-1)} & g_2^{(n-1)} & & \cdots\cdots & g_0^{(n-1)} \end{pmatrix}$$

Hence, for $v = f + (f^{(1)} + c)u + f^{(2)}u^2 + \ldots\ldots + f^{(n-1)}u^{n-1}$, we have

$$v^n = (f, f^{(1)} + c, f^{(2)}, \ldots, f^{(n-1)})\, Y^{n-1} \begin{pmatrix} 1 \\ u \\ \vdots \\ u^{n-1} \end{pmatrix},$$

where

$$Y = \begin{pmatrix} f & f^{(1)}+c & f^{(2)} & \cdots & f^{(n-1)} \\ f & f^{(1)} & f^{(2)}+c & \cdots & f^{(n-1)} \\ & \cdots\cdots\cdots\cdots & & & \\ f & \cdots\cdots\cdots & & & f^{(n-1)}+c \\ f+c & f^{(1)} & f^{(2)} & \cdots\cdots & f^{(n-1)} \end{pmatrix}$$

Put $H = \begin{pmatrix} f & f^{(1)} & \cdots\cdots & f^{(n-1)} \\ f & f^{(1)} & \cdots\cdots & f^{(n-1)} \\ & \cdots\cdots\cdots & & \\ f & f^{(1)} & \cdots\cdots & f^{(n-1)} \end{pmatrix}$

and $C = \begin{pmatrix} 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ & & & & 0 \\ 0 & 0 & \cdots & 0 & c \\ c & 0 & & 0 & 0 \end{pmatrix}$.

Now, we will show that

(1)
$$\begin{aligned} &(f, f^{(1)}+c, f^{(2)}, \ldots\ldots, f^{(n-1)}) \{H + C\}^k \\ &= c^k(f, f^{(1)}, \ldots, f^{(k+1)}+c, \ldots, f^{(n-1)}) \\ &+ c^k(f^{(n-1)}, f, \ldots\ldots\ldots, f^{(n-2)}) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ c^k(f^{(n-k)}, f^{(n-k+1)}, \ldots\ldots\ldots, f^{(n-k-1)}), \end{aligned}$$

for every k, $0 \leq k \leq n-1$, where we identify $f^{(n)}$ with f when $k = n - 1$. We use the induction on k. If $k = 0$, the assertion is clearly valid. Suppose that the following equation holds :

$$\begin{aligned} &(f, f^{(1)}+c, f^{(2)}, \ldots\ldots, f^{(n-1)}) \{H + C\}^{k-1} \\ &= c^{k-1}(f, f^{(1)}, \ldots\ldots, f^{(k)}+c, \ldots, f^{(n-1)}) \\ &+ c^{k-1}(f^{(n-1)}, f, \ldots\ldots\ldots, f^{(n-2)}) \\ &+ \ldots\ldots\ldots\ldots \end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ c^{k-1}(f^{(n-k+1)}, \cdots\cdots\cdots, f^{(n-k)}).$$

Since $f + f^{(1)} + \cdots\cdots + f^{(n-1)} = 0$ and $cf^{(j)} = f^{(j)}c$ for every j, $0 \leqq j \leqq n-1$, by assumption, we get

$(f^{(n-i)}, \ldots\ldots, f^{(n-i-1)})\ H = (0, \ldots\ldots, 0)$ and

$(f^{(n-i)}, \ldots\ldots, f^{(n-i-1)})\ C = c(f^{(n-i-1)}, \ldots, f^{(n-i-2)})$,

for every i, $0 \leq i \leq n-1$.

Therefore we get

$$(f, f^{(1)}+c, \cdots\cdots\cdots, f^{(n-1)})\ \{ H + C\}^{k}$$
$$= c^{k-1}(f, f^{(1)}, \ldots, f^{(k)}+c, \ldots, f^{(n-1)})\ \{ H + C\}$$
$$+ c^{k-1}(f^{(n-1)}, f, \cdots\cdots\cdots, f^{(n-2)})\ \{ H + C\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ c^{k-1}(f^{(n-k+1)}, \cdots\cdots\cdots, f^{(n-k)})\ \{ H + C\}$$

$$= c^{k}(f, f^{(1)}, \ldots, f^{(k+1)}+c, \ldots, f^{(n-1)})$$
$$+ c^{k}(f^{(n-1)}, f, \cdots\cdots\cdots, f^{(n-2)})$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ c^{k}(f^{(n-k)}, f^{(n-k+1)}, \cdots\cdots, f^{(n-k-1)})$$

as desired. When $k = n-1$, the equation (1) implies that

$$(f, f^{(1)}+c, \ldots, f^{(n-1)})\ \{ H + C\}^{n-1} = (c^{n}, 0, \ldots, 0).$$

Hence

$$v^n = (f, f^{(1)}+c, \ldots , f^{(n-1)}) \{ H + C \}^{n-1} \begin{pmatrix} 1 \\ u \\ \vdots \\ u^{n-1} \end{pmatrix} = c^n.$$

This completes the proof.

We here use the same notation as in (3.1). Now, we will give some examples.

Example 1. $G = S_n$, the symmetric group on n symbols $1, 2, \ldots , n$. Write

$A_n = \{ g_1, g_2, \ldots , g_t, g_{t+1}, \ldots, g_{t+S}, (12)g_{t+1}(12), \ldots \ldots, (12)g_{t+s}(12) \}$ where $\{g_1, \ldots , g_t\}$ is the set of all elements of A_n commutative with (12). Put $u = (12)$, $c = 1$ and $f = \sum_{i=1}^{s} g_{t+i} - (12)(\sum_{i=1}^{s} g_{t+i})(12)$. Since $f + (12)f(12)^{-1} = 0$, $v = v((12),f,1)$ is in $U(ZS_n)$ and $v^2 = 1$.

Example 2. $G = D_n$.

Write $D_n = < \sigma, \tau \mid \sigma^n = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} >$.

Set $u = \tau$, $c = 1$ and

$$f = \begin{cases} a_1(\sigma-\sigma^{-1})+ \cdots\cdots + a_{(n-1)/2}(\sigma^{(n-1)/2}-\sigma^{-(n-1)/2}) & :n \text{ odd} \\ a_1(\sigma-\sigma^{-1})+ \cdots\cdots + a_{(n-2)/2}(\sigma^{(n-2)/2}-\sigma^{-(n-2)/2}) & :n \text{ even} \end{cases}$$

Then $f + \tau f\tau^{-1} = 0$, hence $v = v(\tau,f,1)$ is in $U(ZD_n)$

and $v^2 = 1$.

Suppose that n is odd and $a_i = 1$ for each i, $1 \leq i \leq (n-1)/2$. Then

$$v = (\sigma-\sigma^{-1}) + \ldots\ldots + (\sigma^{(n-1)/2}-\sigma^{-(n-1)/2})$$
$$+ \{1 - (\sigma-\sigma^{-1}) - \ldots\ldots - (\sigma^{(n-1)/2}-\sigma^{-(n-1)/2})\}\tau \ .$$

Consider the natural map $\psi : V(ZD_n) \longrightarrow V((Z/2Z)D_n)$. Since $\psi(v) = \sigma + \ldots + \sigma^{n-1} + (1 + \sigma + \ldots + \sigma^{n-1})\tau$, $\psi(v)$ is in the center of $V((Z/2Z)D_n)$. Therefore, τ and v are not conjugate in $V(ZD_n)$. Hence the number of conjugate classes in $V(ZD_n)$ of subgroups of $V(ZD_n)$ of order 2 are more than two.

Remark. Recently, Problem 2 has been solved in each of the following cases([2]) :

(i) $G = D_n$, n an arbitrary positive integer.

(ii) $G = C_m \cdot C_q$, $(q,m) = 1$, the semidirect product of C_m by C_q such that C_q acts faithfully on each Sylow subgroup of C_m.

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ON FROBENIUS EXTENSIONS OF QF-3 RINGS

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Let A be a ring and G a finite group of ring automorphisms of A, and let A^G denote the fixed subring of A relative to G; $A^G = \{a \in A;\ \sigma(a) = a \text{ for all } \sigma \text{ in } G\}$. Then a question arises whether A^G inherites the property of being QF-3 from A. Here a ring is said to be left QF-3 if it has a unique minimal faithful left module; that is, a faithful module which is isomorphic to a direct summand to every faithful module. In general, the answer is negative obviously, so some hypothesis about the relationship between A and A^G is needed. In this note we restricte ourself to the case where A is a G-Galois extension of A^G; that is, there exist $\{x_1,\dots,x_n\}$, $\{y_1,\dots,y_n\} \subset A$ such that $\Sigma_i\ x_i\sigma(y_i) = \delta_{\sigma,1}$ for all σ in G, where the symbol $\delta_{\sigma,1}$ denotes the Kronecker's delta (see [4]). Then we obtain the following.

Theorem. Assume that A is a G-Galois extension of A^G. Then A is left QF-3 iff B is left QF-3.

Remark. Under the same assumption as the theorem, A is QF if A^G is QF but the converse is not necessarily true.

It is the aim of this note to show the above theorem.

Throughout this paper, all rings have a 1, which acts unitally and is preserved by homomorphisms and subrings.

We begin with recalling some definitions which will be employed in the sequel. A left R-module ${}_RM$ is said to be co-finitely generated (co-f.g. in brief) provided that for every set $\{M_i;\ i \in I\}$ of submodules of M if the intersection $\cap_I M_i = 0$, then there exists a finite subset $F \subset I$ such that $\cap_F M_i = 0$. It is well known that ${}_RM$ is co-f.g. iff Soc(M), the socle of ${}_RM$ is f.g. and essential in M (see e.g. [8]). Let A/B be a ring extension. Following Kasch [2], A/B is called a Frobenius extension provided that ${}_BA$ is f.g.projective and ${}_AA_B \simeq {}_A\mathrm{Hom}({}_BA, {}_BB)_B$. It is known that A/B is a Frobenius extension iff there exist $\{r_1,\dots,r_n\}$, $\{l_1,\dots,l_n\} \subset A$ and a B-B-homomorphism h of A to B such that $x = \sum_i h(xr_i)l_i = \sum_i r_i h(l_i x)$ for all x in A (see Onodera [6]). Following Müller [5], A/B is called a left QF extension provided that ${}_BA$ is f.g.projective and ${}_AA_B \mid {}_A\mathrm{Hom}({}_BA, {}_BB)_B$, where the notation ${}_AX_B \mid {}_AY_B$ denotes the fact that X is isomorphic to a direct summand of a direct sum of a finite number of copies of Y as an A-B-module. A right QF extension is defined symmetrically. A Frobenius extension is obviously a left and right QF extension. In case A/B is a left or right QF extension, ${}_BA$ and A_B are both f.g.projective.

The following is well known (see e.g. [1] or [7]).

Lemma 1. For a ring R, the following statements are equivalent.

(1) R is left QF-3.

(2) There exist non-isomorphic, simple left ideals L_i $(i = 1,\dots,t)$ in R such that the injective hull $E(\oplus_i L_i)$ of the module $\oplus_i L_i$ is a faithful left ideal of R.

(3) There exists a left R-module which is f.g.projective, co-f.g.injective and faithful.

Definition. A module which has the proprrty (3) above is called a *-module for convenience.

The next lemma can be seen easily by the definition of co-f.g. module.

Lemma 2. Let ${}_RX$ and ${}_SY$ be R and S-modules, respectively. Let f: $X \to Y$ be an additive injection such that f carries every R-submodule of X to an S-submodule of Y. If ${}_SY$ is co-f.g., then so is ${}_RX$.

Proposition 3. Let A/B be a ring extension. Assume that there exist $a_1,\ldots,a_n \in A$ such that $A = \sum_i Ba_i$, $a_iB = Ba_i$ $(i = 1,\ldots,n)$. Then the following statements hold.

1) If ${}_BM$ is co-f.g., then so is $\mathrm{Hom}({}_BA, {}_BM)$ as a B-module (and hence as an A-module).

2) In case A/B is a left or right QF extension, A is left QF-3 iff B is left QF-3.

Proof. 1): Set $K_i = Ba_i$ and $X_i = \mathrm{Hom}({}_BK_i, {}_BM)$ $(i = 1,\ldots,n)$. For each i, considering a mapping

$$\phi_i\colon X_i \to M, \quad \phi_i(f) = f(a_i)$$

Lemma 2 implies that each X_i is co-f.g. as a B-module. Hence $\mathrm{Hom}({}_BA, {}_BM)$ is co-f.g. as a B-module because the epimorphism

$$\oplus_i K_i \to A, \ (x_i) \to \textstyle\sum_i x_i$$

induces a monomorphism

$$\mathrm{Hom}({}_BA, {}_BM) \to \mathrm{Hom}({}_B\oplus_i K_i, {}_BM) \ (\simeq \oplus_i X_i).$$

2): Let A/B be a left or right QF extension. Assume that A is left QF-3. Let ${}_AU$ be a *-module. Then U is obviously f.g.projective, injective and faithful as a

B-module. To see that U is co-f.g. as a B-module, let $\{S_i\}_{i\in I}$ be a complete set of representatives for the distinct isomorphism classes of simple left B-modules. Setting $X = \oplus_{i\in I} E(S_i)$, X is faithful, and so $Y = \mathrm{Hom}({}_BA, {}_BX)$ is faithful as an A-module. Thus we have ${}_AU|{}_AY$. But, ${}_BA$ being f.g., ${}_AY$ is isomorphic to $\oplus_{i\in I} \mathrm{Hom}({}_BA, {}_BE(S_i))$. Hence there exists a finite subset $F \subset I$ such that U can be imbeded in $\oplus_{i\in F} \mathrm{Hom}({}_BA, {}_BE(S_i))$. Thus U is co-f.g. as a B-module by 1). It follows that U is a *-module as a B-module, which implies that B is left QF-3 by Lemma 1. Conversely assume that B is left QF-3. Let ${}_BV$ be a *-module. Then $\bar{V} = \mathrm{Hom}({}_BA, {}_BV)$ is clearly an injective left A-module. Moreover $\bar{V}$ is co-f.g. by 1). If A/B is a left QF extension, then ${}_AA \otimes {}_BV|{}_A\bar{V}$, and so, to be easily seen, ${}_AA \otimes {}_BV$ is a *-module. If A/B is a right QF extension, then ${}_A\bar{V}|{}_AA \otimes {}_BV$, and so, $\bar{V}$ is a *-module. It follows that A is left QF-3.

Corollary. Let R be any ring and G any finite group. Then the group ring R[G] is left QF-3 iff R is left QF-3.

The following is known or can be seen easily.

Proposition 4. Let e be an idempotent of a ring R such that ${}_RRe$ and eR_R are both faithful R-modules. If R is left QF-3, then so is eRe.

Proposition 5. Let A/B be a Frobenius extension. Let $\{r_1,\dots,r_n\}$, $\{l_1,\dots,l_n\} \subset A$ and let h be a B-B-homomorphism of A to B such that $x = \sum_i h(xr_i)l_i = \sum_i r_ih(l_ix)$ for all x in A. Then the following statements hold.

1) $H = (h(l_i r_j))_{i,j}$ is an idempotent of the n x n matrix ring $(B)_n$ over B. Moreover, $(B)_n H$ and $H(B)_n$ are both faithful as left and right $(B)_n$-modules, respectively.

2) If B is left QF-3, then so is $\mathrm{End}(A_B)$.

Proof. One can see 1) by a direct computation. Noting $\mathrm{End}(A_B) \simeq H(B)_n H$, 2) is a consequence of Proposition 4 and 1).

Let A be a ring and G a finite group of ring automorphisms of A. Let $\Delta = \Delta(A;G)$ be the trivial crossed product of A relative to G: $\Delta = \oplus_{\sigma \varepsilon G} Au_\sigma$, $\{u_\sigma\}_{\sigma \varepsilon G}$ is a free generator for Δ over A; $au_\sigma . bu_\tau = a\sigma(b)u_{\sigma\tau}$. Then A is embeded in Δ as a subring by the mapping $a \to au_1$. It is easy to see that

$$d = \textstyle\sum_\sigma h(du_\sigma)u_{\sigma^{-1}} = \sum_\sigma u_\sigma h(u_{\sigma^{-1}}d) \qquad \text{for all } d \varepsilon \Delta$$

where $h: \Delta \to A$ is defined by $h(\sum a_\sigma u_\sigma) = a_1$. Therefore Δ/A is a Frobenius extension. Furthermore A can be viewed as a left Δ-module by the mapping

$$j: \Delta \to \mathrm{End}(A_B),\ j(\textstyle\sum a_\sigma u_\sigma)(x) = \sum a_\sigma \sigma(x) \qquad (x \varepsilon A)$$

and B is isomorphic to $\mathrm{End}({}_\Delta A)$ in a natural way, where $B = A^G$. If A/B is a G-Galois extension, then the above mapping j is an isomorphism and A/B is a Frobenius extension (see [4]). Noting the mention above, the following is a direct consequence of Proposition 3.

Proposition 6. A is left QF-3 iff Δ is left QF-3.

We are now ready to prove the theorem.

Proof of Theorem. Assume that B is left QF-3. Since A/B is a Frobenius extension, $\Delta(=\mathrm{End}(A_B))$ is left QF-3 by Proposition 5. Thus A is left QF-3 by Proposition 6. Conversely assume that A is left QF-3. Then Δ is left QF-3 by Proposition 6. Let ${}_{\Delta}U$ be a *-module. Then we can see that ${}_{B}\mathrm{Hom}({}_{\Delta}A, {}_{\Delta}U)$ is a *-module (see [3] for details). Hence B is left QF-3.

We shall close this note giving two examples which show some hypothesis about the relationship between A and A^G needed as far as we examine the inheritance of QF-3 property between A and A^G.

Example 1. Let

$$A = \begin{pmatrix} Q & 0 & 0 \\ Q & Z & 0 \\ Q & Q & Q \end{pmatrix}$$

be the subring of the 3 x 3 matrix ring $(Q)_3$, where Q denotes the field of rational numbers and Z the ring of integers. Let σ be the inner automorphism of A determined by the element $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, and let $G = \langle\sigma\rangle$. Then it is easy to see that

$$A^G = \begin{pmatrix} Q & 0 & 0 \\ Q & Z & 0 \\ 0 & 0 & Q \end{pmatrix}.$$

As mentioned in Tachikawa [7], A is left QF-3 as well as right QF-3 but A^G is neither left QF-3 nor right QF-3.

Example 2. Let A be the subring of the 2 x 2 matrix ring $(R)_2$ consisting of all elements of the form $\begin{pmatrix} x & 0 \\ y & x \end{pmatrix}$, $x \in Q$, $y \in R$, where R denotes the field of real numbers. Then A is a commutative ring without idempotents other than

0 and 1, but not self-injective. Thus A is not QF-3. Let σ be the automorphism of A given by $\sigma\begin{pmatrix} x & o \\ y & x \end{pmatrix} = \begin{pmatrix} x & o \\ -y & x \end{pmatrix}$, and let $G = \langle\sigma\rangle$. Then A^G coincides with the field consisting of all elements of the form $\begin{pmatrix} x & o \\ o & x \end{pmatrix}$, $x \in Q$.

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ON QF-2 ALGEBRAS WITH COMMUTATIVE RADICALS

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Group algebras (of finite groups over an algebraically closed field) with commutative radicals have been studied by several authors: D. A. R. Wallace [6, 7, 8], S. Koshitani [1] and K. Motose and Y. Ninomiya [3]. In particular, Wallace has given, in [8], a result which determines the structure of blocks of group algebras of this type. The most important part of his result may be stated in the following form:

Let A be a block of a group algebra of the type mentioned above. If the radical N of A is such that $N^2 \neq 0$, then A is a commutative completely primary algebra.

In this note we shall extend this result to the case of QF-2 algebras in the sense of R. M. Thrall [4], over an arbitrary field K. Besides, in connection with this, we shall also generalize Y. Tsushima's result [5, (2) → (3) of Theorem 4].

Theorem 1. Let A be a QF-2 algebra over a field K and let A be itself a block. Assume that the radical N of A is commutative and N^2 does not vanish. Then A is a completely primary almost symmetric algebra over K such that the residue class algebra A/N is a (commutative) field. Moreover, if the base field K is perfect, then A is a commutative completely primary symmetric algebra over K.

We can show that A is not necessarily commutative unless K is perfect.

Corollary. Let A be a weakly symmetric algebra over a field K and let A be itself a block. Assume that the radical N of A is commutative. Then A is of one of the following three types:

(1) A is a simple algebra over K.

(2) A is a full matrix ring over a completely primary weakly symmetric algebra B over K such that the square of the radical N' (= N $\cap$ B) of B vanishes. (In this case B/N' is a division algebra and N' is one-demensional as a left B/N'-space as well as a right one.)

(3) A is a completely primary almost symmetric algebra over K such that the residue class algebra A/N is a field.

If, in the corollary, we assume moreover that K is perfect, we can say something more: (a) When A is of type (2), there exists a division subalgebra D of B such that $D \cong B/N'$. B is expressible as a direct sum $D \oplus Dm$ (as a left D-space), where m is any (fixed) nonzero element in N'; furthermore, the multiplication in B is given by the rule $m^2 = 0$ and $m\alpha = \sigma(\alpha)m$ $(\alpha \in D)$, σ being an (algebra) automorphism of D. (b) When A is of type (3), then, by Theorem 1, A is a commutative completely primary symmetric algebra.

Now let K be an algebraically closed field and let A be an algebra satisfying the hypothesis of Corollary. Then A satisfies the hypothesis of the next Theorem 2, too. An algebra satisfying this (latter) condition is said to be of LC-type (see [5]). This theorem generalizes [5, (2) $\rightarrow$ (3) of Theorem 4].

Theorem 2. Let A be a finite dimensional algebra over an algebraically closed field and let A be itself a block. If the radical of A is generated over A (i. e. as an ideal of A) by the radical of its center, then A is a full matrix ring over the center of A.

The theorem has been also obtained independently by B. Külshammer [2].

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