PROCEEDINGS OF THE 13TH SYMPOSIUM ON RING THEORY

HELD AT OKAYAMA UNIVERSITY, OKAYAMA

SEPTEMBER 28—30, 1980

EDITED BY

HISAO TOMINAGA

WITH THE COOPERATION OF

SHIZUO ENDO　　MANABU HARADA

TAKASI NAGAHARA　　HIROYUKI TACHIKAWA

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PREFACE

This volume contains the papers presented at the 13th Symposium on Ring Theory held at Okayama University, September 28 - 30, 1980.

The annual Symposium on Ring Theory was founded in 1968. The main aims of the Symposium are to provide a means for the dissemination of recent theories on rings and modules which are not yet widely known and to give algebraists an oppotunity to report on recent progress in the ring theory.

The Symposium was organized by Professors Shizuo ENDO (Tokyo Metropolitan University), Manabu HARADA (Osaka City University), Hiroyuki TACHIKAWA (University of Tsukuba) and Hisao TOMINAGA (Okayama University); the 13th Symposium itself and these Proceedings were supported from the Grant-in-Aid for Scientific Research from the Ministry of Education , Science and Culture (Subject No. 534002) through the arrangements by Professor Yoshikazu NAKAI. I would like to take this oppotunity of thanking him for his arrangements. We hope these Proceedings will serve as a stimulus for the development of the ring theory.

January 1981

H. TOMINAGA

ON MODIFIED CHAIN CONDITIONS

Hiroaki KOMATSU

Throughout the present paper, A will represent a ring without (possibly with) identity, and $N = N(A)$ the prime radical of A. Given a left ideal I of A and an A-submodule M' of a left A-module M, for each positive integer i we set $I^{-i}M' = \{u \in M \mid I^i u \subseteq M'\}$. We note that for A-sbmodules M', M'' of M, $I^i M' \subseteq M''$ is equivalent to $M' \subseteq I^{-i}M''$. Following F. S. Cater [1], we say that a left A-module M is almost Artinian (resp. almost Noetherian) if for each infinite descending (resp. ascending) chain $M_1 \supseteq M_2 \supseteq \ldots$ (resp. $M_1 \subseteq M_2 \subseteq \ldots$) of A-submodules of M there exist positive integers m, q such that $A^q M_m \subseteq M_i$ (resp. $M_i \subseteq A^{-q}M_m$) for all i. If ${}_AA$ is almost Artinian (resp. almost Noetherian), we say that A is an almost left Artinian (resp. almost left Noetherian) ring. Obviously, for s-unital left A-modules (in particular, for completely reducible left A-modules), the concept of "almost Artinian" (resp. "almost Noetherian") coincides with that of "Artinian" (resp. "Noetherian").

In [1], Cater showed that many of well known properties of left Artinian (resp. left Noetherian) rings are also properties of almost left Artinian (resp. almost left Noetherian) rings. The present objective is to give a quick way to the preliminary results in [1], and to improve the main theorems in [1] with some addition. (For the details, refer to [3].)

We begin with improving Propositions 4 and 9 of [1] as follows.

Proposition 1. For a left A-module M, the following are equivalent:

1) M is almost Artinian (resp. almost Noetherian).

2) For each infinite descending (resp. ascending) chain $M_1 \supseteq M_2 \supseteq \dots$ (resp. $M_1 \subseteq M_2 \subseteq \dots$) of A-submodules of M there exists a positive integer p such that $A^pM_p = A^pM_i$ (resp. $A^{-p}M_p = A^{-p}M_i$) for all $i > p$.

3) In each non-empty family $\mathcal{M}$ of A-submodules of M such that $M' \in \mathcal{M}$ implies $AM' \in \mathcal{M}$ (resp. $A^{-1}M' \in \mathcal{M}$), there exists a minimal (resp. maximal) member.

Proof. As is easily seen, 3) $\Rightarrow$ 2) $\Rightarrow$ 1). Now, suppose 3) does not hold for some $\mathcal{M}$. Then we can find successively $M_i \in \mathcal{M}$ $(i = 1, 2, \dots)$ such that $M_{i+1} \subset A^iM_i$ (resp. $A^{-i}M_i \subset M_{i+1}$).

We give here a shorter proof of [1, Proposition 7].

Proposition 2. Let M be a left A-module, and M' an A-submodule of M.

(1) ${}_AM$ is almost Artinian if (and only if) both ${}_AM'$ and ${}_AM/M'$ are almost Artinian.

(2) ${}_AM$ is almost Noetherian if (and only if) both ${}_AM'$ and ${}_AM/M'$ are almost Noetherian.

Proof. (1) Let $M_1 \supseteq M_2 \supseteq \dots$ be an arbitrary descending chain of A-submodules of M. There exists a positive integer p such that $M_p \cap M' \subseteq A^{-p}(M_i \cap M')$ and $M_p + M' \subseteq A^{-p}(M_i + M')$ for all i. Therefore, for every $i > p$ we have $M_p \subseteq A^{-p}M_p \cap (M_p + M') \subseteq A^{-p}M_p \cap A^{-p}(M_i + M') = A^{-p}(M_p \cap (M_i + M')) = A^{-p}(M_i + (M_p \cap M')) \subseteq A^{-2p}M_i$.

(2) Let $M_1 \subseteq M_2 \subseteq \cdots$ be an arbitrary ascending chain of A-submodules of M. There exists a positive integer p such that $A^p(M_i \cap M') \subseteq M_p \cap M'$ and $A^pM_i + M' \subseteq M_p + M'$ for all i. Therefore, for every $i > p$ we have $M_p \supseteq A^pM_p + (M_p \cap M') \supseteq A^pM_p + A^p(M_i \cap M') = A^p(M_p + (M_i \cap M')) = A^p(M_i \cap (M_p + M')) \supseteq A^{2p}M_i$.

Next, we reprove [1, Theorem 1].

Theorem 1. If A is almost left Artinian, then A is semiprimary, namely N is nilpotent and A/N is Artinian (semisimple).

Proof. Suppose contrarily that N is not nilpotent. By the condition 3) of Proposition 1, N contains a minimal non-nilpotent left ideal I. Consider the family of all left subideals I' of N with $II' \neq 0$. Then, again by Proposition 1, the family contains a minimal member I^*. Since $II^* = I^*$, there exists $a^* \in I^*$ such that $Ia^* = I^*$. Hence, $aa^* = a^*(\neq 0)$ with some $a \in I$. Obviously, a is not nilpotent. But this contradicts the fact that N is nil. Hence, N is nilpotent. Thus, it suffices to show that if A is semiprime and almost left Artinian then A is Artinian semisimple. Since, by Proposition 1, every non-zero left ideal of A contains a minimal left ideal, the left socle S of A is essential in ${}_AA$. Since ${}_AS$ is completely reducible and Artinian and every minimal left ideal of A is generated by an idempotent, we see that S itself is generated by an idempotent. Hence S coincides with A, whence we can conclude the assertion.

According to Theorem 1, an almost left Artinian ring A has a principal idempotent e (i.e., $e+N$ is the identity of A/N). We consider the principal Peirce decomposition: $(A,+) = (1-e)Ae \oplus eAe \oplus eA(1-e) \oplus (1-e)A(1-e)$, where $1-e$ is used symbolically. Then the ring A is isomorphic to the generalized matrix ring $\begin{pmatrix} eAe & eA(1-e) \\ (1-e)Ae & (1-e)A(1-e) \end{pmatrix}$. It is easy to see that eAe is a left Artinian ring with identity e and $(1-e)A(1-e)$ is a nilpotent ring. Similarly, the unital left eAe-module $eA(1-e)$ is Artinian and finitely generated. Conversely, let A be a generalized matrix ring $\begin{pmatrix} R & K \\ L & S \end{pmatrix}$, where R is a left Artinian (resp. left Noetherian) ring with identity, S a nilpotent ring, ${}_RK_S$ a finitely generated unital R-module, and ${}_SL_R$ a unital R-module, and the multiplications of K and L are defined by some $(\ ,\ ) : {}_RK \otimes_S L_R \to {}_RN(R)_R$ and $[\ ,\ ] : {}_SL \otimes_R K_S \to {}_SS_S$ such that $(k,\ell)k' = k[\ell,k']$ and $\ell(k,\ell') = [\ell,k]\ell'$ for all $k, k' \in K$ and $\ell, \ell' \in L$. Then A is almost left Artinian (resp. almost left Noetherian). In fact, since $\begin{pmatrix} R & K \\ 0 & 0 \end{pmatrix}$ is an Artinian (resp. Noetherian) left $\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}$-module, $\begin{pmatrix} R & K \\ 0 & 0 \end{pmatrix}$ is a left Artinian (resp. left Noetherian) ring. Now, let $S^n = 0$. Since $A^n = \begin{pmatrix} R & K \\ L & LK \end{pmatrix} = \begin{pmatrix} R & K \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix}\begin{pmatrix} R & K \\ 0 & 0 \end{pmatrix}$, for any left ideal I of A we get $A^nI = \begin{pmatrix} R & K \\ 0 & 0 \end{pmatrix}I \oplus \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix}\begin{pmatrix} R & K \\ 0 & 0 \end{pmatrix}I$. Since $\begin{pmatrix} R & K \\ 0 & 0 \end{pmatrix}I$ is a left ideal of $\begin{pmatrix} R & K \\ 0 & 0 \end{pmatrix}$, A is almost left Artinian (resp. almost left Noetherian) by the condition 2) of Proposition 1.

Recalling that a left Artinian ring with identity is left Noetherian, we readily obtain

Corollary 1. Every almost left Artinian ring is almost left Noetherian.

Next, we state the following that includes Theorems A and B of [1].

Theorem 2. Let I and I_j $(j=1, \ldots, k)$ be left ideals of A, and M a left A-module.

(1) If ${}_A A/I$ is completely reducible and $IM = 0$, then the following are equivalent:

a) ${}_A M$ is almost Artinian.

b) ${}_A AM$ is Artinian.

c) ${}_A AM$ is finitely generated.

d) ${}_A AM$ is Noetherian.

e) ${}_A M$ is almost Noetherian.

(2) If ${}_A A/I_j$ is completely reducible $(j=1, \ldots, k)$ and $I_1 \ldots I_k M = 0$, then the following are equivalent:

a) ${}_A M$ is almost Artinian.

b) ${}_A AM/I_k M$, ${}_A AI_k M/I_{k-1} I_k M$, ... , ${}_A I_2 \ldots I_k M =$ ${}_A AI_2 \ldots I_k M/I_1 \ldots I_k M$ are all finitely generated.

c) ${}_A M$ is almost Noetherian.

(3) Assume that ${}_A A/I$ is completely reducible. If ${}_A M$ is almost Artinian and for any non-zero A-submodule M' of M, $M' \neq IM'$ then ${}_A M$ is almost Noetherian.

(4) Assume that ${}_A A/I$ is completely reducible. If ${}_A M$ is almost Noetherian and for any proper A-submodule M' of M, $M' \neq I^{-1}M'$ then M is almost Artinian.

Proof. (1) It is easy to see that ${}_A AM$ is completely reducible. Therefore, (1) is obvious.

(2) Observe the descending chain

$$M \supseteq I_k M \supseteq I_{k-1} I_k M \supseteq \cdots \supseteq I_2 \cdots I_k M \supseteq I_1 \cdots I_k M = 0.$$

Then the assertion can be proved by (1) and Proposition 2.

(3) By Proposition 1, there exists the smallest member M' among the A-submodules U of M such that ${}_A M/U$ is almost Noetherian. Since ${}_A M'/IM'$ is almost Artinian, ${}_A M'/IM'$ is almost Noetherian by (1). Since $IM' = M'$ by the minimality of M', we get $M' = 0$. Hence ${}_A M$ is almost Noetherian.

(4) By Proposition 1, there exists the greatest almost Artinian A-submodule M' of M. Since ${}_A I^{-1}M'/M'$ is almost Noetherian, ${}_A I^{-1}M'/M'$ is almost Artinian by (1). The maximality of M' yields $I^{-1}M' = M'$, and so $M' = M$. Hence ${}_A M$ is almost Artinian.

The next is an immediate consequence of Theorem 2.

Corollary 2. Let A be a semiprimary ring, and M a left A-module. Then ${}_A M$ is almost Artinian if and only if ${}_A M$ is almost Noetherian.

We consider here almost left Noetherian rings.

Theorem 3. Let A be an almost left Noetherian ring.

(1) A satisfies the ascending chain condition for semiprime ideals.

(2) Every nil subring of A is nilpotent and the nilpotency indices of nil subrings are bounded.

Proof. (1) The proof is straightforward.

(2) There exists a positive integer q such that $A^q r(A^i) \subseteq r(A^q)$, and hence $r(A^i) \subseteq r(A^{2q})$ for all i. Since $A/r(A^{2q})$ is left unital and almost left Noetherian,

it is easy to see that $A/r(A^{2q})$ is a left Goldie ring. According to [2, Corollary 1.7], there exists a positive integer n such that $K^n \subseteq r(A^{2q})$ for all nil subrings K of A. It is immediate that $K^{2q+n} = 0$.

Finally, we shall give necessary and sufficient conditions for a ring to be almost left Artinian. A left ideal I of A is said to be almost maximal if A/I is a sum of minimal left A-modules. If a prime ideal P is an almost maximal left ideal, then ${}_AA/P$ is completely reducible. The following includes Theorems 5, 6 and 11 of [1].

Theorem 4. The following are equivalent:

1) A is almost left Artinian.

2) A is π-regular and almost left Noetherian, and A/N is left s-unital.

3) A is almost left Noetherian and A/N is left Artinian.

4) A is almost left Noetherian and every proper prime ideal of A is an almost maximal left ideal.

5) N is nilpotent, ${}_AAN^{i-1}/N^i$ is finitely generated for all $i > 0$, A satisfies the ascending chain condition for semiprime ideals, and every proper prime ideal of A is an almost maximal left ideal.

6) N is nilpotent and ${}_AAN^{i-1}/N^i$ is Artinian for all $i > 0$.

Proof. 1) $\Leftrightarrow$ 3) $\Leftrightarrow$ 6). Under any of the conditions 1), 3), 6), N is nilpotent and ${}_AA/N$ is completely reducible. Hence, these conditions are equivalent by Theorem 2.

1) $\Rightarrow$ 2). By Theorem 1 and [5, Lemma 2], A is π-regular.

2) $\Rightarrow$3). Since A/N is left unital and almost left Noetherian, it is easy to see that A/N is a left Goldie ring. Therefore, as was claimed in the proof of [6, Theorem 3], A/N contains the identity. Moreover, it is easy to see that every regular element of A/N is a unit. Hence, A/N coincides with its left quotient ring that is Artinian semisimple.

3) $\Rightarrow$4) and 5). Obvious.

4) or 5) $\Rightarrow$3). By [4, Theorem 3], $N = \bigcap_{i=1}^{k} P_i$ with some prime ideals P_i. Since ${}_A A/P_i$ is completely reducible, ${}_A A/P_i$ is Artinian for all i. Hence ${}_A A/N$ is Artinian.

References

[1] F. S. Cater: Modified chain conditions for rings without identity, Yokohama Math. J. 27 (1979), 1-22.

[2] J. W. Fisher: Nil subrings with bounded indices of nilpotency, J. Algebra 19 (1971), 509-516.

[3] H. Komatsu and H. Tominaga: On modified chain conditions, Math. J. Okayama Univ. 22 (1980), 131-139.

[4] H. Tominaga: Some remarks on radical ideals, Math. J. Okayama Univ. 3 (1954), 139-142.

[5] H. Tominaga: Some remarks on π-regular rings of bounded index, Math. J. Okayama Univ. 4 (1955), 135-141.

[6] H. Tominaga: On s-unital rings, Math. J. Okayama Univ. 18 (1976), 117-134.

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CHARACTERIZATIONS OF RINGS WITH TRIVIAL PRERADICAL IDEALS

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Let R be a ring with identity and R-mod the category of all unital left R-modules. A functor σ: R-mod $\longrightarrow$ R-mod is called a preradical if $\sigma(M)$ is a submodule of M for each $M \in$ R-mod and $\sigma(M)\alpha \subset \sigma(N)$ for each morphism α: $M \longrightarrow N$ in R-mod. A preradical σ is called an idempotent preradical (resp. a radical) if $\sigma(\sigma(M)) = \sigma(M)$ (resp. $\sigma(M/\sigma(M)) = 0$) for all $M \in$ R-mod. A preradical is called left exact (resp. cohereditary) if it is kernel preserving (resp. epi-preserving). Every left exact (resp. cohereditary) preradical is idempotent (resp. a radical). A preradical is called a cotorsion radical (resp. an exact radical) if it is an idempotent cohereditary radical (resp. a left exact cohereditary radical).

We call an ideal I of R a preradical ideal if there exists a preradical σ for R-mod such that $\sigma(R) = I$. A preradical ideal of a left exact preradical (resp. a left exact radical) is nothing but a pretorsion ideal (resp. a torsion ideal) in the sence of [4]. From now on, we shall study the rings which have no non-trivial preradical ideals $\sigma(R)$, where we take σ as an idempotent preradical (or an exact radical, etc) for R-mod, and give several characterizations of those rings. Note that, for a preradical σ for R-mod, $\sigma(R) = R$ if and only if $\sigma = 1$, where 1 stands for the identity functor for R-mod. Hence we may rephrase our question as: When the preradical ideals $\sigma(R)$ vanish for various types of preradicals $\sigma \neq 1$ for R-mod ?

To bigin with, we have

Proposition 1. The following properties are equivalent for a ring R:

(1) $\sigma(R) = 0$ for every preradical $\sigma \neq 1$ for R-mod.

(2) $\sigma(R) = 0$ for every radical $\sigma \neq 1$ for R-mod.

(3) $\sigma(R) = 0$ for every cohereditary radical $\sigma \neq 1$ for R-mod.

(4) There exist only two cohereditary radicals for R-mod.

(5) R is a simple ring (i.e. it has exactly two ideals).

(6) Every nonzero (cyclic) left R-module is faithful.

(7) RK = R for every nonzero right ideal K of R.

Definition 1. A ring R is called left G if $\sigma(R) = 0$ for every idempotent preradical $\sigma \neq 1$ for R-mod.

Theorem 2. The following properties are equivalent for a ring R:

(1) R is a left G-ring.

(2) Every left R-module M with $\mathrm{Hom}_R(M, R) \neq 0$ is a generator for R-mod.

(3) Every nonzero torsionless left R-module is a generator for R-mod.

(4) Every nonzero submodule of a projective left R-module is a generator for R-mod.

(5) Every nonzero left ideal of R is a generator for R-mod.

(6) Every nonzero ideal of R is a generator for R-mod.

Remark 1. (1) A property that a ring is left G is Morita invariant.

(2) Every (simple) left R-module is a generator for R-mod if and only if R is simple artinian ([6]).

(3) R is a left G-ring with nonzero (left) socle if and only if R is simple artinian.

(4) Every nonzero left ideal of R is a progenerator for R-mod if and only if R is left hereditary left Noetherian prime ring without non-trivial idempotent ideals ([7]).

(5) If R is left hereditary, then R is left G if and only if every nonzero projective left R-module is a generator for R-mod.

(6) If R is a left G-ring, then the maximal left ring of quotients Q_{max} of R is simple and left self-injective by [4, Prop. 6.2]. In particular Q_{max} is also a left G-ring. If R is a left G-ring and the classical left ring of quotients Q_{cl} of R exists, then Q_{cl} is also a left G-ring.

Now we shall consider some generalizations of left G-rings.

Definition 2. A ring R is called left FGG (resp. left CG) if every finitely generated (resp. cyclic) left ideal of R is a generator for R-mod.

Proposition 3. The following properties are equivalent for a ring R:

(1) R is left FGG.

(2) Every finitely generated left R-module M with $Hom_R(M, R) \neq 0$ is a generator for R-mod.

(3) Every nonzero finitely generated torsionless left R-module is a generator for R-mod.

(4) Every nonzero finitely generated submodule of a projective left R-module is a generator for R-mod.

(5) For each positive integer n, the ring R_n of n×n matrices over R is left CG.

Proposition 4. The following properties are equivalent for a ring R:

(1) R is left CG.

(2) $Ra^{lr} = R$ for every nonzero $a \in R$, where $a^{lr} = \mathrm{Ann}_R^r(\mathrm{Ann}_R^l(a))$.

(3) RK = R for every nonzero annihilator right ideal K (i.e. $K = \mathrm{Ann}_R^r(X)$ for some subset X of R) of R.

(4) Every cyclic left R-module M with $\mathrm{Hom}_R(M, R) \neq 0$ is a generator for R-mod.

(5) Every nonzero cyclic torsionless left R-module is a generator for R-mod.

(6) Every nonzero cyclic submodule of a projective left R-module is a generator for R-mod.

Definition 3. A ring R is called left EG (resp. left SSP [3]) if every essential left ideal of R is a generator for R-mod (resp. cofaithful).

Proposition 5. The following properties are equivalent for a ring R:

(1) R is left EG.

(2) Every ideal which is essential in R as a left ideal is a generator for R-mod.

(3) Every module ${}_RQ$ satisfying that $t_Q(R)$ is an essential left ideal is a generator for R-mod, where $t_Q(R)$

denotes the trace ideal of Q.

Remark 2. (1) Let $R = \oplus_{i=1}^{n} R_i$ be a direct sum of rings $\{R_i\}_{i=1}^{n}$. Then R is left EG if and only if each R_i is left EG.

(2) If R is a left EG-ring and the classical left ring of quotients Q_{cl} of R exists, then Q_{cl} is also a left EG-ring.

Definition 4. A ring R is left R [2] (left SP [4]) (left CTF [4]) if $\sigma(R) = 0$ for every idempotent radical (left exact preradical) (left exact radical) $\sigma \neq 1$ for R-mod.

Proposition 6 ([2, Prop. 1.10]). The following properties are equivalent for a ring R:

(1) R is a left R-ring.

(2) $\mathrm{Hom}_R(I, R/I) \neq 0$ for every non-trivial left ideal I of R.

(3) $\mathrm{Hom}_R(I, M) \neq 0$ for every nonzero left ideal I of R and nonzero $M \in$ R-mod.

Proposition 7 ([4, p2], [8, Theorem 1.7], [9, Theorem 2.1] and [1, Prop. 3.2]). The following properties are equivalent for a ring R:

(1) R is a left SP-ring.

(2) For every finitely generated projective left R-module P and $0 \neq N \subset P$, there exists an embedding $P \longrightarrow N^{(n)}$ for some integer n.

(3) Every nonzero left ideal of R is cofaithful.

(4) Every nonzero left ideal od R generates the injective hull $E({}_RR)$ of ${}_RR$.

(5) R is a left non-singular prime ring, and every non-singular quasi-injective left R-module is injective.

Proposition 8 ([2, Theorem 2.4] and [4, p91]). The following properties are equivalent for a ring R:

(1) R is a left CTF-ring.

(2) For every non-trivial left ideal I of R, there exist $x \in I$, $y \in R \setminus I$ such that $(0:x) \subset (I:y)$.

(3) Every nonzero injective left R-module is faithful.

Remark 3. (1) A ring is left G if and only if it is both left EG and left R.

(2) Every left G-ring R is left SP. The converse holds if R is left self-injective.

Example 1. Every simple ring is a left G-ring, but the converse is not true. The ring Z_n of n×n matrices over the ring Z of integers is a left and right G-ring which is not simple.

Example 2. Every left G-ring is a left R-ring, but the converse is not true. For a counter example, we may take the ring $R = Z/(p^n)$, where p is a prime and n is an integer greater than 1.

Example 3. Every left G-ring is left FGG. Every left FGG-ring is left CG, but the converse is not true. In fact we shall give an example of a left CG-ring R having a finitely generated essential left ideal which is not a generator for R-mod. Let $R = K[x,y]$ be a polynomial ring over a field K. Then R is a (left) CG-ring with an ideal $I = (x,y)$ generated by x and y, which is essential in R and is not a generator for R-mod.

Example 4. Every left CG-ring is left SP, but the converse is not true. Let $D = Z_2[x_1, x_2, x_3, , , ,]$ be the free non-commuting Z_2-algebra on x_i (i=1,2,3,,,,). Let I be the two-sided ideal in D generated by monomials of the form $x_i x_j x_k$ with $i < j < k$. As is shown in [4, p9], R = D/I is left SP. One can check that the cyclic left ideal $A = (Dx_3 + I)/I$ of R is not a generator for R-mod.

Example 5. Every left G-ring is left EG, but the converse is not true. In fact, $R = Z \oplus Z$ is a (left) EG-ring, but is not prime. One may expect that, if R is a left EG-ring, then every (essential submodule of a) projective left R-module is a generator for R-mod. But this is not true. Once again let $R = Z \oplus Z$, and consider the ideal $I = (Z, 0)$ of R. Clearly ${}_RI$ is projective, but an easy verification shows that $t_I(R) = I$, which means ${}_RI$ is not a generator for R-mod.

Definition 5. A ring is called left C2 (resp. left E2) if $\sigma(R) = 0$ for every cotorsion radical (resp. exact radical) $\sigma \neq 1$ for R-mod.

Definition 6. We shall call that an ideal I of a ring R is left strongly idempotent, if $J = IJ$ holds for every left ideal $J \subset I$.

Proposition 9. The following properties are equivalent for a ring R:

(1) R is left C2 (and hence C2)

(2) There exist only two cotorsion radicals for R-mod.

(3) R has no non-trivial idempotent ideals.

Theorem 10. The following properties are equivalent for a ring R:

(1) R is a left E2-ring.

(2) There exist only two exact radicals for R-mod.

(3) If a nonzero injective module ${}_RE$ satisfies the condition that, for a left ideal K, $\mathrm{Hom}_R(R/K, E) = 0$ implies $K + \mathrm{Ann}_R(E) = R$, then E is faithful.

(4) There are no non-trivial ideals I such that $IN = N \cap IM$ for each ${}_RN \subset {}_RM$.

(5) There are no non-trivial (idempotent) ideals I such that $(R/I)_R$ are flat.

(6) R has no non-trivial left strongly idempotent ideals.

Remark 4. (1) A property that a ring is left E2 is Morita invariant.

(2) A ring is simple if and only if it is left E2 and left weakly regular.

(3) Every left E2-ring is indecomposable as a ring.

(4) Put $T = \{{}_RM \mid M$ is projective and completely reducible$\}$. It is known that T is a TTF-class. Hence if R is left E2, we have $T =$ R-mod or $T = \{0\}$. Thus if R is not simple artinian, then every simple left R-module is not projective.

Example 6. Every left R-ring is C2, but the converse is not true. For a counter example, consider $S = Z \times Q$, where Z is the ring of integers and Q the field of rational numbers. Define the addition on S by component wise and the multiplication on S by

$$(z_1, q_1) * (z_2, q_2) = (z_1 z_2, z_1 q_2 + z_2 q_1)$$

Then S becomes a commutative ring without non-trivial idempotent ideals, but as is shown in [2] S is not an R-ring.

Example 7. Clearly every left strongly idempotent ideal is idempotent, but the converse is not true. Let R be the ring of 2×2 upper triangular matrices over a field K. One can check that $\begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$ is idempotent but not left strongly idempotent. On the other hand, $\begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$ is left strongly idempotent.

Example 8. There is a right E2-ring which is not left E2. Let $D = F[x,y]$ be the free non-commuting algebra on $\{x,y\}$ over a field F. Then $DxD = \oplus_{i=0}^{\infty} y^i xD \cong \oplus_{i=0}^{\infty} D_D$. The ring $R = \mathrm{End}(DxD_D)$ is right SP ([4, Example 13.2]) and so is right E2. But R contains a non-trivial left strongly idempotent ideal $K = \oplus_{i=0}^{\infty} e_i R$, where e_i denotes the matrix with 1 in the (i,i) position, 0 elsewhere.

Example 9. If R is a left CTF-ring, then every non-zero flat right R-module is faithful ([4 Prop. 13.9]). If R has this property, then R is left E2 by Theorem 10. But the converse is not true. Let

$$R = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & a \end{pmatrix} \,\middle|\, a,b,c,d,e \in K \right\},$$

where K is a field. One can check that there are only two non-trivial idempotent ideals

$$I_1 = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ d & e & a \end{pmatrix} \,\middle|\, a,b,d,e \in K \right\} \quad \text{and} \quad I_2 = \begin{pmatrix} 0 & 0 & 0 \\ K & K & 0 \\ K & K & 0 \end{pmatrix}$$

Put $J_1 = \begin{pmatrix} 0 & 0 & 0 \\ K & 0 & 0 \\ K & K & 0 \end{pmatrix} \subset I_1$ and $J_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ K & 0 & 0 \end{pmatrix} \subset I_2$. Then $J_i \neq$

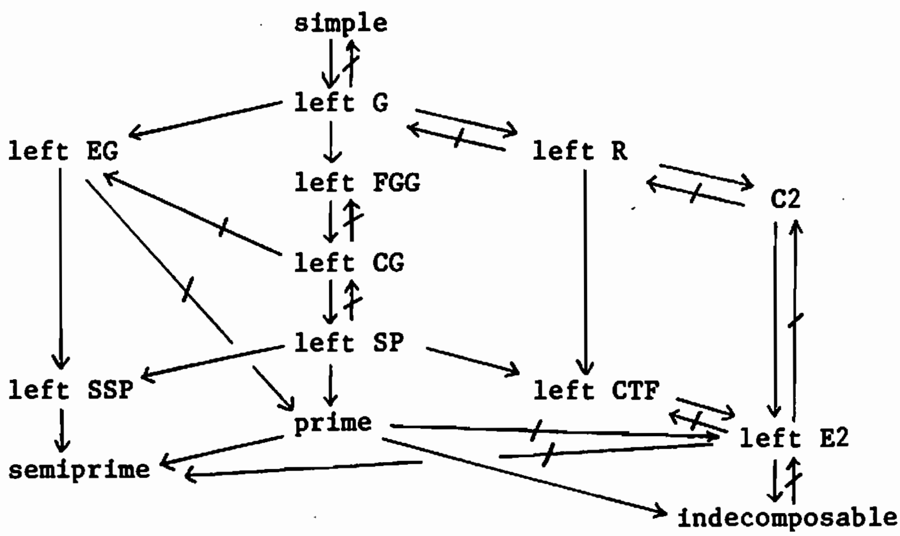
I_iJ_i (i=1,2). Thus I_i (i=1,2) are not left strongly idempotent ideals. This gives an example of left E2-ring which is not C2. The same argument shows R is also a right E2-ring. Now put

$$A = \begin{pmatrix} 0 & 0 & 0 \\ K & K & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ d & e & a \end{pmatrix} \;\middle|\; a,d,e \in K \right\}.$$

Then $R = A \oplus B$, and so A_R is flat. But A_R is not faithful. Finally, we remark that R is not semiprime.

Example 10. We give an example of a prime ring which is not left E2. Let V_D be an infinite dimensional vector space over a division ring D. Put $R = \mathrm{End}(V_D)$. Then R is a regular and prime ring. Put $I = \mathrm{soc}(R)$, then I consists of $f \in R$ such that $\mathrm{Im}(f)_D$ is finite dimensional. Thus I is a non-trivial (left) strongly idempotent ideal. One may remark that ${}_RI$ is not cofaithful, and so R is not a left SSP-ring.

A table of rings



References

[1] J. A. Beachy and W. D. Blair: Rings whoso faithful left ideals are cofaithful, Pacific J. Math. 58 (1975), 1-13.

[2] L. Bican, P. Jambor, T. Kepka and P. Němec: On rings with trivial torsion parts, Bull. Austral. Math. Soc. 9 (1973), 275-290.

[3] D. Handelman: Strongly semiprime rings, Pacific J. Math. 60 (1975), 115-122.

[4] D. Handelman, K. R. Goodearl and J. Lawrence: Strongly prime and completely torsionfree rings, Carleton Mathematical Series 109, Carleton University, 1974.

[5] H. Katayama: A study of rings with trivial preradical ideals, preprint.

[6] F.F. Mbuntum and K. Varadarajan: Half-exact preradicals, Comm. Alg. 5 (1977), 555-590.

[7] J. C. Robson: A note on Dedekind prime rings, Bull. London Math. Soc. 3 (1971). 42-46.

[8] R. A. Rubin: Absolutely torsion-free rings, Pacific J. Math. 46 (1973), 503-514.

[9] J. Viola-Prioli: On absolutely torsion-free rings, Pacific J. Math. 56 (1975), 275-283.

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WEAKLY REGULAR MODULES

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This note is an abstract of the author's paper [2]. The principal objective of [2] is to give the conditions that some modules in the injective hull of a weakly regular module become weakly regular modules, and to give ones that some rings in the injective hull of the endomorphism ring of a weakly regular module become left weakly regular rings.

1. Preliminaries. Throughout this note, R will represent an associative ring with identity, and M a unitary right R-module. We set $M^* = \mathrm{Hom}_R(M,R_R)$ and $S = \mathrm{Hom}_R(M,M)$. Every (right or left) module A is unitary, and furthermore $E(A)$, $Z(A)$, $J(A)$, T_A and $d(A)$ denote the injective hull of A, the singular submodule of A, the Jacobson radical of A, the trace ideal of A and the Goldie dimension of A, respectively. $T = T_M$ is the trace ideal of M. Next, we set $\sqrt{M} = \cap \mathrm{Ker}\, f$, where f runs over all elements in $\mathrm{Hom}_R(E(M), E(M) \oplus E(R_R))$ with $f(M) = 0$, and $\sqrt{S} = \mathrm{Hom}(\sqrt{M},\sqrt{M})$. Finally, we set $Q(M,R) = \cap \mathrm{Ker}\, g$, where g runs over all elements in $\mathrm{Hom}_R(E(R_R),E(M) \oplus E(R_R))$ with $g(R) = 0$.

A right R-module M is called a weakly regular module (abbr. w.regular module) if $m \in S(m)M^*(m)$ for every $m \in M$, and a ring Y is called a left (right) weakly regular ring if $z \in YzYz$ $(z \in zYzY)$ for every $z \in Y$.

2. Weakly regular modules. In this section we state the main results of [2] without proof.

Proposition 1. If M_R is w.regular, then there hold the following:

(a) $Z(M) = 0$.

(b) $MT = M$ and $T^2 = T$.

(c) $mT \neq 0$ for every non-zero $m \in M$.

(d) $J(S_S) = 0$.

(e) $Z(S_S) = 0$.

(f) $E(S_S) \underset{f}{\cong} \mathrm{Hom}_R(E(M), E(M))$ (ring isomorphism), and $f|S$ is an identity map.

(g) $J(M) = 0$.

Proposition 2. Let M_R be a w.regular module, and W a ring with $S \subseteq W \subseteq E(S_S)$. Let a be a right W-submodule in $E(S_S)$. Then there hold the following:

(a) W is a semiprime ring.

(b) $d_W(a_W) = d_S(a_S)$.

(c) $Z_W(W_W) = 0$.

Theorem 3. Let M_R be a w.regular module, and W a ring with $S \subseteq W \subseteq E(S_S)$. If $G(W) = \{a \in W \mid d_W(aW_W) < \infty\}$, then there hold the following:

(a) $G(W)$ is a two-sided ideal of W.

(b) $G(W)$ is a left weakly regular ring.

Corollary 4. Let M_R be a w.regular module with $d_R(M) < \infty$, and W a ring with $S \subseteq W \subseteq E(S_S)$. Then $W = W_1 \oplus \dots \oplus W_p$, where W_i is W-W-simple.

Proposition 5. (a) $Q(M,R)$ is an intermediate ring between R and its maximal right quotient ring (see [3, Proposition 1.3]).

(b) $\sqrt{M}$ is a right $Q(M,R)$-module.

(c) If N is a right R-module with $M \subseteq N \subseteq \overline{M}$, then $\mathrm{Hom}_R(N, \overline{M}) \subseteq \sqrt{S}$, and especially $S \subseteq \sqrt{S}$.

(d) $\mathrm{Hom}_R(M,R) \subseteq \mathrm{Hom}_{Q(M,R)}(\sqrt{M},Q(M,R))$.

(e) $\sqrt{S} = \mathrm{Hom}_{Q(M,R)}(\sqrt{M},\sqrt{M})$.

Furthermore, if W is a ring with $S \subseteq W \subseteq \sqrt{S}$, then

(f) $M \subseteq W(M) \subseteq \sqrt{M}$.

Proposition 6. If W is a ring with $S \subseteq W \subseteq \sqrt{S}$, and $Q(W(M)) = \{x \in Q(M,R) \mid W(M)x \subseteq W(M)\}$, then there hold the following:

(a) $Q(W(M))$ is an intermediate ring between R and $Q(M,R)$.

(b) $W(M)$ is a right $Q(W(M))$-module.

(c) $\mathrm{Hom}_R(M,R) \subseteq \mathrm{Hom}_{Q(W(M))}(W(M),Q(W(M)))$.

(d) $S \subseteq \mathrm{Hom}_{Q(W(M))}(W(M),W(M))$.

Furthermore, if M_R is finitely generated projective (abbr. f.g. projective), then

(e) $W(M) = MQ(W(M))$.

(f) The trace ideal of $W(M)$ as a right $Q(W(M))$-module is $Q(W(M))TQ(W(M))$.

Theorem 7. Let M_R be a w.regular module, and W a ring with $S \subseteq W \subseteq \sqrt{S}$. If $G(W(M)) = \{x \in W(M) \mid d_R(xR) < \infty\}$ and $Q(G(W(M)) = \{x \in Q(W(M)) \mid G(W(M))x \subseteq G(W(M))\}$, then there hold the following:

(a) $G(W(M))$ is a left W- and right $Q(G(W(M)))$-module.

(b) $G(W(M))$ is a w.regular module as a right $Q(G(W(M)))$-module.

Corollary 8. If M_R is a w.regular module, $d(R_R) < \infty$

and W is a ring with $S \subseteq W \subseteq \sqrt{S}$, then $W(M)$ is a w.regular module as a right $Q(W(M))$-module (see Proposition 6 (b)).

Corollary 9. Let M_R be a w.regular module, $d(R_R) < \infty$, and W a ring with $S \subseteq W \subseteq \sqrt{S}$. If $W = \mathrm{Hom}_R(W(M), W(M))$, then the center of W is a regular ring.

Proposition 10. If M_R is a f.g.w.regular module and W is a ring with $S \subseteq W \subseteq \sqrt{S}$, then the following conditions are equivalent:

(a) $W(M)$ is a w.regular module as a right $Q(W(M))$-module (see Proposition 6 (b)).

(b) $Q(W(M))TQ(W(M))\mathfrak{a} = Q(W(M))TQ(W(M))\mathfrak{a}^2$ for every left ideal $\mathfrak{a}$ of $Q(W(M))$ (see Proposition 6 (f)).

(c) W is a left weakly regular ring and $x \in WxQ(W(M))TQ(W(M))$ for every $x \in W(M)$.

References

[1] T. Mabuchi: Weakly regular modules, Osaka J. Math. 17 (1980), 35-40.

[2] T. Mabuchi: Weakly regular modules II, to appear.

[3] A. I. Kasu: Bicommutators of fully divisible modules, Math. Sbornik 29 (1976), 431-440.

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REGULAR MODULES AND V-MODULES

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The notion of regularity has been extended to modules by D. Fieldhouse [5], R. Ware [18] and J. Zelmanowitz [19]. In this paper, following Zelmanowitz [19], we call a right R-module M regular if given any $m \in M$ there exists $f \in \mathrm{Hom}_R(M,R)$ with $mf(m) = m$. G. Michler and O. Villamayor [14] have shown that the following are equivalent: (1) every simple right R-module is injective, (2) every right R-module is semisimple and (3) every right ideal of R is an intersection of maximal right ideals. If a ring R satisfies these equivalent conditions, R is called a right V-ring. The notion of V-rings has been extended to modules by V. S. Ramamurthi [15] and H. Tominaga [17]. In this paper, following Tominaga [17], we call a module M_R a V-module if every R-submodule is an intersection of maximal R-submodules. Such a module M_R has also been called "co-semisimple" by K. R. Fuller [8]. The connections between the class of regular rings and the class of V-rings are studied by many authors (see the references of [6]). In this paper, we consider the connections between the class of regular modules and the class of V-modules, and we study the relationship between these modules and their endomorphism rings.

Throught this paper, R will denote a ring with identity and all modules considered are unitary right R-modules. For any module M, M^* denotes $\mathrm{Hom}_R(M,R)$, and $S = S(M)$ denotes $\mathrm{End}_R(M)$. We denote by $Z(M)$ and $J(M)$ the singular submodule of M and the Jacobson radical of M, respectively. And we say that M is semisimple if $J(M) = 0$. We set $\mathrm{Ann}_R(M) = \{r \in R \mid Mr = 0\}$. The homomorphisms $(\, , \,) : M^* \otimes_S M \longrightarrow R$ with $(f,m) = f(m)$ and $[\, , \,] : M \otimes_R M^* \longrightarrow S$ with $[m,f] = mf$ are R-R-linear and S-S-linear respectively. As is well known, (S, M^*, M, R) with these homomorphisms forms a Morita context. The images (M^*,M) and $[M,M^*]$ will be denoted by T and , respectively. We denote by $U({}_SM_R)$ (resp. $U(M_R)$) the lattice of S-R-submodules (resp. R-submodules) of M, and by $U_T({}_RR)$ (resp. $U_T({}_RR_R)$) the lattice of all left ideals (resp. ideals) I of R with $TI = I$. Further, $U_\Delta(S_S)$ (resp. $U_\Delta({}_SS_S)$) denotes the lattice of all right ideals (resp. ideals) K of S with $K\Delta = K$. Given R-modules M and N, we set $T_M(N) = \sum\{\mathrm{Im}(f) \mid f \in \mathrm{Hom}_R(M,N)\}$.

1. Preliminaries. Let R' be a ring (with or without identity). Following Tominaga [17], we say that a right R'-module $M \neq 0$ is s-unital if $u \in uR'$ for any $u \in M$. As was shown in [17], if $M_{R'}$ is s-unital then for any finite subset F of M there exists an element e in R' such that $xe = x$ for all $x \in F$. Following B. Zimmerman-Huisgen [22], we say that M is locally projective if M satisfies the following condition: For all diagrams

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \longrightarrow & 0 \\ & & \uparrow g & & \\ F & \hookrightarrow & M & & \end{array}$$

with exact upper row and a finitely generated submodule F of M there is $g' \in \mathrm{Hom}_R(M,A)$ such that $g|F = fg'|F$. It is known that M is locally projective if and only if M is s-unital as a left Δ-module (see [22]). As is easily seen, if N is an S-R-submodule of a locally projective module M, then M/N is a locally projective R/Ann_R

(M/N)-module. A module M is called a self-generator if $T_M(K) = K$, for all R-submodules K of M. We also call a module M a $\sum$-self-generator if $T_M(N) = N$, for all R-submodules N of M^n and all positive integers n (see [21]). We begin with the following proposition.

Proposition 1.1. The following are equivalent:

1) M is s-unital as a right T-module.

2) The lattice homomorphism $U_\Delta(S_S) \to U(M_R)$; $I \to IM$, is an isomorphism.

3) M is a self-generator and $MT = M$.

If M ,is locally projective, we may add:

4) Every simple homomorphic image of any submodule of M_R is a homomorphic image of M_R.

Proof. See the proof of [9, Proposition 1.1].

A ring R is said to be fully right idempotent if $I^2 = I$ for every right ideal I of R. And R is fully idempotent if $I^2 = I$ for every ideal I of R. Let us call a module M a fully right idempotent (resp. fully idempotent) module if for every $m \in M$, $m \in [m,M^*]mR$ (resp. $m \in S[m,M^*]mR$). A ring R is fully right idempotent or fully idempotent according as R_R is so. Here, we give several characterizations of a fully right idempotent module and a fully idempotent module.

Proposition 1.2 (cf. [12, Theorem 7]). (1) The following statements are equivalent:

1) M is a fully right idempotent module.

2) For every R-submodule N of M, $N = [N,M^*]N$.

3) M_T is s-unital and $I^2 = I$ for every $I \in U_\Delta(S_S)$.

4) M_T is s-unital and $N \cap IM = IN$ for every S-R-submodule N of M and every right ideal I of S.

5) M_T is s-unital and ${}_SM/N$ is flat for each S-R-submodule N of M.

(2) The following statements are equivalent:

1) M is a fully idempotent module.

2) For every S-R-submodule N of M, $N = [N,M^*]N$.

3) The lattice homomorphism $U_T({}_RR_R) \to U({}_SM_R)$; $I \to MI$, is an isomorphism and $I^2 = I$ for every $I \in U_T({}_RR_R)$.

4) The lattice homomorphism $U_\Delta({}_SS_S) \to U({}_SM_R)$; $K \to KM$, is an isomorphism and $K^2 = K$ for every $K \in U_\Delta({}_SS_S)$.

Proof. See the proof of [9, Proposition 1.2].

The proofs of the following two propositions are similar to those of corresponding assertions in [13], [19] and [22] and hence omitted.

Proposition 1.3. (1) $M = \bigoplus_{\alpha \in A} M_\alpha$ is fully right idempotent (resp. fully idempotent) if and only if each M_α is fully right idempotent (resp. fully idempotent).

(2) If R is a fully right idempotent ring, then every locally projective module over R is fully right idempotent.

(3) If R is a fully idempotent ring, then every projective module over R is fully idempotent.

Proposition 1.4. If M is fully idempotent, then there hold the following:

(1) S is a semiprime ring.

(2) The center of S is a regular ring.

(3) If $S = S_1 \oplus S_2 \oplus \ldots \oplus S_n$ with two-sided simple rings S_i, then $M_i = S_iM$ is S-R-simple and $M = M_1 \oplus \ldots \oplus M_n$.

If M is finitely generated projective, then $\Delta = S$. Hence by Proposition 1.2 we have the following

Proposition 1.5. Let M be a finitely generated projective module. If M is fully right idempotent (resp. fully idempotent), then S_S is fully right idempotent (resp. fully idempotent).

2. Regular modules. We begin this section with the following characterizations of a regular module.

Theorem 2.1 ([9], [19]). The following statements are equivalent:

1) M is a regular module.

2) For every $m \in M$, mR is projective and is a direct summand of M.

3) For every $m_1, \ldots, m_t \in M$, $\sum_{i=1}^{t} m_iR$ is projective and is a direct summand of M.

4) M is locally projective and every homomorphic image of M is flat.

5) M is locally projective and for any submodule N of M and any left R-module L, the natural homomorphism $N \otimes_R L \to M \otimes_R L$ is a monomorphism.

6) M is locally projective and $MI \cap N = NI$ for every submodule N of M and every left ideal I of R.

Examples of regular modules 2.2. (1) Any locally projective module over a regular ring is a regular module ([22]).

(2) Any completely reducible projective module is a regular module.

(3) An ideal of a ring R is called a regular ideal if it is regular as a subring. It is known that every ring possess a unique largest regular ideal. As is easily seen, every regular ideal is a regular module.

Proposition 2.3 (Zelmanowitz [19]). (1) If M is a regular module, then $J(M) = 0 = Z(M)$.

(2) $M = \oplus_{\alpha \in A} M_\alpha$ is regular if and only if each M_α is regular.

A module M is prime (resp. semiprime) if for every non-zero elements m, m_1 in M there holds $m(M^*, m_1) \neq 0$ (resp. $m(M^*, m) \neq 0$) (see [20]).

The following theorem is an extension of [7, Corollary 1.3] to modules.

Theorem 2.4. The following are equivalent:

1) M is a regular module.

2) M is locally projective and fully idempotent, and for each prime ideal P of R, M/MP is a regular R/P-module.

3) M is locally projective and fully idempotent, and each prime factor module $M/N_{\bar{R}}$ $(N \subseteq {}_S M_R)$ is a regular $\bar{R}$-module, where $\bar{R} = R/\mathrm{Ann}_R(M/N)$.

Proof. 1) $\Rightarrow$ 2). By the definition, a regular module M is s-unital as a left Δ-module and hence M is locally projective. Let I be an ideal of R. Then, since each

$f \in M^*$ induces an element in $\mathrm{Hom}_{R/I}(M/MI, R/I)$, M/MI is a regular R/I-module.

2) $\Rightarrow$ 3). If $\bar{M} = M/N_{\bar{R}}$ is prime for an S-R-submodule N, then $\mathrm{Ann}_R(\bar{M})$ is a prime ideal by [20, Proposition 1.1]. By Proposition 1.2 (2), it is easy to see that $N = M(\mathrm{Ann}_R(\bar{M}))$. Hence M/N is a regular $R/\mathrm{Ann}_R(\bar{M})$-module by 2).

3) $\Rightarrow$ 1). We have to show that for each $m \in M$ there exists an $f \in M^*$ such that $m = mf(m)$. Assume, to the contrary, that there exists an m in M such that $m = mx(m)$ has no solution in M^*. Then, by making use of the fact that M is locally projective and Zorn's lemma, we can choose an S-R-submodule N of M which is maximal with respect to the property that $\bar{m} = \bar{m}x(\bar{m})$ has no solution in $\mathrm{Hom}_{\bar{R}}(\bar{M}, \bar{R})$ where $\bar{M} = M/N$ and $\bar{R} = R/\mathrm{Ann}_R(\bar{M})$, i.e. $m - mx(m)$ is not in N for every $x \in M^*$. By hypothesis, $\bar{M}$ is not prime. Therefore there exist non-zero elements $\bar{m}_1$ and $\bar{m}_2$ in $\bar{M}$ such that $[m_1, \mathrm{Hom}_R(\bar{M}, \bar{R})]m_2 = 0$. Since M is fully idempotent, as is easily seen, the $\bar{R}$-module $\bar{M}$ is also fully idempotent and so semiprime. Thus, we have $\overline{Sm_1R} \cap \overline{Sm_2R} = 0$. By the choice of N and the fact that M is locally projective, there exist x and y in M^* with $m - m(x,m) \in Sm_1R + N$ and $m - m(y,m) \in Sm_2R + N$. Thus $m - m(x + y - x[m,y])m$ is in $(Sm_1R + N) \cap (Sm_2R + N) = N$. This contradicts the choice of N. Consequently M is regular and the proof is complete.

Corollary 2.5. Let R be a ring all of whose prime factor rings are regular. Then every locally projective fully idempotent module is regular.

The endomorphism ring of a regular module need not be regular. Indeed, Cukerman [3] and Ware [18] have noted that the endomorphism ring of an infinitely generated free module over a regular ring R is regular if and only if R is artinian.

Theorem 2.6 (Ware [18]). If M is a finitely generated regular module, then S is a regular ring.

Even if the endomorphism ring of a projective module M is regular, M need not be regular. The following example is due to Ware [18].

Example 2.7. A cyclic projective module M which is not regular but such that S is a field: Let K be a field and $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$. Let $M = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$. Then M_R is a cyclic projective module and $S \cong K$. Since $\begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ is a cyclic submodule of M which is not a direct summand, M cannot be a regular module.

Proposition 2.8. Let M be a locally projective module. If S is a regular ring and M is a self-generator, then M is regular.

Proof. By Proposition 1.1, for any $m \in M$ there is an I in $U_\Delta(S_S)$ with $mR = IM$. Then $m = \sum a_i m_i$ with some $a_i \in I$ and $m_i \in M$. If we set $I' = \sum a_i S$, it is easy to see that $mR = I'M$. Since S is regular, the right ideal I' is generated by an idempotent e. Then mR $(= eM)$ is a direct summand of M and is projective. Thus we conclude that M is regular by Theorem 2.1.

Ware [18, Theorem 3.8] proved that if M is a projective module over a commutative ring R and S is a regular ring, then M is regular. More generally, we have the following

Corollary 2.9. Let R be a commutative ring. If M is a locally projective module and S is a regular ring, then M is a regular module.

Proof. A locally projective module M over a commutative ring R is a self-generator by [21, 2.3, 3)]. Thus, by Proposition 2.8 M is a regular module.

3. V-modules. Let M be a right R-module. A right R-module N is defined to be M-injective in case for each monomorphism $f : K_R \to M_R$ and each homomorphism $g : K_R \to N_R$ there is an R-homomorphism $g' : M_R \to N_R$ such that $g = g'f$:

$$\begin{array}{ccccc} 0 & \longrightarrow & K_R & \xrightarrow{f} & M_R \\ & & {\scriptstyle g}\downarrow & \swarrow {\scriptstyle g'} & \\ & & N_R & & \end{array}$$

The following theorem characterizes a V-module.

Theorem 3.1 (Fuller [8]). The following are equivalent:

1) M is a V-module.

2) Every simple right R-module is M-injective.

3) Every finitely cogenerated factor module of M is completely reducible.

In case we restrict our attension to locally projective modules, we obtain the following

Corollary 3.2. Let M be a locally projective module.

Then the following are equivalent:

1) M is a V-module.

2) M is a self-generator and every simple homomorphic image of M is M-injective.

3) M is a self-generator and for any simple right R-module X, $\mathrm{Hom}_R(M,X)_S$ is injective.

Proof. 1) $\Rightarrow$ 2). Since every simple homomorphic image of any submodule of M is a homomorphic image of M (Theorem 3.1), M is a self-generator by Proposition 1.1.

2) $\Rightarrow$ 1). Obvious by Theorem 3.1.

2) $\Leftrightarrow$ 3). Since M is a Σ-self-generator by [21, Theorem 2.4], the equivalence of 2) and 3) is a consequence of [21, Corollary 1.5] and Theorem 3.1.

If $0 \to M' \to M \to M'' \to 0$ is exact and a module U is M-injective, then U is injective relative to both M' and M''. If U is injective relative to each of the R-modules M_α $(\alpha \in A)$, then U is $\oplus_{\alpha\in A} M_\alpha$-injective (see [1, p.188]). Hence, the following proposition is immediate from Theorem 3.1.

Proposition 3.3 (1) Every submodule and every homomorphic image of a V-module is also a V-module.

(2) $\oplus_{\alpha\in A} M_\alpha$ is a V-module if and only if every M_α is a V-module.

Corollary 3.4. Let R be a commutative ring, and M a finitely generated V-module. Then $R/\mathrm{Ann}_R(M)$ is a V-ring (and hence a regular ring).

The following theorem corresponds to [6, Theorem 14].

For the another proof see [9].

Theorem 3.5. If M is a fully right idempotent module and M/MP_R is a V-module for each primitive ideal P of R, then M is a V-module. If M is a locally projective module, then the converse is also true.

Proof. To prove the first assertion, it suffices to show that for any submodule N and any $m \in M \setminus N$, there exists a maximal submodule X of M such that $X \supseteq N$ and $m \notin X$. Now, let $m \notin N$, and let Y be maximal among the submodules K of M such that $m \notin K \supseteq N$. Let D denote the intersection of all submodules Q of M with $Q \supsetneq Y$. Obviously, m is in D, and D/Y is a simple module. By Proposition 1.2, $D = IM$ for some right ideal I of S. If we set $P = \mathrm{Ann}_R(D/Y)$, P is a primitive ideal of R. Since $D \cap (Y + MP) = Y + D \cap MP = Y + IMP = Y + DP \subseteq Y$ by Proposition 1.2, we conclude $m \notin Y + MP$. By the maximality of Y, there holds $Y \supseteq MP$. Therefore, M/Y is naturally considered as an R/P-module. Hence, by hypothesis, Y must be a maximal submodule of M.

Next, we assume that M is a locally projective V-module. If there is an $m \in M$ such that $m \notin [m,M^*]mR$, then we have a maximal submodule N of M such that $[m,M^*]mR \subseteq N$ and $mR \nsubseteq N$. Since N is a maximal submodule, we have $M = mR + N$, and so $\Delta = [M,M^*] = [m,M^*] + [N,M^*]$. Hence we have $mR \subseteq \Delta mR = [m,M^*]mR + [N,M^*]mR \subseteq N$. This is a contradiction.

The endomorphism ring of a V-module need not be a V-ring. For example, a left vector space W over a field K is an injective right $\mathrm{End}_K W$-module if and only if

$[W:K] < \infty$ ([4, p.88]).

Theorem 3.6. Let M be a finitely generated projective module. Then the following are equivalent:

1) M is a V-module.

2) M is a self-generator and S is a right V-ring.

Proof. Recall first that every locally projective V-module is a self-generator (Corollary 3.2). Since M is finitely generated projective, we have that $\Delta = S$. Assume that M is a self-generator. Then, by Proposition 1.1, the lattice $U(S_S)$ is isomorphic to the lattice $U(M_R)$. Therefore S is a right V-ring if and only if M is a V-module.

Corollary 3.7. If M is a finitely generated projective module over a right V-ring R, then the endomorphism ring S is a right V-ring.

If M is a projective module such that $\mathrm{End}_R(M)$ is a V-ring, is M a V-module? Example 2.7 shows the answer is negative. But we have the following proposition.

Proposition 3.8. Assume that M is quasi-projective or $MT = M$. If M is a self-generator and S is a right V-ring, then M is a V-module.

Proof. By [21, Theorem 2.4], M is a Σ-self-generator. If a module X is simple then $\mathrm{Hom}_R(M,X)_S$ is simple or zero by [1, p.191] and by [21, Theorem 4.5], and so, by [21, Corollary 1.5], X is M-injective. Therefore, M is a V-module by Theorem 3.1.

The next corresponds to Corollary 2.9.

Corollary 3.9. Let R be a commutative ring, and M a locally projective module. If S is a right V-ring, then M is a V-module.

Proof. This is clear by Proposition 3.8 and the assertion in the proof of Corollary 2.9.

4. Regular modules versus V-modules. First, we shall extend [6, Theorem 16] to modules. We say that R is a P.I.-ring if R satisfies a polynomial identity with coefficients in the centroid and at least one coefficient is invertible.

Theorem 4.1. Let R be a P.I.-ring, and M a right R-module. Then the following are equivalent:

1) M is a regular module.
2) M is a locally projective V-module.
3) M is locally projective and fully right idempotent.

Proof. 1) ⇒ 3) ⇒ 2) are clear by Theorem 3.5 and the fact that every primitive factor ring of a P.I.-ring R is simple artinian (Kaplansky [10]).

3) ⇒ 1). Cleary, if M is fully right idempotent and N an S-R-submodule, then M/N is a fully right idempotent $R/\mathrm{Ann}_R(M/N)$-module. If $M/N_{\bar{R}}$ is prime, then $\bar{R} = R/\mathrm{Ann}_R(M/N)$ is prime by [20, Proposition 1.1]. Hence, according to Theorem 2.4, it is sufficient to show that a faithful, prime and fully right idempotent module over a prime P.I.-ring R is regular. Let C be the center of R. First we shall show that M is C-torsion-free. Assume, to the contrary,

that there exist a non-zero $m' \in M$ and a non-zero $c' \in C$ such that $m'c' = 0$. Since M_R is faithful, there is a non-zero $m'' \in M$ such that $m''c' \neq 0$. Then we have $m'(M^*, m''c') = m'c'(M^*, m'') = 0$. This contradicts the primeness of M. Since M is fully right idempotent, for each $m \in M$ and each non-zero $c \in C$, there are $f_1, \ldots, f_n \in M^*$ and $r_1, \ldots, r_n \in R$ such that $mc = \sum mcf_i(mc)r_i = (\sum mf_i(m)r_i)c^2$. Hence, we can define $mc^{-1} = \sum mf_i(m)r_i$, and then M has the Q-module structure, where Q is the ring of central quotients of R. By [16, Corollary 1], Q is a simple artinian ring. Since M_Q is completely reducible, by Proposition 2.3 we may assume that M is an irreducible Q-module. Since $End_R(M) \cong End_Q(M)$ is a division ring by Shur's lemma, M is a regular module by Proposition 2.8.

Corollary 4.2. Let R be a P.I.-ring. Then a locally projective module M is completely reducible if and only if every completely reducible module is M-injective.

Proof. One way is clear. To prove the other way, assume that every completely reducible right R-module is M-injective. Let m be an arbitrary element of M. Since M is regular by Theorem 4.1, mR is a regular module and every completely reducible module is mR-injective. Now, we shall show that mR contains no infinite direct sum of non-zero submodules. Assume, to the contrary, that mR contains an infinite direct sum $N = \oplus_{\alpha \in A} M_\alpha$ with $M_\alpha \neq 0$. Since each M_α is a V-module, it contains a maximal submodule M'. Then $N' = \oplus_{\alpha \in A} M_\alpha / M'_\alpha$ is completely reducible, and hence mR-injective. Thus the canonical homomorphism $N_R \to N'_R$ can be extended to an homomorphism $f : mR \to N'$.

Noting that $f(m) \in \oplus_{\alpha \varepsilon A'} M_\alpha/M'_\alpha$ with a finite subset A' of A, we obtain $N' = f(N) \subseteq f(mR) \subseteq \oplus_{\alpha \varepsilon A'} M_\alpha/M'_\alpha$, which is a contradiction. Thus, we see that mR is isomorphic to a finite direct sum of minimal right ideals by [19, Theorem 1.8], concluding that M is a sum of simple submodules.

A ring is said to be normal if every idempotent is central. For example, reduced rings and left and right duo rings are normal.

Proposition 4.3. Let R be normal. If M is a regular module, then every simple homomorphic image of M is injective. In particular, M is a V-module.

Proof. If M is regular, then for every $m \varepsilon M$, mR is projective and is a direct summand of M by Theorem 2.1. Hence we may assume that M is cyclic (and projective). Since R is normal, $M \simeq eR$ for some central idempotent e in R. Since the ring eR is regular and normal, it is a strongly regular ring, and hence a right V-ring by Chiba-Tominaga [2]. The second assertion is clear by Corollary 3.2.

For a locally projective module M over a commutative ring R, we have

Theorem 4.4. Let R be a commutative ring. Then the following are equivalent:

1) M is a regular module.

2) M is a locally projective V-module.

3) M is fully right idempotent.

4) M is locally projective and every simple homomorphic image of M is injective.

5) M is locally projective and every simple homomorphic image of M is M-injective.

Proof. 1) $\Rightarrow$ 2). By Theorem 4.1.

2) $\Rightarrow$ 3). By Theorem 3.5.

3) $\Rightarrow$ 1). Since M is fully right idempotent, for each $m \in M$ we have that $m \in [m,M^*]mR$. Since R is commutative, the right multiplication of any element of R is in S. Therefore $m \in [m,M^*]Sm = [m,M^*]m$. Consequently, M is regular.

1) $\Rightarrow$ 4). By Proposition 4.3.

4) $\Rightarrow$ 5). Trivial.

5) $\Rightarrow$ 2). Recall that M is a self-generator (see the proof of Corollary 2.9). Therefore M is a V-module by Corollary 3.2.

Remark. For a projective module M, Ware [18, Proposition 2.5] has proved 1) $\Rightarrow$ 4), Ramamurthi [15, Theorem 4] has proved that 4) $\Rightarrow$ 3) $\Rightarrow$ 1), and Maoulaoui [13, Proposition 1] has proved 4) $\Rightarrow$ 1).

In case R is a P.I.-ring, the implication 1) $\Rightarrow$ 5) in Theorem 4.4 does not remain valid (in spite of the assertion in [13, Proposition 2]). Here is an example.

Example 4.5. Ler R be the ring in Example 2.7. Clearly, R is a P.I.-ring. If we set $I = \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}$, I is a minimal right ideal and is a direct summand of R_R. Hence I is a regular module. However, I is not injective, because the homomorphism $f : \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix} \to I$ defined by $f\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$ can not be extended to a homomorphism of R to I.

Finally, we shall improve [9, Proposition 4.5]. We shall begin with some preparations. Let M be a right R-module. An element m of M is called regular (in M) if there exists an element f of M^* such that $mf(m) = m$. A submodule N of M_R is called a regular submodule of M if every element of N is regular in M. As is easily seen, if a regular module N is a direct summand of M, then N is a regular submodule of M. We state here without proof the following two theorems. (The proofs given in [11, p.112] for rings are available with only minor changes.)

Theorem 4.6. Let M_R be a locally projective module. Let P N be S-R-submodules of M, and $\bar{R} = R/\mathrm{Ann}_R(M/P)$. Then N is a regular submodule of M if and only if P is a regular submodule of M and N/P is a regular submodule of $M/P_{\bar{R}}$.

Theorem 4.7. Let M_R be a module. Then there exists a unique maximal N among the regular submodules of M_R which are S-R-submodules, and $M/N_{\bar{R}}$ has no non-zero regular $\bar{R}$-submodules which is an S-$\bar{R}$-submodule, where $\bar{R} = R/\mathrm{Ann}_R(M/N)$.

A module M is said to be semi-artinian if every non-zero homomorphic image of M has the non-zero socle. We are now ready to prove the following generalization of [9, Proposition 4.5].

Theorem 4.8. If M_R is semi-artinian, then the following are equivalent:

1) M is a regular module.

2) M is a locally projective, fully idempotent module.

Proof. It is enough to prove that 2) implies 1). Let N be as in Theorem 4.7. If $N \neq M$, then $X = \mathrm{Soc}(M/N_R)$ is non-zero. We shall show that X is a regular submodule of $\bar{M}_{\bar{R}}$, where $\bar{M} = M/N$ and $\bar{R} = R/\mathrm{Ann}_R(\bar{M})$. Now, let Y be a simple submodule of $\bar{M}$. Since $\bar{M}_{\bar{R}}$ is semiprime (see the proof of Theorem 2.4), there exists $f \in (\bar{M}_{\bar{R}})^*$ such that $Yf(Y) \neq 0$, and hence $fY = e\bar{R}$ with some idempotent e in $\bar{R}$. Since a projective minimal right ideal fY is a regular submodule of $\bar{R}_{\bar{R}}$, for each $y \in Y$ there is a $g \in (\bar{R}_{\bar{R}})^*$ such that $fy = (fy)g(fy)$. Hence we have $y = ygf(y)$ and $gf \in (\bar{M}_{\bar{R}})^*$, and therefore y is regular in $\bar{M}_{\bar{R}}$. Thus Y is a direct summand of $\bar{M}$. Now, we can easily see that any finite sum of simple submodules is a direct summand of $\bar{M}$. Thus, for each $m \in X$, $m\bar{R}$ is a direct summand of $\bar{M}$, whence it follows that X is a regular submodule of $\bar{M}_{\bar{R}}$. But this contradicts Theorem 4.7. Hence, $M = N$, namely M is regular.

Since every locally projective V-module is fully right idempotent by Theorem 3.5, we readily obtain the following

Corollary 4.9. If M is a locally projective, semi-artinian V-module, then M is a regular module.

References

[1] F. W. Anderson and K. R. Fuller: Rings and Categories of Modules, Springer-Verlag, 1973.

[2] K. Chiba and H. Tominaga: On strongly regular rings, Proc. Japan Acad. 49 (1973), 435-437.

[3] G. M. Cukerman: Rings of endomorphisms of free modules, Sibirsk. Mat. Ž. 7 (1966), 1161-1167.

[4] C. Faith: Algebra II Ring Theory, Springer-Verlag, 1976.

[5] D. J. Fieldhouse: Pure theories, Math. Ann. 184 (1969), 1-18.

[6] J. W. Fisher: Von Neumann regular rings versus V-rings, Ring Theory: Proc. Univ. Oklahoma Conference, Marcel Dekker, 1974, 101-119.

[7] J. W. Fisher and R. L. Snider: On the von Neumann regularity of rings with regular prime factor rings, Pacific J. Math. 54 (1974), 135-144.

[8] K. R. Fuller: Relative projectivity and injectivity classes determind by simple modules, J. London Math. Soc. 5 (1972), 423-431.

[9] Y. Hirano: Regular modules and V-modules, Hiroshima Math. J. 11 No,1 (1981), to appear.

[10] I. Kaplansky: Rings with a polynomial identity, Bull. Amer. Math. Soc. 54 (1948), 575-580.

[11] I. Kaplansky: Fields and Rings, Univ. of Chicogo Press, Chicago, 1969.

[12] T. Mabuchi: Weakly regular modules, Osaka J. Math. 17 (1980), 35-40.

[13] Z. Maoulaoui: Sur les modules règuliers, Arch. Math. 30 (1978), 469-472.

[14] G. Michler and O. Villamayor: On rings whose simple modules are injective, J. Algebra 25 (1973), 185-201.

[15] V. S. Ramamurthi: A note on regular modules, Bull. Austral. Math. Soc. 11 (1974), 359-364.

[16] L. Rowen: Some results on the center of a ring with polynomial identity, Bull. Amer. Math. Soc. 79 (1973),

[17] H. Tominaga: On s-unital rings, Math. J. Okayama Univ. 18 (1976), 117-134.

[18] R. Ware: Endomorphism rings of projective modules, Trans. Amer. Math. Soc. 155 (1971), 233-256.

[19] J. Zelmanowitz: Regular modules, Trans. Amer. Math. Soc. 163 (1972), 341-355.

[20] J. Zelmanowitz: Semiprime modules with maximum conditions, J. Algebra 25 (1973), 554-574.

[21] B. Zimmermann-Huisgen: Endomorphism rings of self-generators, Pacific J. Math. 61 (1975), 587-602.

[22] B. Zimmermann-Huisgen: Pure submodules of direct products of free modules, Math. Ann. 224 (1976), 233-245.

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ON AUTOMORPHISMS IN SEPARABLE EXTENSIONS OF RINGS

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In order to generalize the theory of Azumaya algebras we introduced a notion of a special type of separable extensions called H-separable extension, and found that many properties which hold in Azumaya algebras hold also in this type of separable extensions (See for example [4], [7], [12] and [13]). In this report we will summarize and improve the results which are obtained in [14] and [15]. In §1 we will find some sufficient conditions for an H-separable extension A of B to have the property that all automorphisms of A which fix all elements of B are inner ones. On the other hand, in [15] we showed that in the case of algebras over a commutative ring R H-separable Galois extensions of R are same as central Galois extensions of R (See Prop. 2.2 [15]). In §2 we will show some necessary and sufficient conditions for Galois extensions to be H-separable extensions, for H-separable extensions to be Galois extensions and also for the orders of Galois groups of H-separable Galois extensions to be units. It is well known that in the case of central Galois extension the order of Galois group is always unit. But we can give a counter example in the case of H-separable Galois extension.

1. Automorphisms in separable extensions.

Throughout this report A is always a ring with 1 and B is a subring of A which contains 1 of A. For a subset S of A we will denote by $V_A(S)$ the subring $\{a \in A \mid sa = as$ for all

$s \in S\}$. Furthermore, we will denote the center of A and $V_A(B)$ by C and D, respectively. We also assume that all A-modules are unitary, and all two sided A-modules are associative. For a two sided A-module M, we always let $M^A = \{m \in M \mid ma = am \text{ for all } a \in A\}$. First we shall recall the definition.

THEOREM 1.1. Let A, B, C and D be as above. Then the following conditions are equivalent.

(1) $A \otimes_B A$ is isomorphic to a direct summand of some finite direct sum of the copies of A as A-A-module.

(2) D is C-finitely generated projective, and the map η of $A \otimes_B A$ to $\mathrm{Hom}({}_C D, {}_C A)$ defined by $\eta(a \otimes b)(d) = adb$, for $a, b \in A$, $d \in D$, is an A-A-isomorphism.

(3) For any A-A-module M, the map g_M of $D \otimes_C M^A$ to M^B defined by $g_M(d \otimes m) = dm$, for $d \in D$, $m \in M^A$, is an isomorphism.

(4) $1 \otimes 1 \in (A \otimes_B A)^A D$ in $A \otimes_B A$.

Proof. See [3], [8], [6] or [11].

DEFINITION. A is an H-separable extension of B if and only if A and B satisfy the conditions of Theorem 1.1.

REMARK. Theorem 1.1 (4) shows that A is H-separable over B if and only if there exist $\sum_j x_{ij} \otimes y_{ij}$ in $(A \otimes_B A)^A$ and d_i in D with $1 \otimes 1 = \sum_{i,j} (x_{ij} \otimes y_{ij}) d_i$. We call $\{\sum x_{ij} \otimes y_{ij}, d_i\}$ an H-system for $A|B$. (See [6]).

THEOREM 1.2. Let A be an H-separable extension of B. Then every endomorphism of A which fixes all elements of B is an automorphism and fixes all elements of $B' = V_A(V_A(B))$.

Proof. Let σ be an automorphism of A which fixes all elements of B. Since $\sigma \in \mathrm{Hom}({}_A B, {}_A B) = D \otimes_C A$, where the latter isomorphism was proved in Prop. 3.1 [4], there exists $\sum d_j \otimes a_j$ in $D \otimes_C A$ such that $\sigma(x) = \sum d_j x a_j$ for all x in A. Then $\sigma(1) =$

$\Sigma d_j a_j = 1$, and $\sigma(b) = \Sigma d_j b a_j = b\Sigma d_j a_j = b$ for any b in B'. Thus σ fixes all elements of B'. Since $C \subset B'$, and A is H-separable over B' by Theorem 1.3' [8], we can assume that $C \subset B$ from the beginning. Let $\{\Sigma x_{ij} \otimes y_{ij}, b_i\}$ be an H-system for $A|B$. Then, since σ is a B-B-map, and $x \otimes 1 = \Sigma x(x_{ij} \otimes y_{ij})d_i = \Sigma(x_{ij} \otimes y_{ij})xd_i$, we have $x \otimes 1 = x \otimes \sigma(1) = \Sigma x_{ij} \otimes \sigma(y_{ij} x d_i)$ and $x = \Sigma x_{ij}\sigma(y_{ij})\sigma(x)\sigma(d_i)$. Therefore, $\sigma(x) = 0$ implies $x = 0$, and we have that $\ker\sigma = 0$. Now let $\bar{A} = \sigma(A)$ and $\bar{D} = \sigma(D)$. Then $\bar{A}$ is an H-separable extension of B, and $\bar{D} = V_{\bar{A}}(B) \cong D$, which are C-finitely generated projective. Furthermore, since $C \subset B$ by assumption, the center of $\bar{A} = \sigma(C) = C$. Let $V_A(\bar{A}) = \bar{C}$, and $\underline{m}$ an arbitrary maximal ideal of C. Then, $D = V_A(B) \cong \bar{D} \otimes_C V_A(\bar{A}) = D \otimes_C \bar{C}$ by Theorem 1.1 (3). Hence $\bar{D}/\underline{m}\bar{D} \otimes_{C/\underline{m}} \bar{C}/\underline{m}\bar{C} = D/\underline{m}D$, and we have $[\bar{C}/\underline{m}\bar{C} : C/\underline{m}] = 1$, because $D \neq \underline{m}D$ and $\bar{D} \neq \underline{m}\bar{D}$. Thus we see that $\bar{C} = \underline{m}\bar{C} + C$ for any maxmal ideal $\underline{m}$ of C. On the other hand, $\bar{C}$ is C-finitely generated, since $\bar{C} = C \otimes_C \bar{C} \lessdot \oplus \bar{D} \otimes_C \bar{C} \cong D$. Hence $\bar{C} = C$ by Nakayama's Lemma, and we have $D = \bar{D} \subset \bar{A}$. Now, it is easy to see that each $\Sigma x_{ij} \otimes \sigma(y_{ij})$ belongs to D, since σ is a B-B-map, and $\Sigma x_{ij} \otimes y_{ij} \in (A \otimes_B A)^A$. Then, for any x in A, we have $x = \Sigma x_{ij}\sigma(y_{ij})\sigma(xd_i) \in D\bar{A} \subset \bar{A}$. Thus $\sigma(A) = A$, and we have proved that σ is an automorphism.

Now, let σ be any automorphism of A which fixes all elements of B, and $J_\sigma = \{a \in A \mid xa = a\sigma(x)$ for all x in A$\}$. By A and σ we can construct as usual a new A-A-module A_σ, namely, $A_\sigma = A$ as left A-module, but the right A-module structure is defined by $a \cdot x = a\sigma(x)$ for a A_σ and x A. Then we see that $(A_\sigma)^A = J_\sigma$ and $(A_\sigma)^B = D$.

LEMMA 1.1. Let σ be an automorphism of A such that $\sigma|B = 1_B$. Then, σ is inner if and only if $J_\sigma = Cu$ for some unit u of D.

Proof. Clear.

LEMMA 1.2. Let A be an H-separable extension of B, and an automorphism of A such that $\sigma|B = 1_B$ Then, we have

(1) The map g_σ of $D\otimes_C J_\sigma$ to D defined by $g_\sigma(d\otimes a) = da$, for $d\in D$ and $a\in J_\sigma$, is an isomorphism. Consequently, we have $DJ_\sigma = J_\sigma D = D$ and $AJ_\sigma = J_\sigma A = A$.

(2) J_σ is rank 1 C-projective.

(3) $J_\sigma J_{\sigma^{-1}} = J_{\sigma^{-1}}J_\sigma = C$.

(4) σ is an inner automorphism if and only if $J_\sigma = Cu$ for some u (not necessarily unit) in D.

Proof. (1). Clear by $(A_\sigma)^B = D$, $(A_\sigma)^A = J_\sigma$ and Theorem 1.1 (3). (2). By (1) we have $D\otimes_C J_\sigma \cong D$. But D is C-finitely generated projective, and consequently, C is a C-direct summand of D. Then it follows that J_σ is rank 1 C-projective. (3). Clearly $J_\sigma J_{\sigma^{-1}}$ is an ideal of C. By (1) we have $DJ_\sigma = D$ and $DJ_\sigma J_{\sigma^{-1}} = DJ_{\sigma^{-1}} = D$. But C is a C-direct summand of D. Hence we have $J_\sigma J_{\sigma^{-1}} = DJ_\sigma J_{\sigma^{-1}}\cap C = D\cap C = C$. (4). If $J_\sigma = Cu$ for some $u\in D$, $D = DJ_\sigma = Du$. Then, u is a unit. Now, we can apply Lemma 1.1. The converse is obvious by Lemma 1.1.

Theorem 1.3. Let A be an H-separable extension of B, and C and C' the centers of A and B, respectively. Then, all automorphisms which fix all elements of B are inner automorphisms, if one of the following conditions is satisfied.

(1) C is a semilocal ring.

(2) C' is a semilocal ring, and $V_A(V_A(B)) = B$.

Proof. Suppose (1), and let $\underline{m}_1, \underline{m}_2, \cdots, \underline{m}_r$ be the set of all maximal ideals of C. Now we can follow the same lines as the proof of Lemma 1 [5]. Let σ be any automorphism of A such that $\sigma|B = 1_B$. Then, for any i $(1\leqq i\leqq r)$ we have $\underline{m}_i J_\sigma \not\supset \underline{m}_1\cdots\underline{m}_{i-1}\underline{m}_{i+1}\cdots\underline{m}_r J_\sigma$, since $J_\sigma J_{\sigma^{-1}} = C$ by lemma 1.2 (3).

Hence there exists a_i in $\underline{m}_1 \cdots \underline{m}_{i-1}\underline{m}_{i+1} \cdots \underline{m}_r J_\sigma$ such that $a_i \notin \underline{m}_i J_\sigma$. Let $a = \Sigma a_i$. Then, $a \in J_\sigma$ and $a \notin \underline{m}_i J_\sigma$ for any i. But J_σ is rank 1 C-projective by Lemma 1.2 (2). Hence $[J_\sigma/\underline{m}_i J_\sigma : C/\underline{m}_i] = 1$, and $J_\sigma/\underline{m}_i J_\sigma = (a+\underline{m}_i J_\sigma)C/\underline{m}_i$. Thus we have $J_\sigma = aC+\underline{m}_i J_\sigma$ for each maximal ideal $\underline{m}_i$ of C. Hence $J_\sigma = aC$ by Nakayama's Lemma. Then by Lemma 1.2 (4) σ is an inner automorphism. Next suppose (2), and let J be the radical of C'. Then, we see that C'/J is semisimple, and $C' = V_B(B) = D \cap B = V_D(D) \supseteq C$. Hence D is finitely generated as C'-module. Then D/JD is artinean, and we see that JD is contained in every maximal left ideal of D by Nakayama's Lemma. Hence D is also a semilocal ring. On the other hand, C is a C-direct summand of D. Hence we have $\underline{a}D \cap C = \underline{a}$ for any ideal $\underline{a}$ of C. This implies that every proper ideal of C is contained in at least a maximal left ideal of D. If $\underline{m}$ and $\underline{m}'$ are any two maximal ideals of C which are contained in a maximal left ideal $\underline{L}$ of D, then $1 \notin \underline{m} + \underline{m}' \subset \underline{L}$. Hence $\underline{m} = \underline{m}'$. Thus we see that C is a semilocal ring. Then, we can apply (1).

REMARK. The proof of Theorem 1.2 would have been completed, if it had been proved that $\sigma|C$ = identity. But the author gave the complete proof in this report for the convenience to readers.

2. On H-separable Galois extensions.

In this section there is no new result, we will only summarize the results obtaied in §2 [14] and [15].

In this section G is always a finite group of automorphisms of A, and Δ is the crossed product $\Delta(A;G)$ with the trivial factor set. Thus $\Delta = \Sigma AU_\sigma$ with $\{U_\sigma\}_{\sigma \in G}$ a left free base over A such that $U_\sigma U_\tau = U_{\sigma\tau}$ and $U_\sigma a = \sigma(a)U_\sigma$ for $a \in A$

and $\sigma,\tau\in G$. Now let us recall the definition of Galois extension. Let $A^G = \{a\in A \mid \sigma(a) = a \text{ for any } \sigma\in G\}$. In case $B\subset A^G$, there is a ring homomorphism j of Δ to $\mathrm{Hom}(A_B,A_B)$ such that $j(aU_\sigma)(x) = a\sigma(x)$ for a, $x\in A$ and $\sigma\in G$.

DEFINITION. A is a Galois extension of B relative to G if and only if the following three conditions are satisfied. (1) $B = A^G$. (2) A is right B-finitely generated projective. (3) j is an isomorphism.

Let t_G be a map of A to A^G such that $t_G(x) = \Sigma_{\sigma\in G}\sigma(x)$ for every $x\in A$. Clearly t_G is an A^G-A^G-homomorphism.

LEMMA 2.1. Let A be a Galois extension of B relative to G. Then we have

(1) There exists c in C such that $t_G(c) = 1$, if and only if ${}_BB_B\lessdot\oplus\,{}_BA_B$ (B-B-direct summand).

(2) Suppose furthermore $C\subset B$. Then $|G| = n$ is a unit of C, if and only if ${}_BB_B\lessdot\oplus\,{}_BA_B$.

Proof. (1). If there is $c\in C$ such that $t_G(c) = 1$, we obtain an A-A-map f of A to A such that $f(x) = xc$ for $x\in A$. Then $(t_G\circ f)|B = 1_B$, and we see that ${}_BB_B\lessdot\oplus\,{}_BA_B$. Conversely, suppose ${}_BB_B\lessdot\oplus\,{}_BA_B$. Then since A is right B-fibitely generated projective, $\mathrm{Hom}(A_B,A_B)$ is a separable extension of A by Theorem 7 [10]. Hence Δ is a separable extension of A. But by direct computations we can see that Δ is separable over A if and only if there exists $c\in C$ such that $t_G(c) = 1$. (2) is obvious by (1), since $t_G(c) = 1$ with $c\in C$ implies $nc = 1$.

THEOREM 2.1. Let A be an H-separable Galois extension of B relative to G. Then we have $B = V_A(V_A(B))$. Furthermore, the following three conditions are equivalent.

(1) $|G|$ is a unit.

(2) ${}_B B_B \lessdot \oplus {}_B A_B$.

(3) D is a separable C-algebra.

Proof. By Theorem 1.2 each σ in G fixes all elements of $B' = V_A(V_A(B))$. Hence $B \subset B' \subset A^G = B$, and we have $B' = B$. Then $C \subset B$. Therefore, (1) and (2) are equivalent by Lemma 2. 1. (2)$\Rightarrow$(3) follows from Prop. 4.7 [4]. Suppose (3). Then, $B = B' = V_A(D) = \mathrm{Hom}({}_D D_D, {}_D A_D) \lessdot \oplus \mathrm{Hom}({}_D D \otimes_C D_D, {}_D A_D) = A$ as B-B-module. Then, we have (2).

THEOREM 2.2. Let A be an H-separable extension of B, and G, Δ, and j as above with $B \subset A^G$. Then we have

(1) j is an isomorphism if and only if $D = \Sigma^{\oplus}_{\sigma \in G} J_\sigma$. In this case we have $A^G = B'$ $(= V_A(V_A(B)))$.

(2) In the case where A is right B-finitely generated projective, A is a Galois extension of B relative to G, if and only if $D = \Sigma^{\oplus}_{\sigma \in G} J_\sigma$.

Proof. Denote the opposite ring of D by D^o, and let $X^o = \{x^o \mid x \in X\}$ for $X \subset D$. By Prop. 3.1 [4] there is a ring isomorphism η_r of $A \otimes_C D^o$ to $\mathrm{Hom}(A_B, A_B)$ such that $\eta_r(a \otimes d^o)(x) = axd$ for a, $x \in A$ and $d \in D$, while for each σ in G we have a left A-isomorphism g'_σ of $A \otimes_C J^o_\sigma$ to AU_σ such that $g'_\sigma(a \otimes d^o_\sigma) = ad_\sigma U_\sigma$ for $a \in A$ and $d_\sigma \in J_\sigma$ by Lemma 1.2 (1). Let $g = \Sigma g'_\sigma$. g is an isomorphism of $\Sigma^{\oplus}_{\sigma \in G} A \otimes_C J^o_\sigma$ to Δ. Now suppose that $D = \Sigma^{\oplus}_{\sigma \in G} J_\sigma$. Then g is an isomorphism of $A \otimes_C D^o$ $(= \Sigma^{\oplus} A \otimes_C J^o_\sigma)$ to Δ such that $jg = \eta_r$. Hence j is an isomorphism. Conversely, if j is an isomorphism, we have $\Sigma^{\oplus} J_\sigma U_\sigma = \Delta^A \cong [\mathrm{Hom}(A_B, A_B)]^A = \mathrm{Hom}({}_A A_B, {}_A A_B) \cong D$. Next suppose $x \in A^G$. Then for any $d \in D$, we have $d = \Sigma d_\sigma$ for $d_\sigma \in J_\sigma$, and $xd = \Sigma x d_\sigma = \Sigma d_\sigma \sigma(x) = \Sigma d_\sigma x = dx$. Hence $A^G \subset B'$, and we have $A^G = B'$. Thus we have proved (1). Since $A \otimes_C D^o = \mathrm{Hom}(A_B, A_B)$, $B' = \mathrm{Hom}({}_A A_D, {}_A A_D)$ is the double centralizer of A_B. Hence if A_B is finitely generated projective, $A_{B'}$ is also finitely generated projective. Then (2) follows from (1).

REMARK. Note that the 'only if' part of Theorem 2.2 (1) holds without the assumption that A is H-separable over B.

THEOREM 2.3. Let A be a Galois extension of B relative to G. Then the following three conditions are equivalent

(1) A is an H-separable extension of B.

(2) The map g_σ of $D \otimes_C J_\sigma$ to D defined in Lemma 1.2 (1) is an isomorphism for each $\sigma \in G$.

(3) $J_\sigma J_{\sigma^{-1}} = C$ for each σ G.

Proof. First note that $D = \Sigma^{\oplus}_{\sigma \in G} J_\sigma$ by Prop. 1 [5] or by the obove remark. Suppose (2). Then from $DJ_\tau = D$, we have $D = \Sigma_{\sigma \in G} J_\sigma J_\tau \subset \Sigma^{\oplus} J_{\tau\sigma} = D$ with $J_\sigma J_\tau \subset J_{\tau\sigma}$. Hence $J_\sigma J_\tau = J_{\tau\sigma}$ for each $\sigma, \tau \in G$. Especially, $J_\sigma J_{\sigma^{-1}} = J_{\sigma^{-1}\sigma} = C$. Next suppose (3). Then we have $1 = \Sigma a_j b_j$ with $a_j \in J_{\sigma^{-1}}$, $b_j \in J_\sigma$. for each $\sigma \in G$. Then if $g_\sigma(\Sigma d_i \otimes e_i) = 0$ for $d_i \in D$, $e_i \in J_\sigma$, we have that $\Sigma d_i \otimes e_i = \Sigma d_i \otimes e_i a_j b_j = \Sigma d_i e_i a_j \otimes b_j = \Sigma 0 \otimes b = 0$, since $e_i a_j \in C$. Hence $\ker g_\sigma = 0$. Clearly g_σ is onto. Hence we have (3). Thus we have proved (2)$\Leftrightarrow$(3). Suppose (3). Then each J_σ is invertible and C-finitely generated projective of rank 1. Hence $D = \Sigma^{\oplus} J_\sigma$ is C-finitely generated projective. On the other hand by (2), we obtain an isomorphism g of $A \otimes_C D^o (= \Sigma^{\oplus} A \otimes_C J^o_\sigma)$ to Δ such that $g(x \otimes d^o_\sigma) = x d_\sigma U_\sigma$ for $x \in A$, $d_\sigma \in J_\sigma$. Clearly, $jg = \eta_r$ with j an isomorphism. Hence η_r is an isomorphism. Then by Corollary 3 [10], A is H-separable over B, since A is right B-finitely generated projective. Thus we showed that (3) and (2) imply (1). Conversely by Lemma 1.2, we see that (1) implies (2) and (3).

COROLLARY 2.1. Let A be a Galois extension of B relative to G. Suppose that all elements of G are inner automorphisms. Then A is an H-separable extension of B, and D = $\Sigma^{\oplus}_{\sigma\in G} Cu_\sigma$, where each u_σ is a unit which induces σ.

Proof. Let $\sigma(x) = u_\sigma^{-1} x u_\sigma$ for $x \in A$. Then $J_\sigma = Cu_\sigma$, and $D = \Sigma^{\oplus} J_\sigma = \Sigma^{\oplus} Cu_\sigma$. Furthermore since $u_\sigma u_{\sigma^{-1}}$ is a unit of C, $J_\sigma J_{\sigma^{-1}} = Cu_\sigma u_{\sigma^{-1}} = C$. Hence A is H-separable over B by Theoem 2.3.

EXAMPLE. Let R be an arbitrary ring with 1, A = $(R)_2$ 2x2-full matrix ring over R and B = $\{\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in R\}$. Put $I = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and let $\varepsilon(\underline{A}) = \underline{A}$ amd $\sigma(\underline{A}) = I^{-1}\underline{A}I$ for $\underline{A} \in A$. Then $G = \{\varepsilon, \sigma\}$ is a group, and $B = A^G$. Furthermore, for the matrix units $e_{i,j}$ (i,j = 1,2) of A, we have $\Sigma e_{i,1} e_{1,i}$ = E and $\Sigma e_{i,1} \sigma(e_{1,i}) = 0$. Hence A is a Galois extension of B (See Theorem 1.1 [1]). Then by Corollary 2.1, A is an H-separable extension of B. Note that $|G| = 2$ is not always unit.

3. Separability of trivial extensions.

In this section the author wants to show some results which he obtained recently and will appear in [16]. But the proof is different from that which is shown in [16]. For an A-A-module M we can introduce to $A \oplus M$ a ring structure by $(a + m)(b + n) = ab + (an + mb)$ for $a, b \in A$ and $m, n \in M$. This ring is called the trivial extension of A with recpect to M. Now we shall show that any trivial extension can never be a separable extension. First we will prove the next

PROPOSITION 3.1. Let A be a separable extension of B. and suppose that $A = B \oplus M$ for some B-B-submodule M of A such that $M^2 \subset M$. Then M is generated by a central idempotent.

Proof. Since $M^2 \subset M$, M is an ideal of A. Now let R be a subring of $C \cap B$. Then A and B are R-algebras, and $A \otimes_R A^o$ is a separable extension of $\iota(B \otimes_R B^o) = \mathrm{Im}(B \otimes_R B^o \to A \otimes_R A^o)$ by Prop. 2.7 [2]. Hence $A \otimes_R A^o$ is a semisimple extension of $\iota(B \otimes_R B^o)$ in the sense of [2] by Prop. 2.6 [2]. Then since M is a left $A \otimes_R A^o$-submodule of A and an $\iota(B \otimes_R B^o)$-direct summand, M is an $A \otimes_R A^o$-direct summand of A. This means that M is an A-A-direct summand of A.

THEOREM 3.1. Any trivial extension can not be a separable extension.

Proof. Let A be a trivial extension of B. Then $A = B \oplus M$ as B-B-module with $M^2 = 0 \subset M$. If A is a separable extension of B, $M = Ae$ for some $e = e^2 \in C$. Then $M^2 = M \neq 0$, a contradiction. Hence A is not a separable extension of B.

COROLLARY 3.1. Let A be a separable extension of B such that $A = B \oplus M$ as B-B-module with $B \supset M^2$. Then we have $M = M^3$, and M^2 is an idempotent ideal of B.

Proof. Let $M^2 = \alpha$. α is clearly an ideal of B, and $\alpha A = A\alpha = \alpha \oplus M^3$ is an ideal of A. Then by Prop. 2.4 [2] $A/\alpha A = B/\alpha \oplus M/M^3$ is separable over B/α. But $(M/M^3)^2 = 0$. Hence $M/M^3 = 0$ by Theorem 3.1. Then $M = M^3$ and $M^2 = M^4$.

PROPOSITION 3.2. Let A and B satisfy the same conditions as Corollary 3.1. Furthermore let B be a local ring and A be finitely generated as left or right B-module. Then we have $A = B[X,\sigma]/(X^2 - a)$ for some automorphism σ of B and a unit a of B such that $xa = a\sigma^2(x)$ for all $x \in B$.

Proof. Let J be the radical of B. Since $\alpha A = \alpha + M$, $A = B + \alpha A$. Hence we have $0 \neq \alpha \not\subset J$ by Nakayama's Lemma. Hence $M^2 = B$. Then M_B and ${}_BM$ are invertible and free of rank 1, since

B is local. Then by direct computations we have that M = Bm = mB and A = $B[X,\sigma]/(X^2 - m^2)$, where bm = mσ(b) for all b $\in$ B. See [16] for detail.

References

[1] M. Harada and T. Kanzaki; Galois theory of ring (in Japanese), Sugaku 18 (1966), 144-159, Iwanami.

[2] K. Hirata and K. Sugano; On semisimple extensions and separable extensions over non commutative rings, J. Math. Soc. Japan, 18 (1966), 360-373.

[3] K. Hirata; Some types of separable extensions of rings, Nagoya Math. J., 33 (1968), 107-115.

[4] K. Hirata; Separable extensions and centralizers of rings, Nagoya Math J., 35 (1969), 31-45.

[5] T. Kanzaki; On Golois algebra over a commutative ring, Osaka J. Math., 2 (1965), 309-317

[6] T. Nakamoto; On QF-extensions in an H-separable extension, Proc. Japan Acad., 50 (1974), 440-443.

[7] T. Nakamoyo and K. Sugano; Note on H-separable extensions, Hokkaido Math. J., 4 (1975), 295-299.

[8] K. Sugano; Note on semisimple extensions and separable extensions, Osaka J. Math., 4 (1967), 265-270.

[9] K. Sugano; Separable extensions and Frobenius extensions, Osaka J. Math., 7 (1970), 291-299.

[10] K. Sugano; Note on separability of endomorphism rings, J. Fac. Sci. Hokkaido Univ., 21 (1971), 196-208.

[11] K. Sugano; On some commutor theorem of rings, Hokkaido Math. J., 1 (1972) 242-249.

[12] K. Sugano; Separable extensions of quasi-Frobenius rings, Algebra-Berichite, 28 (1975), Uni-Druck, Munchen.

[13] K. Sugano; On projective H-separable extensions, Hokkaido Math. J., 5 (1976), 44-54.

[14] K. Sugano; Note on automorphisms in separable extensions, to appear in Hokkaido Math. J..

[15] K. Sugano; On a special type of Galois extensions, to appear in Hokkaido Math. J..

[16] K. Sugano; Note on separable extensions over a local ring, to appear.

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ON SEPARABLE POLYNOMIALS IN SKEW POLYNOMIAL RINGS

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Throughout the present paper, every ring has identity 1, its subring contains 1, and every module over a ring is unital. A ring homomorphism means such one sending 1 to 1. In what follows, B will represent a ring, ρ an automorphism of B, D a ρ-derivation of B (i.e. an additive endomorphism of B such that $D(ab) = D(a)\rho(b) + aD(b)$ for all $a, b \in B$). Let $R = B[X; \rho, D]$ be the skew polynomial ring in which the multiplication is given by $aX = X\rho(a) + D(a)$ $(a \in B)$. In particular, we set $B[X; \rho] = B[X; \rho, 0]$, $B[X; D] = B[X; 1, D]$. By $R_{(0)}$ we denote the set of all monic polynomials g in R with $gR = Rg$.

A ring extension B/A is called a separable extension if the B-B-homomorphism of $B\otimes_A B$ onto B defined by $a\otimes b \to ab$ splits, and B/A is called an H-separable extension if $B\otimes_A B$ is B-B-isomorphic to a direct summand of a finite direct sum of copies of B. As is well known, an H-separable extension is a separable extension (Hirata). A polynomial g in $R_{(0)}$ is called a separable (resp. H-separable) polynomial if R/gR is a separable (resp. H-separable) extension of B.

Now, let G be a finite group of automorphism of a ring B, and $A = B^G = \{b \in B \mid \sigma(b) = b\ (\sigma \in B)\}$. If there exist $x_i, y_i \in B$ such that $\sum_i x_i\sigma(y_i) = \delta_{1,\sigma}$ $(\sigma \in G)$, then B/A is called a G-Galois extension.

We shall use the following convensions:

C(A) = the center of a ring A.

$V_B(A)$ = the centralizer of A in B for a ring extension B/A.

u_ℓ (resp. u_r) = the left (resp. right) multiplication effected by $u \in B$.

$B^\rho = \{a \in B \mid \rho(a) = a\}$, $B^D = \{a \in B \mid D(a) = 0\}$.

$\rho^* : B[X; \rho] \to B[X; \rho]$ is the ring automorphism defined by $\rho^*(\sum_i X^i d_i) = \sum_i X^i \rho(d_i)$.

$D^* : B[X; D] \to B[X; D]$ is the inner derivation defined by $D^*(\sum_i X^i d_i) = \sum_i X^i D(d_i)$.

For several years, separable polynomials in skew polynomial rings are extensively studied by Kishimoto [9,10], Nagahara [13,14,15,16,17], Miyashita [12], and by the author [4,5,6].

The main result of this paper is the following: Let B be a commutative ring, $A = B^\rho$, and $f \in R_{(0)} = B[X; \rho]_{(0)}$. If R/fR is an Azumaya A-algebra, then the order of ρ is equal to the degree of f and B/A is a Galois extension with Galois group $G = \langle\rho\rangle$, and f is of the from $X^m + a_0$ with a unit a_0 in A (Theorem 2.2). Conversely, if B/A is a Galois extension with a cyclic automorphism group $G = \langle\rho\rangle$ of order m, then for every unit a in A, $B[X; \rho]/(X^m + a)B[X; \rho]$ is an Azumaya A-algebra (Corollary 2.3). The present study contains also some sharpenings of G. Szeto[19,20] and Y. F. Wong [21], and some results concerning skew polynomial rings of derivation type (§ 3).

In our study, H-separable polynomials in skew polynomial rings play important rôles. Therefore, § 1 is devoted to giving preliminary results concerning H-separable polynomials.

We shall use freely the results of [6].

1. The present section is devoted to giving preliminary results concerning H-separable polynomials in $B[X; \rho]$ and $B[X; D]$, which play important rôles in the subsequent study.

First, we shall prove the following

Proposition 1.1 ([7]). Let S be a ring with center C, and B an intermediate ring of S/C. If S is an Azumaya C-algebra and S_B (or ${}_BS$) is (f.g.) projective, then S/B is an H-separable extension.

Proof. Since S/C is separable, there exists $\sum_i a_i'\otimes a_i''$ in $S\otimes_C S$ such that $\sum_i a_i'a_i'' = 1$ and $\sum_i xa_i'\otimes a_i'' = \sum_i a_i'\otimes a_i''x$ for all $x \in S$. Further, since S_B is f.g. projective, there exist $a_j \in S$ and $f_j \in \mathrm{Hom}(S_B, B_B)$ such that $\sum_j a_jf_j(x) = x$ for all $x \in S$. Consider the map $\theta : S\otimes_C S \to S\otimes_C S$ defined by $x\otimes y \to \sum_j xa_j\otimes f_j(y)$. Then, for all $x \in S$,

$$\begin{aligned}\sum_{i,j} a_i'a_j\otimes f_j(a_i''x) &= \theta(\textstyle\sum_i a_i'\otimes a_i''x)\\ &= \theta(\textstyle\sum_i xa_i'\otimes a_i'') = \sum_{i,j} xa_i'a_j\otimes f_j(a_i''),\end{aligned}$$

and hence, for all $a, x, y \in S$,

$$\sum_{i,j} a_i'a_j\otimes f_j(a_i''ax)y = \sum_{i,j} aa_i'a_j\otimes f_j(a_i''x)y.$$

This proves the map $\phi : S\otimes_B S \to S\otimes_C S$ defined by $x\otimes y \to \sum_{i,j} a_i'a_j\otimes f_j(a_i''x)y$ is an S-S-homomorphism. Obviously, the canonical map $\psi : S\otimes_C S \to S\otimes_B S$ is an S-S-homomorphism and $\psi\phi$ is the identity map of $S\otimes_B S$. Hence, ${}_SS\otimes_B S_S <\oplus\, {}_SS\otimes_C S_S$. As is well known, there exists a positive integer m such that ${}_SS\otimes_C S_S <\oplus\, {}_SS_S^m$. From those above, it is immediate that ${}_SS\otimes_B S_S <\oplus\, {}_SS_S^m$, namely S/B is an H-separable extension.

In the rest of this section, let $f = X^m + X^{m-1}a_{m-1} + \dots + Xa_1 + a_0$ be in $B[X; \rho, D]$. Now, we shall state the following theorem which has been proved by Miyashita in a different form.

Theorem 1.2 ([12, Theorem 1.9]). Let f be in $R_{(0)} = B[X; \rho, D]_{(0)}$, and $I = fR$. If f is an H-separable polynomial in R, then there exist $y_i, z_i \in R$ with $\deg y_i < m$ and $\deg z_i < m$ such that $ay_i = y_i a$, $\rho^{m-1}(a)z_i = z_i a$ $(a \in B)$ and $\sum_i y_i X^{m-1} z_i \equiv 1 \pmod I$, $\sum_i y_i X^k z_i \equiv 0 \pmod I$ $(0 \leqq k \leqq m-2)$, and conversely.

As easy consequences of Theorem 1.2, we have the following usefull lemmas.

Lemma 1.3. If f is an H-separable polynomial in $R = B[X; \rho]$, then a_0 is a unit in B, $\rho^m = (a_0^{-1})_\ell (a_0)_r$, and f is in $C(B^\rho)[X]$.

Proof. By Theorem 1.2, $1 \equiv \sum_i y_i X^{m-1} z_i \equiv X^{m-1} \sum_i \rho^{*m-1}(y_i) z_i \equiv \sum_i y_i \rho^{*-(m-1)}(z_i) X^{m-1} \pmod{fR}$. We put here $x = X + fR$. Then, x is invertible in R/fR. Hence there exist $d_j \in B$ $(0 \leqq j \leqq m-1)$ such that

$$x(x^{m-1}d_{m-1} + \dots + d_0) = (x^{m-1}d_{m-1} + \dots + d_0)x = 1 .$$

Since $x^m = -x^{m-1}a_{m-1} - \dots - a_0$ and $\{1, x, \dots, x^{m-1}\}$ is a free basis of R/fR_B, we have $-a_0 d_{m-1} = -a_0\rho(d_{m-1}) = 1$. Then, since $aa_0 = a_0\rho^m(a)$ $(a \in B)$ ([6, Lemma 1.3 a)]), a_0 is a unit in B and $\rho^m = (a_0^{-1})_\ell (a_0)_r$. Hence, $\rho(d_{m-1}) = d_{m-1}$, and therefore, $\rho(a_0) = a_0$. Thus, f is in $C(B^\rho)[X]$ by [6, Proposition 3.1].

Lemma 1.4. If f is an H-separable polynomial in

$R = B[X; \rho]$, then $1, \rho, \ldots, \rho^{m-1}$ $(\varepsilon\ {}_B\mathrm{Hom}({}_B B_B\rho\,, {}_B B_B\rho))$ are linearly independent over B.

Proof. Assume that $\sum_{j=0}^{m-1} \beta_j \rho^j = 0$ $(\beta_j \in B)$. Then, it is easily verified that $\sum_{j=0}^{m-1} \rho^{\nu}(\beta_j)\rho^j = 0$ $(\nu \geqq 0)$. Hence we have $\sum_{j=0}^{m-1} \beta_j \rho^{*j} = 0$. Let y_i, $z_i \in R$ be as in Theorem 1.2. Then we have

$$\begin{aligned}\rho^{-(m-1)}(\beta_{m-1}) &\equiv \sum_{j=0}^{m-1} \rho^{-(m-1)}(\beta_j)X^{m-1} \sum_i \rho^{*j}(y_i)z_i \\ &\equiv X^{m-1} \sum_{j=0}^{m-1} \beta_j \sum_i \rho^{*j}(y_i)z_i \\ &\equiv X^{m-1} \sum_i (\sum_{j=0}^{m-1} \beta_j \rho^{*j}(y_i))z_i \\ &\equiv 0 \pmod{fR}.\end{aligned}$$

Therefore, we have $\beta_{m-1} = 0$. Now, by $\sum_{j=0}^{m-2} \beta_j \rho^{*j} = 0$, we have $\sum_{j=0}^{m-2} \beta_j \rho^{*j+1} = 0$. By the similar way as above, we have $\beta_{m-2} = 0$. Repeating this, we conclude that $\beta_j = 0$ $(0 \leqq j \leqq m-1)$.

Next, we consider the case $R = B[X; D]$. Then it is easily verified that the condition $\sum_i y_i X^{m-1} z_i \equiv 1 \pmod I$, $\sum_i y_i X^k z_i \equiv 0 \pmod I$ $(0 \leqq k \leqq m-2)$ in Theorem 1.2 is equivalent to the one $\sum_i D^{*m-1}(y_i)z_i \equiv 1 \pmod I$, $\sum_i D^{*k}(y_i)z_i \equiv 0 \pmod I$ $(0 \leqq k \leqq m-2)$. Thus, we have the following

Lemma 1.5. Let f be in $R_{(0)} = B[X; D]_{(0)}$, and $I = fR$. If f is H-separable in R, then there exist y_i, $z_i \in R$ with $\deg y_i < m$ and $\deg z_i < m$ such that $ay_i = y_i a$, $az_i = z_i a$ $(a \in B)$ and $\sum_i D^{*m-1}(y_i)z_i \equiv 1$ $\pmod I$, $\sum_i D^{*k}(y_i)z_i \equiv 0 \pmod I$ $(0 \leqq k \leqq m-2)$, and conversely.

Corresponding to Lemma 1.4, we have the following

Lemma 1.6. If f is an H-separable polynomial in $R = B[X; D]$, then $1, D, \dots, D^{m-1}$ ($\varepsilon\ {}_B\mathrm{Hom}(B_B D, B_B D)$) are linearly independent over B.

Proof. Assume that $\sum_{j=0}^{m-1} \beta_j D^j = 0$ ($\beta_j \varepsilon B$). Then, for all $a \varepsilon B$, $0 = D(\sum_{j=0}^{m-1} \beta_j D^j(a)) = \sum_{j=0}^{m-1} D(\beta_j)D^j(a) + \sum_{j=0}^{m-1} \beta_j D^j(D(a))$. Hence we have $\sum_{j=0}^{m-1} D(\beta_j)D^j = 0$. An easy induction shows that $\sum_{j=0}^{m-1} D^{\nu}(\beta_j)D^j = 0$ ($\nu \geq 0$). By this fact, we can easily verified that $\sum_{j=0}^{m-1} \beta_j D^{*j} = 0$. Now, let y_i, z_i be as in Lemma 1.5. We have then,

$$\beta_{m-1} = \sum_{j=0}^{m-1} \sum_i \beta_j D^{*j}(y_i)z_i \equiv 0 \pmod{fR}.$$

Therefore, we obtain $\beta_{m-1} = 0$. Since $\sum_{j=0}^{m-2} \beta_j D^{*j} = 0$, we have $\sum_{j=0}^{m-2} \beta_j D^{*j+1} = 0$. By the similar way as above, we have $\beta_{m-2} = 0$. Repeating this, we conclude that $\beta_j = 0$ ($0 \leq j \leq m-1$).

2. Throughout this section, B is a commutative ring, $R = B[X; \rho]$, G the cyclic group generated by ρ, and $A = B^G = B^{\rho}$.

Proposition 2.1. Assume that the order of G is m. If B/A is G-Galois, then $X^m + b_0$ is an H-separable polynomial in R for every unit b_0 in A.

Proof. Since B/A is G-Galois, there exist α_i, β_i εB such that $\sum_i \alpha_i\beta_i = 1$ and $\sum_i \rho^k(\alpha_i)\beta_i = 0$ ($1 \leq k \leq m-1$). We put here $y_i = \alpha_i$ and $z_i = -Xb_0^{-1}\beta_i$. Obviously, $ay_i = y_ia$ and $\rho^{m-1}(a)z_i = z_ia$ ($a \varepsilon B$). Since

$X^m \equiv -b_0 \pmod{(X^m + b_0)R}$, it is easily seen that $\sum_i y_i X^{m-1} z_i \equiv 1$ and $\sum_i y_i X^k z_i \equiv 0$ $(0 \leqq k \leqq m-2)$. Thus, $X^m + b_0$ is H-separable in R by Theorem 1.2.

We are now in a position to state our first main theorem

Theorem 2.2. Let $R = B[X; \rho]$. Let $f = X^m + X^{m-1}a_{m-1} + \dots + Xa_1 + a_0$ be in $R_{(0)}$, and $S = R/fR$. Then S is an Azumaya A-algebra if and only if f is an H-separable polynomial in R. When this is the case, we have the following:

(a) The order of ρ is m and B/A is G-Galois.

(b) $f = X^m + a_0$ (i.e. $a_{m-1} = \dots = a_1 = 0$), and a_0 is a unit in A.

(c) B is a maximal commutative A-subalgebra of S with $B \otimes_A S \cong M_m(B)$ and $S \otimes_A B \cong M_m(B)$.

(d) If m is invertible in B, then $H = A[X]/fA[X]$ is a separable splitting ring for S.

Proof. Assume that S is an Azumaya A-algebra. Since $S \supseteqq B$ and S_B is free, f is H-separable in R by Proposition 1.1.

Next, assume that f is H-separable in R. Then by Lemmas 1.3 and 1.4, we see that the order of ρ is m and a_0 is a unit in A. Since $fR = Rf$, we have $aa_i = a_i\rho^{m-i}(a)$ $(a \in B,\ 0 \leqq i \leqq m-1)$ ([6, Lemma 1.3 a)]). Hence, $a_i 1 - a_i\rho^{m-i} = 0$, which implies $a_i = 0$ $(1 \leqq i \leqq m-1)$ by Lemma 1.4. Now, we shall prove that B/A is G-Galois. Let $y = X^{m-1}d_{m-1} + \dots + Xd_1 + d_0$ be in R such that $ay = ya$ $(a \in B)$. Then there holds $\rho^{m-i}(a)d_i = d_i a$ $(0 \leqq i \leqq m-1)$. Hence, by Lemma 1.4, we have $d_i = 0$ $(1 \leqq i \leqq$

$\leq$ m-1), i.e. $y = d_0$. This shows $V_S(B) = B$. Let $z = X^m c_{m-1} + \ldots + c_0$ be in R such that $\rho^{m-1}(a)z = za$ $(a \in B)$. Then in the similar way as above, we have $z = Xc_1$. Let y_i, $z_i \in R$ be as in Theorem 1.2. According to the above, we may assume $y = \alpha_i$ and $z_i = X\beta_i$ $(\alpha_i, \beta_i \in B)$. Since $X^m \equiv -a_0$ (mod fR), we have

$$1 \equiv \sum_i y_i X^{m-1} z_i \equiv -\sum_i a_0 \alpha_i \beta_i \pmod{fR}$$

and

$$0 \equiv \sum_i y_i X^k z_i \equiv X^{k+1} \sum_i \rho^{k+1}(\alpha_i)\beta_i \pmod{fR}$$

$(0 \leq k \leq m-2)$. Then, we have $\sum_i \alpha_i(-a_0\beta_i) = 1$ and $\sum_i \rho^{k+1}(\alpha_i)(-a_0\beta_i) = 0$ $(0 \leq k \leq m-2)$. Thus, B/A is G-Galois. Since $V_S(B) = B$, it is now clear that C(S) = A. Hence S is an Azumaya A-algebra, because both S/B and B/A are separable extensions. Now the latter half of (c) is immediate by [3, Proposition 3.1] or [18, Lemma 1(3)]. Finally, we shall prove (d). If m is invertible in B, then $f = X^m + a_0$ is a separable polynomial in A[X] by [6, Theorem 2.2]. Moreover, H may be considered as an A-subalgebra of S, and then $V_S(H) = H$. Thus, H is a separable splitting ring for S by [2, Theorem 5.5 p.64]. This complites the proof.

Now, by making use of Proposition 2.1 and Theorem 2.2, we can improve the results of G. Szeto [19,20] and G. Szeto and Y. F. Wong [21]. First, the following contains [20, Lemma 3.1 and Theorem 3.2] (or [19, Lemma 2.1 and Theorem 2.2]).

Corollary 2.3. The following are equivalent:

(a) The order of ρ is m and B/A is G-Galois.

(b) R contains an H-separable polynomial of degree m.

(b') $X^m + a_0$ is an H-separable polynomial in R for some a_0 in A.

(b") $X^m + a_0$ is an H-separable polynomial in R for every unit a_0 in A.

(c) $R_{(0)}$ contains a polynomial f of degree m such that R/fR is an Azumaya A-algebra.

(c') $R/(X^m + a_0)R$ is an Azumaya A-algebra for some a_0 in A.

(c") $R/(X^m + a_0)R$ is an Azumaya A-algebra for every unit a_0 in A.

When this is the case, $(R/(X^m + a_0)R)\otimes_A B \cong M_m(B)$ and $B\otimes_A(R/(X^m + a_0)R) \cong M_m(B)$ for every unit a_0 in A.

The following is an sharpening of [20, Theorem 3.5] (or [19, Theorem 2.5]) and [21, Theorem 3.6].

Corollary 2.4. Assume that the order of ρ is m. Let $X^m + a_0$ be in $R_{(0)}$. If m is invertible in B and $B\otimes_A(R/(X^m + a_0)R)$ is an Azumaya B-algebra, then B/A is G-Galois.

Proof. Since m is invertible in B, A is a direct summand of B. Hence, by [2, Corollary 1.10 p.45], $R/(X^m + a_0)R$ is an Azumaya A-algebra. Thus, B/A is a G-Galois extension by Corollary 2.3.

The next sharpens [21, Theorem 3.5].

Corollary 2.5. Let $f = X^m + X^{m-1}a_{m-1} + \dots + a_0$

be in $A[X] \cap R_{(0)}$. If $(A[X]/fA[X]) \otimes_A (R/fR)$ is an Azumaya $A[X]/fA[X]$-algebra, then the order of ρ is m and B/A is a G-Galois extension.

In the rest of this section, we assume that the order of ρ is $m \geqq 2$, and B/A is a G-Galois extension. Then the following is well known.

Lemma 2.6. Let $b \in B$, and $1 \leqq j \leqq m-1$. If $(\rho^j(a) - a)b = 0$ for all $a \in B$, then $b = 0$.

Now, we are able to determine all the separable polynomials in R.

Proposition 2.7. Let g be any separable polynomial in R. Then the following hold:

(a) If $\deg g = 1$, then $g = X$.

(b) If $\deg g \geqq 2$, then there exists a separable polynomial $g_0(t) = t^k + t^{k-1}c_{k-1} + \dots + c_0$ in $A[t]$ such that c_0 is a unit in A, and $g = g_0(X^m)$ or $g = Xg_0(X^m)$.

Proof. Let $g = X^n + X^{n-1}d_{n-1} + \dots + d_0$. Since $gR = Rg$, we have $ad_i = d_i\rho^{n-i}(a)$ $(a \in B)$. If $g = X + d_0$, then $g = X$ by Lemma 2.6. Hence, we may assume $n \geqq 2$. By [4, Lemma 1], there exist $\alpha, \beta \in B$ such that $d_1\alpha - d_0\beta = 1$. Hence we have $d_1 \neq 0$ or $d_0 \neq 0$. If $d_0 \neq 0$, then $(\rho^n(a) - a)d_0 = 0$ $(a \in B)$ implies that $\rho^n = 1$ (Lemma 2.6), and therefore $m \mid n$. We put here $n = km$. Then by Lemma 2.6, we can easily see that $g = X^{mk} + X^{m(k-1)}c_{k-1} + \dots + X^m c_1 + c_0$ $(c_i = d_{mi})$, and c_0 is a unit in A. Similarly, if $d_1 \neq 0$ then we can write $g = X(X^{mj} + X^{m(j-1)}c'_{j-1} + \dots + c'_0)$, and c'_0 is a unit in A $(c_i' = d_{mi+1})$. We put here $h = X^{mj} + X^{m(j-1)}c_{j-1}' + \dots + c_0'$. Then, $g = Xh = hX$ and $hR = Rh$. Hence, h is a separable polynomial in R by [12, Theorem 1.10]. Therefore, we may consider the case $g = X^{mk} + X^{m(k-1)}c_{k-1} + \dots + c_0$. We put $S = R/fR$. Since S/B and B/A are separable extensions, S is a separable A-algebra. It is easily verified that $C(S) = (A[X^m] + gR)/gR \cong A[t]/g_0A[t]$, where $g_0 = t^k + t^{k-1}c_{k-1} + \dots + c_0 \in A[t]$. Since S is a separable A-algebra, S is an Azumaya $C(S)$-algebra and $C(S) \cong A[t]/g_0A[t]$ is a separable A-algebra by [2, Theorem 3.8 p.55]. Thus, g_0 is a separable polynomial in $A[t]$ and c_0 is a unit in A.

Concerning the converse of Proposition 2.7, we have the following

Proposition 2.8. Let $g_0(t) = t^k + t^{k-1}c_{k-1} + \dots + c_0$ be a separable polynomial in $A[t]$ such that c_0 is a unit in A. Then, both $g_0(X^m)$ and $Xg_0(X^m)$ are separable polynomials in R.

Proof. Obviously, both $g_0(X^m)$ and $Xg_0(X^m)$ are contained in $R_{(0)}$. Since $R/Xg_0(X^m)R \cong B \oplus (R/g_0(X^m)R)$, it suffices to prove the separability of $g_0(X^m)$. We put $g = g_0(X^m)$ and $S = R/gR$. Since c_0 is a unit in A, $u = X^m + gR$ is also unit in S. It is easily verified that $V_S(B) = B[u] \cong B[t]/g_0B[t]$ and $C(S) = A[u] \cong A[t]/g_0A[t]$. Let $\hat{\rho} : B[u] \to B[u]$ be the map defined by $\hat{\rho}(\sum_i u^i b_i) = \sum_i u^i \rho(b_i)$. Then, $\hat{\rho}$ is an $A[u]$-automorphism of order m. Since B/A is G-Galois, $B[u]/A[u]$ is also a $\langle\hat{\rho}\rangle$-Galois extension. Consider the skew polynomial ring $B[u][Y; \hat{\rho}]$ defined by $\beta Y = Y\hat{\rho}(\beta)$ $(\beta \in B[u])$.

Then, since $Y^m - u$ is contained in $B[u][Y; \rho]_{(0)}$ and u is a unit in $A[u]$, $B[u][Y; \rho]/(Y^m - u)B[u][Y; \rho]$ is an Azumaya $A[u]$-algebra by Corollary 2.3. It is easily seen that S is $B[u]$-ring isomorphic to $B[u][Y; \rho]/(Y^m - u)B[u][Y; \rho]$. Hence S is a separable $A[u]$-algebra. Since $A[u]$ is a separable A-algebra, S is a separable A-algebra. Therefore, S/B is a separable extension, which means that g is a separable polynomial in R.

3. Throughout this section, B will mean a commutative ring, $R = B[X; D]$, and $A = B^D$. First, we state the following lemma.

Lemma 3.1. Let f be in $R_{(0)}$, $\deg f = m$, and $S = R/fR$. Then the following are equivalent:

(a) S is an Azumaya A-algebra.

(b) f is an H-separable polynomial in R.

(c) There exist $y_i, z_i \in B$ such that $\sum_i D^{m-1}(y_i)z_i = 1$ and $\sum_i D^k(y_i)z_i = 0$ $(0 \leqq k \leqq m-2)$.

When this is the case, B is a maximal commutative A-subalgebra of S with $B \otimes_A S \cong M_m(B)$ and $S \otimes_A B \cong M_m(B)$.

Proof. (a) $\Rightarrow$ (b). Since $S \supseteqq B$ and S_B is free, f is an H-separable polynomial in R by Proposition 1.1.

(b) $\Leftrightarrow$ (c). By Lemma 1.6, we can easily see that $V_S(B) = B$. Hence the assertion is obvious by Lemma 1.5.

(b) $\Rightarrow$ (a). By Lemma 1.6, we have $V_S(B) = B$, and hence $C(S) = A$. Then by [18, Lemma 1(3)], ${}_AB$ is f.g. projective, $B \otimes_A S \cong M_m(B)$ and $S \otimes_A S \cong M_m(B)$. Since ${}_AB$ is f.g. projective faithful, A is a direct summand of B. Thus, S is an Azumaya A-algebra by [2, Corollary 1.10 p.45].

As an immediate consequence of Lemma 3.1, we have the following

Corollary 3.2. Let f be in $R_{(0)}$. If f is an H-separable polynomial in R, then so is $f + a$ for every $a \in A$.

Now, we shall state the main theorem of this section.

Theorem 3.3. Let $R = B[X; D]$. Then, the following are equivalent:

(a) R contains an H-separable polynomial of degree m.

(b) $R_{(0)}$ contains a polynomial f of degree m such that R/fR is an Azumaya A-algebra.

(c) $R_{(0)}$ contains a polynomial f of degree m, and there exist $y_i, z_i \in B$ such that $\sum_i D^{m-1}(y_i)z_i = 1$ and $\sum_i D^k(y_i)z_i = 0$ $(0 \leqq k \leqq m-2)$.

(d) ${}_AB$ is a finitely generated projective module of rank m and $\mathrm{Hom}({}_AB, {}_AB) = B[D]$ (i.e. $\mathrm{Hom}({}_AB, {}_AB)$ is generated by the set $\{b_\ell \mid b \in B\}$ and D as ring).

Proof. (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c). These equivalences have been proved in Lemma 3.1.

(b), (c) $\Rightarrow$(d). Since $fR = Rf$, we have $D^m + a_{m-1}D^{m-1} + \dots + a_1D = 0$ and $a_i \in A$ by [6, Lemma 1.6], where $f = X^m + X^{m-1}a_{m-1} + \dots + Xa_1 + a_0$. Then the map $f_i : B \to B$ defined by $f_i(b) = \sum_{j=0}^{m-1} a_{j+1}D^j(by_i)$ $(a_m = 1)$ is in $\mathrm{Hom}({}_AB, {}_AA)$, since $D(f_i(b)) = 0$. According to (c), we have

$$\sum_i f_i(b)z_i = \sum_i (\sum_{j=0}^{m-1} a_{j+1}D^j(by_i))z_i$$

$$= \sum_i \sum_{j=0}^{m-1} a_{j+1}(\sum_{\nu=0}^{j} \binom{j}{\nu} D^{j-\nu}(b)D^{\nu}(y_i))z_i$$
$$= \sum_{j=0}^{m-1} a_{j+1} \sum_{\nu=0}^{j} \binom{j}{\nu} D^{j-\nu}(b)(\sum_i D^{\nu}(y_i)z_i)$$
$$= a_m b \sum_i D^{m-1}(y_i)z_i = b \quad (b \in B).$$

Hence, ${}_AB$ is f.g. projective. On the other hand,

$$f_i(b) = \sum_{j=0}^{m-1} a_{j+1}D^j(by_i)$$
$$= \sum_{j=0}^{m-1} a_{j+1}(\sum_{\nu=0}^{j} \binom{j}{\nu} D^{j-\nu}(y_i)D^{\nu}(b))$$
$$= \sum_{\nu=0}^{m-1} (\sum_{j=\nu}^{m-1} a_{j+1}\binom{j}{\nu} D^{j-\nu}(y_i))D^{\nu}(b).$$

Hence, $f_i = \sum_{\nu=0}^{m-1} (\sum_{j=\nu}^{m-1} a_{j+1}\binom{j}{\nu} D^{j-\nu}(y_i))D^{\nu} \in B[D]$. Since $\phi = \sum_i \phi(z_i)f_i$ ($\phi \in \mathrm{Hom}({}_AB, {}_AB)$), and $1, D, \dots, D^{m-1}$ are linearly independent over B (Lemma 1.6), we have $\mathrm{Hom}({}_AB, {}_AB) = B[D] = B \oplus BD \oplus \dots \oplus BD^{m-1}$. Now, we shall show that ${}_AB$ is of rank m. Let P be a prime ideal of A. Assume that ${}_{A_P}B_P$ is of rank n. Let $\hat{D}$ be the natural extension of D to B_P (i.e. $\hat{D} = D \otimes 1$). Then we have $\mathrm{Hom}({}_{A_P}B_P, {}_{A_P}B_P) = B_P[\hat{D}] = B_P \oplus B_P\hat{D} \oplus \dots \oplus B_P\hat{D}^{m-1}$. The A_P-rank of the left hand side is n^2 and that of the right hand side is nm. Thus, we have $n = m$, which means ${}_AB$ is of rank m.

(d) $\Rightarrow$ (b). Since $Db = bD + (D(b))_\ell$ $(b \in B)$, the map $\psi : B[X; D] \to \mathrm{Hom}({}_AB, {}_AB) = B[D]$ defined by $\psi(\sum_i X^i d_i) = \sum_i (-D)^i (d_i)_\ell$ is a B-ring epimorphism. Then we have $R/\mathrm{Ker}\,\psi \cong \mathrm{Hom}({}_AB, {}_AB)$. Since ${}_AB$ is projective of rank m, $\mathrm{Hom}({}_AB, {}_AB)$ is an Azumaya A-algebra of rank m^2. Hence $R/\mathrm{Ker}\,\psi$ is a projective B-module of rank m (cf. [1]). Then by [11, Theorem 3], there exist a polynomial f in $R_{(0)}$ such that $\mathrm{Ker}\,\psi = fR$. Now, it is obvious that the degree of f is m. This proves (b).

Remark 3.4. Assume that B is of prime characteristic p. Let f be an H-separable polynomial in R of degree m. Then, in virtue of Lemma 1.6, it is easily seen that $m = p^e$ and f is a p-polynomial of the form $\sum_{i=0}^{e} X^{p^i} b_{i+1} + b_0$. Hence, Theorem 3.3 contains the result of S. Yuan [22, Theorem 2.4].

Corresponding to Proposition 2.7 and 2.8, we have the following theorem.

Theorem 3.5. Assume that R contains an H-separable polynomial $f = X^m + X^{m-1}a_{m-1} + \dots + Xa_1$. Let $\psi : A[t] \to R$ be befined by $\psi(\sum_i t^i d_i) = \sum_i f^i d_i$.

(a) ψ induces a one-to-one correspondence between $A[t]_{(0)}$ and $R_{(0)}$.

(b) Let g_0 be in $A[t]_{(0)}$. Then g_0 is a separable polynomial in $A[t]$ if and only if $R/\psi(g_0)R$ is a separable A-algebra.

(c) Let g_0 be in $A[t]_{(0)}$. Then $\psi(g_0)$ is an H-separable polynomial in R if and only if $\deg g_0 = 1$.

Proof. Obviously, ψ induces an injective mapping of $A[t]_{(0)}$ into $R_{(0)}$. Let g be in $R_{(0)}$. Since $g, f \in A[X]$, there exist $h, r \in A[X]$ such that $g = hr + r$ and drg $r < m$. We shall show that $h \in R_{(0)}$ and $r \in A$. Assume that there exists $b \in B$ such that $bh - hb \neq 0$. Since $bg = gb$ and $bf = fb$, we have $(bh - hb)f = rb - br$. However, since f is monic and $\deg r < m$, we have $\deg (bh - hb)f \geqq m$ and $\deg (rb - br) < m$, which is a contradiction. Hence h is in $R_{(0)}$, and so $rc = cr$ $(c \in B)$. Then since $1, D, D^2, \dots, D^{m-1}$ are linearly independent over B (Lemma 1.6) and $\deg r < m$, we have

$r \in A$. By those above, we can easily verify that there exist $c_i \in A$ such that $g = f^k + f^{k-1}c_{k-1} + \dots + fc_1 + c_0$. Thus, ψ maps $A[t]_{(0)}$ onto $R_{(0)}$. This proves (a).

Let g_0 be in $A[t]_{(0)}$. We put $g = \psi(g_0)$, $S = R/gR$ and $u = f + gR$. Then, it is easily verified that $V_S(B) = B[u] \cong B[t]/g_0B[t]$ and $C(S) = A[u] \cong A[u]/g_0A[t]$. Let $\hat{D} : B[u] \to B[u]$ be the map defined by $\hat{D}(\sum_i u^i b_i) = \sum_i u^i D(b_i)$. Then, $\hat{D}$ is a derivation of $B[u]$ and $B[u]^{\hat{D}} = A[u]$. Consider the skew polynomial ring $B[u][Y; \hat{D}]$ defined by $\beta Y = Y\beta + \hat{D}(\beta)$ $(\beta \in B[u])$. Then $h = Y^m + Y^{m-1}a_{m-1} + \dots + Ya_1 - u$ is contained in $B[u][Y; \hat{D}]_{(0)}$. Since ${}_AB$ is f.g. projective of rank m and $\mathrm{Hom}({}_AB, {}_AB) = B[D]$ (Theorem 3.3), we see that ${}_{A[u]}B[u]$ is f.g. projective of rank m and $\mathrm{Hom}({}_{A[u]}B[u], {}_{A[u]}B[u]) = B[u][\hat{D}]$. It is easily seen, S is $B[u]$-ring isomorphic to $B[u][Y; \hat{D}]/hB[u][Y; \hat{D}]$. Thus, S is an Azumaya $A[u]$-algebra by Theorem 3.3. Now, assume that g_0 is separable in $A[t]$. Then, since $A[u] \cong A[t]/g_0A[t]$ and S is an Azumaya $A[u]$-algebra, S is a separable A-algebra. Conversely, if S is a separable A-algebra, then $A[u]$ is separable over A by [2, Theorem 3.8 p.55], and so g_0 is separable in $A[t]$. This proves (b). Finally, (c) is obvious by Corollary 3.2 and Lemma 1.6.

References

[1] N. Bourbaki: Éleméŉts de mathématique, Algèbre Commutative, Chaps. 1-2, Hermann, Paris, 1961.

[2] F. Demeyer and E. Ingraham: Separable algebras over a commutative ring, Lecture notes in Math. 181, Springer, Berlin, 1971.

[3] K. Hirata: Separable extensions and centralizers of rings, Nagoya Math. J. 35 (1969), 31-45.

[4] S. Ikehata: On a theorem of Y. Miyashita, Math. J. Okayama Univ. 21 (1979), 49-52.

[5] S. Ikehata: A note on separable polynomials in skew polynomial rings of derivation type, Math. J. Okayama Univ. 22 (1980), 59-60.

[6] S. Ikehata: On separable polynomials and Frobenius polynomials in skew polynomial rings, Math. J. Okayama Univ. 22 (1980), 115-129.

[7] S. Ikehata: A note on Azumaya algebras and H-separable extensions, to appear.

[8] S. Ikehata: Azumaya algebras and skew polynomial rings, to appear.

[9] K. Kishimoto: A classification of free quadratic extensions of rings, Math. J. Okayama Univ. 18 (1976), 139-148.

[10] K. Kishimoto: A classification of free extensions of rings of automorphism type and derivation type, Math. J. Okayama Univ. 19 (1977), 163-169.

[11] Y. Miyashita: Note on an ideal of a positively filtered ring over a commutative ring, Math. J. Okayama Univ. 19 (1976), 61-63.

[12] Y. Miyashita: On a skew polynomial ring, J. Math. Soc. Japan 31 (1979), 317-330.

[13] T. Nagahara: On separable polynomials of degree 2 in skew polynomial rings,'Math. J. Okayama Univ. 19 (1976), 65-95.

[14] T. Nagahara: Supplements to the previous paper "On separable polynomials of degree 2 in skew polynomial rings", Math. J. Okayama Univ. 19 (1977), 159-161.

[15] T. Nagahara: On separable polynomials of degree 2 in skew polynomial rings II, Math. J. Okayama Univ. 21 (1979), 166-177.

[16] T. Nagahara: On separable polynomials of degree 2 in skew polynomial rings III, Math. J. Okayama Univ. 22 (1980), 61-64.

[17] T. Nagahara: A note on separable polynomials in skew polynomial rings of automorphism type, Math. J. Okayama Univ. 22 (1980), 73-76.

[18] T. Nakamoto: On QF-extensions in an H-separable extension, Proc. Japan Acad. 50 (1974), 440-443.

[19] G. Szeto: On Galois extensions over commutative rings, Proceedings of the 1978 Antwerp Conference, Lecture notes pure and applied Math. 51, Mercel Dekker, New York- Basel, 1979.

[20] G. Szeto: A characterization of a cyclic Galois extension of commutative rings, J. Pure and Applied algebra 16 (1980), 315-322.

[21] G. Szeto and Y. F. Wong: On quaternion algebras over a commutative ring, Math. Japonica 25 (1980), 55-59.

[22] S. Yuan: On logalithmic derivatives, Bull. Soc. Math. France 96 (1968), 41-52.

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ON AUTOMORPHISMS OF SKEW POLYNOMIAL RINGS

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1. Throughout this paper, A will mean a ring with identity element 1, ρ an automorphism of A, D a derivation of A, and N the union of all nilpotent ideals of A. Further, $R = A[X;D]$ (resp. $R = A[X;\rho]$) will means the skew polynomial ring $\{\sum_{i=0}^{\infty} X^iA\}$ whose multiplication is given by $aX = Xa + D(a)$ (resp. $ax = X\rho(a)$) for $a \in A$.

In [3, Th. 1], M. Rimmer proved the following theorem: The A-linear map $A[X;\rho] \to A[X;\rho]$ defined by $X^k \to (\sum_{i=0}^{m} X^ia_i)^k$ induces an A-automorphism if and only if

(1) $a_i\rho(a) = \rho^i(a)a_i$ for any $a \in A$ and $i = 0,1,\dots,n$,

(2) a_1 is a central unit in A,

(3) a_i are nilpotent for $i \geq 2$.

In this paper, corresponding to the above, we shall study on conditions for the A-linear map $A[X;D] \to A[X;D]$ defined by $X^k \to (\sum_{i=0}^{m} X^ia_i)^k$ to be an A-automorphism. For the details refer to [2].

2. Let $S = \{s_i;\ 1 \leq i \leq k\}$ be a set of nilpotent elements of A. If $s_iA \subseteq A_i = \sum_{j=i}^{k} As_j$, then A_i is a two-sided ideal. Further A_k is a nilpotent ideal. Since $\bar{A}_i = \overline{As_i}$ is a nilpotent ideal of $\bar{A} = A/A_{i+1}$, we can see that A_i is nilpotent, by induction method. In particular, we have $S \subseteq N$.

Let N be the set of natural numbers. It is known that $(N,N) = \{(n,m);\ n, m \in N\}$ has a linear order such that $(i,j) > (i',j')$ if 1) $i+j > i'+j'$ or 2) $i+j =$

$i'+j'$ and $i > i'$. Thus we have the following

Lemma 1. (1) Let $S=\{s_i;\ 1 \le i \le k\}$ be a set of nilpotent elements of A. If $s_iA \subseteq \sum_{j=i}^{k} As_j$ for all i, then $S \subseteq N$.

(2) Let $S=\{s_{ij};\ 1 \le i \le h,\ 1 \le j \le k\}$ be a set of nilpotent elements of A. If $s_{ij}A \subseteq \sum_{p=i}^{h}\sum_{q=j}^{k} As_{pq}$, then $S \subseteq N$.

3. In the rest of this paper, we assume that Φ is an A-linear map $A[X;D] \to A[X;D]$ defined by $\Phi(X^k) = (\sum_{i=0}^{m} X^i a_i)^k$. It is easy to see that Φ induces a ring endomorphism if and only if $a\Phi(X)=\Phi(aX)$ for all $a \in A$. Since

$$a\Phi(X) = \sum_{i=0}^{m} X^i(\sum_{k=i}^{m} \binom{k}{i} D^{k-i}(a)a_k) \quad \text{and}$$

$$\Phi(aX) = (Xa+D(a)) = \sum_{i=0}^{m} X^i a_i a + D(a),$$

we have the following

Proposition 2. Φ induces a ring endomorphism if and only if

$$\text{(i)} \qquad \begin{aligned} a_i a &= \sum_{k=i}^{m} \binom{k}{i} D^{k-i}(a)a_k \quad (i \ge 1) \\ a_o a + D(a) &= \sum_{i=0}^{m} D^i(a)a_i. \end{aligned}$$

Suppose now that Φ is a ring automorphism. Then there exists $\sum_{j=0}^{n} X^j c_j$ ($c_j \in A$) such that $\Phi(\sum_{j=0}^{n} X^j c_j) = X$. Then $\{c_j;\ j=0,1,\dots,n\}$ satisfies the same identity as (i). By (i) we can see that a_m (and c_n) are central. Now if $\Phi(X)=a_0+Xa_1$, then $X = \Phi^{-1}\Phi(X) = a_0+c_0a_1+Xc_1a_1+X^2c_2a_1+\dots+X^nc_na_1$ shows that a_1 is a unit in A. Combining this with (i), we can prove the following

Proposition 3. Let $\Phi(X) = a_0 + Xa_1$. Then Φ is an automorphism if and only if a_1 is a central unit in A and $a_0a - aa_0 = D(a)(a_1 - 1)$ for all $a \in A$.

Henceforth, we assume $m \geq 2$. By (i), an easy induction shows

(ii) $D^r(a_i)a = \sum_{k=i}^{m} \binom{k}{i} D^{k-i}(a)D^r(a_k)$ $(i \geq 1,\ r \geq 0)$.

Since c_j also satisfies (ii), we have

(iii) $D^r(a_i)D^s(c_j)a = \sum_{k=i}^{m} \sum_{h=j}^{n} \binom{k}{i}\binom{h}{j} D^{k+h-(i+j)}(a)\cdot D^r(a_k)D^s(c_h)$,

and hence

(iv) $\sum_{i=1}^{m} AD^r(a_i)$, $\sum_{j=1}^{n} AD^s(c_j)$, $\sum_{i=1}^{m}\sum_{j=1}^{n} AD^r(a_i)\cdot D^s(c_j)$ are twosided ideals $(r,\ s \geq 0)$.

4. In this section, we assume that Φ is a ring automorphism and $D(N) \subseteq N$. Then, by making use of Lemma 1, (ii), (iii) and (iv), we can prove the following lemma.

Lemma 3. (1) If a_ic_j $(i \geq 1,\ j \geq 1)$ are nilpotent whenever $i + j \geq h$, then $D^r(a_i)Ac_j \subseteq N$ $(r \geq 0)$.

(2) a_ic_j $(i,\ j \geq 1)$ are nilpotent whenever $i + j \geq 3$).

By Lemma 3 and the equality $X = \sum_{j=0}^{n} (\sum_{i=0}^{m} X^i a_i)^j c_j$, we can see that $1 = a_1c_1 + d$ with $d \in N$. Hence we obtain a_1 and c_1 are units. Again by Lemma 3, we have $(\overline{a_ic_1})^t = \bar{a}_i^t\bar{c}_1^t = 0$ in $\bar{A} = A/N$ for $i \geq 2$. Thus we obtain the following main theorem of this section.

Theorem 4. Suppose $D(N) \subseteq N$. If Φ is an automorphism then

(1) (i) is fulfilled,

(2) a_1 is a unit,

(3) a_i are nilpotent for $i \geq 2$, and therefore $\{a_i ; i \leq 2\} \leq N$.

5. In this section, we assume that Φ is a ring endomorphism and $D(N) \subseteq N$. We assume further that a_1 is a unit. Then we can easily see that $A[X;D] = A[Y;\rho,E]$, where $Y = (X - a_0)a_1^{-1}$, $\rho : a \to a_1 a a_1^{-1}$ and E is a $(\rho, 1)$-derivation defined by $a \to \sum_{i=1}^{m} D^i(a)a_i a_1^{-1}$. We set $d_i = a_i a_1^{-1}$. Then $\Phi(Y) = \sum_{i=1}^{m} X^i d_i$. Now, by N_0 we denote the ideal gen-rated by $\{D^r(a_i); i \geq 2, r \geq 0\}$. Then $N_0 = \sum_{r=0}^{\infty} \sum_{i=2}^{m} AD^r(a_i)$ by (ii). Hence if a_i are nilpotent for $i \geq 2$, then we have $D^r(d_i)(= D^r(a_i a_1^{-1})$ $D(N_0) \subseteq N_0 \subseteq N$ $(r \geq 0)$. Moreover, for any finite subset $\{s_j; 2 \leq j \leq t\}$ of A, $\sum_{j=2}^{t} \Phi(Y)^j s_j = \sum_{i=2}^{t} X^i s_i + \sum_{i=2}^{u} X^i (\sum_{j=2}^{t} d_{ij} s_j)$ where $d_{ij} \in N_0$. Noting these above, we can prove the following

Theorem 5. (1) If a_i are nilpotent for $i \geq 2$, then is a monomorphism.

(2) If N_0 is nilpotent, then Φ is an A-ring automorphism.

Proof. (1) Let $\sum_{j=0}^{t} Y^j s_j \in R$ and $\Phi(\sum_{j=0}^{t} Y^j s_j) = 0$. Then we have $s_0 = s_1 = 0$ and $0 = s_i + \sum_{j=2}^{t} d_{ij} s_j = \sum_{j=2}^{t} (1 + d_{ij}) s_j$ for some $d_{ij} \in N$ $(i \geq 2)$. Hence we can see that $s_2 = \ldots = s_t = 0$.

(2) According to (1), it remains to show that $\Phi(R) = R$. Since RN_0 is an ideal of R and $X = \Phi(Y) \pmod{RN_0}$,

we have $R = \sum_{i=0}^{\infty} \Phi(Y^i)A + RN_0$. Hence it is known that $R = \sum_{i=0}^{\infty} \Phi(Y^i)A$.

Combining Lemma 1 with Th. 5 (2), we obtain

Corollary 6. If A is Noetherian and a_i are nilpotent for $i \geq 2$, then Φ is an A-ring automorphism.

6. In this section, we prove the following proposition.

Proposition 7. If A is torsion free, then $D(N) \subseteq N$.

Proof. Let I be an ideal of A with $I^n = 0$. Obviously, $D(I) + I$ is an ideal of A. If $s_1, s_2, \ldots, s_n$ are elements of I, then $0 = D^n(s_1 s_2 \cdots s_n) = n!D(s_1)D(s_2)\cdots D(s_n) - s$ for some $s \in I$. Hence $n!D(I)^n \subseteq I$. Thus, we obtain $(D(I) + I)^{n^3} = 0$.

By the aid of Prop. 7, we can prove the following

Theorem 8. Assume that A is torsion free. Assume further that (i) is fulfilled, a_1 is a unit and that a_i $(i \geq 2)$ are central nilpotent elements. Then Φ is an A-ring automorphism.

Remark. Let A and B be rings with an isomorphism $\varphi: A \to B$, and let ρ and η be automorphisms of A and of B, respectively. By making use of the characterization of an A-ring automorphism of $A[X; \ ]$, M. Rimmer proved the following theorem [3, Th. 3]: The map $X \to \sum_{i=0}^{m} X^i b_i$

($b_i \in B$) extends φ to an isomorphism $A[X;\rho] \to B[X;\eta]$ if and only if

(1) $b_i\varphi\rho(a) = \eta^i\varphi(a)b_i$ for any $a \in A$ and $i = 0,1, \ldots ,n$

(2) b_1 is a unit

(3) b_i are nilpotent for $i \geq 2$.

In a similar way, we can give conditions for the map $X \to \sum_{i=0}^{m} X^i b_i$ to extend φ to an isomorphism $A[X;D] \to B[X;E]$ where E is a derivation of B. For the details, refer to [1].

References

[1] G. Baumvol and M. Rerrero: On isomorphisms between skew polynomial rings, Bol. Soc. Brasil, to appear.

[2] M. Ferrero and K. Kishimoto: On automorphisms of skew polynomial rings of derivation type, Math. J. Okayama Univ. 22 (1980), 21-26.

[3] M. Rimmer: Isomorphisms between skew polynomial rings, J. Austral. Math. Soc. 25 A (1978), 314-321.

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ON PROBLEMS OF NINOMIYA AND OF TSUSHIMA

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Let G be a p-solvable group with a p-Sylow subgroup P of order p^a, K an algebraically closed field of characteristic p, KG the group algebra of G over K, and $t(G)$ the nilpotency index of the radical $J(KG)$ of KG.

D. A. R. Wallace [7] proved an inequality: $t(G) \geq a(p - 1) + 1$. Along with this inequality, Y. Ninomiya [3] presented the next problem: If $t(G) = a(p - 1) + 1$, then is P elementary ? Recently, Y. Tsushima [5] presented the following problem: If E is a p'-subgroup contained in the center of G, then is $t(G)$ equal to $t(G/E)$?

In case $p = 2$, these problems were answered in the negative (see [3] and [2]). In this paper, we shall give negative answers to these problems for every p.

We set $q = p^r$ and $\ell = (q^p - 1)/(q - 1)$. Let F be a finite field of q^p elements, λ a generator of the multiplicative group of F, $\eta = \lambda^{q-1}$, and $A = \langle\eta\rangle$. We define subgroups of the symmetric group on F as follow:

$H = \{x \to ax^{q^k} + b \mid a \in A,\ b \in F,\ k = 0, 1, \dots , p - 1\}$ and $M = \{x \to ax^{q^k} \mid a \in A,\ k = 0, 1, \dots , p - 1\}$.

The following Proposition 1 gives a negative answer to Ninomiya's problem [3].

Proposition 1. $t(H) = (rp + 1)(p - 1) + 1$ and a p-Sylow subgroup of H is not regular.

Let S be a direct product of two cyclic groups $\langle s\rangle$

and <t> which have the same order ℓ. Let ϕ and ψ be automorphisms of S defined by $\phi(s^n t^m) = s^{n+m} t^m$ and $\psi(s^n t^m) = s^n t^{qm}$. Then $\phi^\ell = 1$, $\psi^p = 1$ and $\phi\psi = \psi\phi^q$. Hence $v_\eta \to \phi$, $\sigma \to \psi^{-1}$ defines a homomorphism of M into the automorphism group of S, which can be regarded as that of H. Let G be a semi-direct product of S by H with respect to this homomorphism.

The next gives a negative answer to Tsushima's problems [5, Problems 3, 4, 5].

Proposition 2. s is a p'-element contained in the center of G and $t(G) - t(G/\langle s \rangle) \geq r$.

References

[1] K. Morita: On group rings over a modular field which radicals expressible as principal ideals, Sci. Report of T. B. D. 4(1951), 177-194.

[2] K. Motose: On the nilpotency index of the radical of a group algebra II, Math. J. Okayama Univ., 22(1980), 141-143.

[3] Y. Ninomiya: On the nilpotency index of the radical of a group algebra, Symposium on algebra (1974) (in Japanease).

[4] D. S. Passman: Radicals of twisted group rings, Proc. London Math. Soc. 20(1970), 409-437.

[5] Y. Tsushima: Problems on the radical of a finite group ring, Proc. 10th Symposium on Ring Theory (Shinshu Univ., August 18-20, 1977), Okayama, 1978, 116-120.

[6] Y. Tsushima: Some notes on the radical of a finite

group ring II, Osaka J. Math. 16(1979), 35-38.

[7] D. A. R. Wallace: Lower bounds for the radical of the group algebra of a finite p-soluble group, Proc. Edinburgh Math. Soc. 16(1968/69), 127-134.

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ON THE COHOMOLOGY OF FINITE GROUPS

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We shall be concerned with the principal block B_0 of a group algebra kG, where k is a field and G is a finite group. Throughout of this note M means a finitely generated left kG-module. $\hat{H}^i(G,M)$ will be the Tate group, while $H^i(G,M)$ the ordinary cohomology. In this note we shall study three problems on the cohomology of finite groups:

(A) Suppose M is simple and lies in B_0. Is M projective, if $H^i(G,M) = 0$ for all $i > 0$?

(B) Is M projective, if $H^i(G,Hom_k(M,M)) = 0$ for all $i > 0$?

(C) Suppose M is indecomposable and lies in B_0. Is M projective, if $H^i(G,M) = 0$ for all $i > 0$?

Main object of this note is to show that the answer to (A) is affirmative, if G is p-solvable with a Sylow p-subgroup G_p abelian(Theorem 1). Moreover, given M as in (A), we shall compute $H^i(G,M)$, when G is an extension of a cyclic group K by an abelian group A (theorem 2).

Now we explain the background of these problems. Let R be a finite-dimensional algebra over k and L be a finitely generated left R-module. Nakayama [5] proposed the following conjecture:

(N) If dom dim $R = \infty$, then R is QF.

Tachikawa [8] proposed another conjecture which would be a consequence of (N):

(T) If R is QF and $\mathrm{Ext}_R^i(L,L) = 0$ for all $i > 0$, then L is projective.

Elementary diagram chasing then tells us

Proposition 1. The answer to (B) is affirmative, if (T) holds when $R = kG$ and $L = M$.

1. Now we make a few elementary remarks concerning (A), (B), (C). We will assume $\mathrm{char}(k) = p > 0$ and $G_p \neq 1$, for otherwise M is always projective. We recall that there is a kG-projective resolution lying entirely within B_0. Therefore, if M is indecomposable and not in B_0, we have $H^i(G,M) = 0$ for all integers i. From this we obtain

Lemma 1. The following statements are equivalent:

(a) M is projective.

(b) Every indecomposable direct summand of M lying in B_0 is projective.

(c) $\hat{H}^0(G,\mathrm{Hom}_k(M,M)) = 0$.

Consequently we have

Proposition 2. If the answer to (C) is affirmative, then the same is said of (A) and (B).

Tachikawa [8] showed that (T) is true for R which has finite representation type. Hence, from Proposition 1, the answer to (B) is affirmative if G_p is cyclic. Including this we have

Proposition 3. If G_p is cyclic or generalized quarternion, the answer to (B) is affirmative.

A proof is given by using the periodesity of the cohomology of G and Lemma 1.

Further Tachikawa showed that, $\mathrm{Ext}^1_{kG}(M,M) = 0$ implies that M is projective if G is a p-group. As a refinement of this we have

Proposition 4. Suppose G is p-nilpotent and M is indeconposable lying in B_0. If $H^i(G,M) = 0$ for some integer i, then M is projective.

To prove this, by the five-term exact sequence of Hochschild and Serre, we may assume that G is a p-group; in this case M is free.

Corollary 1. Answers to (A), (B), (C) are affirmative for a p-nilpotent group.

This follows immediately from Proposition 2. For (A) see also (3) in Section 2.

In the calculation of cohomology (Theorem 2) we will need the following.

Lemma 2. Suppose G is an extension of a group K by a p-nilpotent group A. Let (E,H) be the Hochschild and Serre spectral sequence such that $E_2^{r,s} = H^r(A,H^s(K,M))$ and $H^n = H^n(G,M)$. Given $n \geq 1$, if $E_2^{0,s} = 0$ for all $0 < s < n$, then we have $H^i \simeq E_2^{i,0}$ for all $0 \leq i < n$ and there is an exact sequence $0 \longrightarrow E_2^{n,0} \longrightarrow H^n \longrightarrow E_2^{0,n} \longrightarrow E_2^{n+1,0} \longrightarrow H^{n+1}$.

This follows from Proposition 4 and [3, XV, 5.12].

Corollary 2. Further, if $H^0 = 0$, then we have $H^i = 0$ for all $0 \leq i < n$ and $H^n \simeq E_2^{0,n}$.

2. From now on we consider only (A). k means $GF(p)$ and M will be given as in (A). (A) is also a restatement of a problem of Stammbach [6] :

Problem. Is there an $i > 0$ such that $H^i(G,M) \neq 0$?

A few results have been known about this problem:

(1) A theorem of Swan [7] shows that $H^i(G,k) \neq 0$ for an infinite number of values of $i > 0$, where k is regarded as a trivial kG-module.

(2) Gaschütz (cf. [6]): If M is an abelian complemented p-chief factor of G, then $H^1(G,M) \neq 0$.

(3) If G is p-nilpotent, then B_0 consists of k only and $H^1(G,k) \cong \mathrm{Hom}(G,k) \neq 0$.

Theorem 1. If G is p-solvable with G_p abelian, then there is an i $(1 \leq i \leq |G:O_{p',p}(G)|)$ such that $H^{2i}(G,M) \neq 0$. Further, if G_p is an elementary abelian 2-group, H^{2i} can be replaced by H^i.

To prove this it suffices to consider the following typical situation. Given a faithful representation V of p'-group H over k, let G be the semi-direct product of V by H. Then M is a simple kH-module. Since $V^* = \mathrm{Hom}_k(V,k)$ is a faithful representation of H over k, there is an i $(1 \leq i \leq |H|)$ such that M^* is a kH-sub-module of $S^i(V^*)$ a symmetric power of V^*. (This was noted by Professor S.Endo. In fact kH is a direct summand of $\bigoplus_{i=1}^{|H|} S^i(V^*)$.) It also follows from [2, Theorem 9.2 and Theorem 10.2] that $S^i(V^*)$ is a kH-submodule of $H^{2i}(V,k)$. Thus $\mathrm{Hom}_k(M,M) \simeq M^* \otimes M \leq H^{2i}(V,k) \otimes M \simeq H^{2i}(V,M)$. Considering the set of invariant elements of these groups, we have $0 \neq \mathrm{Hom}_{kH}(M,M) \subseteq H^{2i}(V,M)^H$. Since $V = G_p \lhd G$, there is an isomorphism $H^{2i}(V,M)^H \simeq H^{2i}(G,M)$ [3] and therefore $H^{2i}(G,M) \neq 0$.

Remark. The case where G_p is cyclic shows that the bound $|G:O_{p',p}(G)|$ cannot be improved (see Theorem 2).

3. We end this note with the following calculations of cohomology.

Let G be an extension of a cyclic group K by an abelian group A. Suppose $K_p \neq 1$. $K_p/(K_p)^p$ may be regarded as a 1-dimensional kG-module, with G-action given by inner automorphisms. We denote this by M. Let m be the least positive integer such that $M^{(j)} = M \otimes \cdots {}_j\otimes M \simeq k$. Then we can directly show that simple kG-modules lying in B_0 are $M^{(1)}, \ldots, M^{(m)}$, which is a special case of Basmaji [1].

Theorem 2. Notations being as above, the cohomology of G with coefficients in $M^{(1)}, \ldots, M^{(m)}$ is given as follows:

(i) The case where $A_p = 1$. For $1 \leq j \leq m$ and for all integers i, we have

$$\hat{H}^i(G,M^{(j)}) \simeq \begin{cases} k & [i \equiv 2j-1, 2j \pmod{2m}] \\ 0 & \text{otherwise} \end{cases}.$$

(ii) The case where $A_p \neq 1$. For $1 \leq j \leq m-1$ we have $H^i(G,M^{(j)}) \simeq \begin{cases} 0 & (0 \leq i \leq 2j-2) \\ k & (i = 2j-1) \end{cases}$ and $H^1(G,k) \neq 0$.

To compute $H^i(G,M^{(j)})$, we may assume $K = K_p$ by the five-term exact sequence of Hochschild and Serre. From the definition of the conjugation homomorphism we have the following isomorphisms of kA-modules: for $1 \leq j \leq m$ and for all integers i, $\hat{H}^{2i-1}(K,M^{(j)}) \simeq \hat{H}^{2i}(K,M^{(j)}) \simeq M^{(t)}$, where $1 \leq t \leq m$ and $t \equiv j-i \pmod m$. Then (i) follows from [3], since $K = G_p \triangleleft G$. If $1 \leq j \leq m-1$, (ii) is a direct consequence of Corollary 2. Finally $H^1(G,k) \neq 0$ is clear.

Remark on (ii). If generators and relations of G are given, $H^1(G,k)$ is easily computed. Also, by using the Hamada resolution [4], we have determined $H^i(G,M^{(j)})$ for all $1 \leq j \leq m$ and for all $0 \leq i \leq 2m-1$, when A is cyclic.

Detailed proofs will be given in a subsequent paper.

References

[1] B. G. Basmaji: Moduler representations of metabelian groups, Trans. Amer. Math. Soc. 169 (1972), 389-399.

[2] H. Cartan: Seminare Cartan 1954-55, Algèbres d'Eilenberg

Maclane et homotopie, Benjamin, New York, 1967.

[3] H. Cartan and S. Eilenberg: Homological Algebra, Princeton Univ. Press, Princeton, N.J., 1956.

[4] S. Hamada: Cohomology of generalized dihedral groups (in Japanese), Sûgaku 16 (1964/65), 106-107.

[5] T. Nakayama: On algebras with complete homology, Abh. Math. Sem. Univ. Hamburg 22 (1958), 300-307.

[6] U. Stammbach: Cohomological characterizations of finite solvable and nilpotent groups, J. Pure and Applied Algebra 11 (1977), 293-301.

[7] R. G. Swan: The nontriviality of the restriction map in the cohomology of groups, Proc. Amer. Math. Soc. 11 (1960), 885-887.

[8] H. Tachikawa: Quasi-Frobenius Rings and Generalizations, Lecture Note in Math. 351, Springer-Verlag, Berlin, 1973.

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ON SOME INVARIANT SUBRINGS OF POLYNOMIAL RINGS IN POSITIVE CHARACTERISTICS

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Introduction. Let k be a field of characteristic p and G a finite subgroup of GL(V) where V is a finite dimensional vector space over k. Then G acts naturally on the symmetric algebra k(V) of V. It is well known that the ring $k(V)^G$ consisting of all invariant polynomials in k(V) under this action of G is an affine normal domain.

We now consider a chain of conditions

polynomial ring $\Rightarrow$ hypersurface $\Rightarrow$ complete intersection $\Rightarrow$ Gorenstein $\Rightarrow$ Cohen-Macaulay $\Rightarrow$ Buchsbaum

on affine rings. We have already known necessary and sufficient conditions on G for the first, fourth, fifth or sixth condition to hold for $k(V)^G$ if $p = 0$ or $(|G|, p) = 1$ (cf. [2, 3, 11, 16, 19]). For hypersurfaces or complete intersections, there are partial results which relate to semi-invariants of finite groups (cf. [17, 18, 20]). But we do not have such characterizations in the case where the order of G is divisible by p. For example there exist many finite groups G such that $k(V)^G$ are not Buchsbaum rings.

In this paper we shall try to study the local properties and $a(\cdot)$ of invariant subrings in positive characteristics (for definition of $a(\cdot)$, see [8]).

§1. Cyclic quotient singularities. Let $G = \langle g\rangle \cong Z/p^mZ$ be a group of automorphisms of a finite dimensional vector space V over an algebraically closed field k of characteristic $p > 0$. Then G acts naturally on an affine variety V which is associated with the k-space V and we denote by V/G the quotient variety of V under this action of G. Obviously V/G has a singularity if and only if $\dim V^G \leqq \dim V - 2$. Furthermore we can show the following theorem :

Theorem 1.1 (cf. [5]). If x is the closed point of V/G induced from the origin of V, then

$$\text{depth}\ \mathcal{O}_{V/G,x} = \min\left\{\dim V^G + 2,\ \dim V\right\}.$$

Especially V/G is a Cohen-Macaulay (consequently a Gorenstein) variety if and only if $\dim V^G \geqq \dim V - 2$.

Proof. We show this by induction on $\dim V$. Let $\overline{V}$ be the dual module of V and choose a nonzero element y from $\overline{V}^G$. Then, from the short exact sequence

$$0 \longrightarrow k(\overline{V}) \xrightarrow[\cdot y]{} k(\overline{V}) \longrightarrow k(V') \longrightarrow 0$$

of kG-modules, we get a long exact sequence

$$(*)\quad \begin{array}{l} 0 \longrightarrow k(\overline{V})^G \longrightarrow k(\overline{V})^G \longrightarrow k(V')^G \longrightarrow H^1(G, k(\overline{V})) \\ \longrightarrow H^1(G, k(\overline{V})) \longrightarrow H^1(G, k(V')) \longrightarrow \cdots\cdot \end{array}$$

of cohomology groups, where V' denotes the quotient module $\overline{V}/ky$. This can be regarded as a sequence in the category of finitely generated $k(\overline{V})^G$-modules. On the other hand, since G is unipotent, we can construct a transitive action of A_k^r on the set consisting of closed points in the closed subvariety of V defined by the ideal of $k(\overline{V})$ generated by $(g - 1)\overline{V}$ which is compatible with the action of G. Here $r = \dim V - \dim (g - 1)\overline{V}$. Thus it is easy to see that

$$H^i(G, k(V'))_w \cong H^i(G, k(V'))_{w'}\ (i \geqq 1)$$

for $w, w' \in \operatorname{supp} H^i(G, k(V')) \cap \{\text{closed points of } V/G\}$ and

$$H^i(G, k(\overline{V}))_z \cong H^i(G, k(\overline{V}))_{z'}\ (i \geqq 1)$$

for $z, z' \in \operatorname{supp} H^i(G, k(\overline{V})) \cap \{\text{closed points of } V/G\}$.
Because Cohen-Macaulay loci are open with respect to Zariski topology, $H^i(G, k(V'))$ and $H^i(G, k(\overline{V}))$ $(i \geq 1)$ are Cohen-Macaulay $k(\overline{V})^G$-modules. Let us consider the long exact sequences of local cohomology modules

$$\cdots \longrightarrow H^i_x(k(\overline{V})^G) \longrightarrow H^i_x(k(\overline{V})^G) \longrightarrow H^i_x(C_1) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^i_x(C_1) \longrightarrow H^i_x(k(V')^G) \longrightarrow H^i_x(C_2) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^i_x(C_2) \longrightarrow H^i_x(H^1(G,k(\overline{V}))) \longrightarrow H^i_x(C_3) \longrightarrow \cdots$$

$$\cdots\cdots\cdots$$

with support $\{x\}$, where C_i are naturally defined by cutting the sequence (✱). Then our assertion follows from this, since

$$\operatorname{depth} k(V')^G = \min\{\dim V'^G + 2, \dim V\}$$

by the induction hypothesis.

Remark 1.2. The completion $\hat{\mathcal{O}}_{V/G,x}$ is a unique factorization domain, if V is the regular representation of G (cf. [6]). Griffith has proved in [9] that

$$\mathrm{Cl}(\mathcal{O}_{U/H,x}) \cong \mathrm{Cl}(\hat{\mathcal{O}}_{U/H,x})$$

if a linearly reductive affine algebraic group H acts linearly on an affine space U defined over an algebraically closed field.

Proposition 1.3 (cf. [5]). If H is a finite group of automorphisms of a vector space V, then

$$\operatorname{depth} \mathcal{O}_{V/H,x} \geq \dim V^H + 2,$$

where x denotes the closed point of V/H induced from the origin of V.

Proof. It is easy to see that

$$\mathcal{O}_{V/H,x} \cong \mathcal{O}_{V/H,y}$$

for any closed point y which is a specialization of the point of V/H induced from the ideal of $k(\overline{V})$ generated by W. Here $\overline{V}$ is the dual module of V and W is a kH-submodule of $\overline{V}$ such that $\overline{V}/W$ is a trivial kH-module with $\dim \overline{V}/W = \dim V^H$. Hence, using the upper semi-continuous theorem, we deduce from the Serre condition S_2 of the normal variety V/H the inequality of (1.3).

Recently Almkvist and Fossum ([1]) have determined the Hilbert series of $k(\overline{V})^G$ when $G \cong Z/pZ$ and $\overline{V}$ are indecomposable.

§2. Unipotent groups. In this section k stands for the prime field of characteristic $p > 0$ and R denotes the symmetric algebra $k(V)$ of an n-dimensional vector space V over k.

(V, G), which is called a couple, means a pair of a G-faithful kG-module V such that V/V^G is a non-zero trivial kG-module (consequently G is an elementary abelian p-group). The dimension of a couple (V, G) is defined to be $\dim V/V^G$. We say that (U, H) is a subcouple of (V, G), if H is a subgroup of G and U is a kH-submodule of V. Furthermore, if subcouples (V_i, G_i) $(1 \leq i \leq m)$ of (V, G) satisfy $G = \oplus_{1\leq i\leq m} G_i$, $V^G \subseteq V_i \subseteq V^{G_j}$ for all $1 \leq i, j \leq m$ with $i \neq j$ and $V/V^G = \oplus_{1\leq i\leq m} V_i/V^G$, we say that (V, G) decomposes to them. The decomposable or indecomposable couples are defined in the natural way.

Lemma 2.1. If a couple (V, G) decomposes to $(V_i,$

G_i) ($1 \leq i \leq m$), then the following conditions (1), (2) are equivalent :

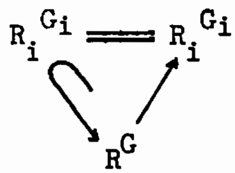
(1) R^G is a polynomial ring.

(2) $R_i^{G_i}$ ($1 \leq i \leq m$) are polynomial rings where each R_i is the symmetric algebra of V_i.

If (1) and (2) are satisfied, then

$$a(R^G) = \sum_{1 \leq i \leq m} a(R^{G_i}) + m(n - 1) - \dim V^G .$$

Proof. For any couple (U, H), we can easily show that $k(U)^H/(U^H)^H$ is a polynomial ring. On the other hand the natural kG_i-epimorphism $V \longrightarrow V_i$ induces the commutative diagram



of k-algebras. The assertion follows from these facts.

The next theorem, which has been proved in [12] , is fundamental in the theory of invariants of reducible groups over finite fields.

Theorem 2.2. Let (V, G) be an indecomposable couple. Then R^G is a polynomial ring if and only if (V, G) is one dimensional.

Using (2.2), we can completely determine abelian groups G such that R^G are polynomial rings (cf. [12]).

Theorem 2.3. Suppose that G is an abelian group generated by pseudo-reflections in $GL(V)$ and the order of G is divisible by p. Then R^G is a polynomial ring if and only if the kG_p-module V defines a couple (V, G_p) which decomposes to one dimensional subcouples. Here G_p denotes the p-part of G.

R^G is not always a Cohen-Macaulay ring even if G is an abelian group generated by pseudo-reflections in $GL(V)$.

Example 2.4. Let $S = k(\oplus_{1 \leq i \leq 2d-1} kX_i)$ $(d \geq 1)$ and let e_{ij} $(i \neq j)$ denote elementary matrices in $GL_{2d-1}(k)$. Suppose that G is a subgroup of $GL_{2d-1}(k)$ generated by the set

$$\{e_{d+1\,1}, e_{d+2\,2}, \dots, e_{2d-1\,d-1}, e_{d+1\,d} + \dots + e_{2d-1\,d}\}.$$

The elements of G act on S in the following way ;

$$\begin{pmatrix} g(X_1) \\ \vdots \\ g(X_{2d-1}) \end{pmatrix} = (g_{ij}) \begin{pmatrix} X_1 \\ \vdots \\ X_{2d-1} \end{pmatrix}$$

for $g = (g_{ij})$ of G. Clearly G is an elementary abelian p-group generated by pseudo-reflections in $GL(\oplus_{1 \leq i \leq 2d-1} kX_i)$, but S^G is a Cohen-Macaulay ring if and only if $d < 4$. For the sake of simplicity let us consider the case of $d = 4$. We assume that S^G is a Cohen-Macaulay ring and will show a contradiction. Put $P = SX_1 + SX_2$ and N (resp. H) denotes the inertia group of P (resp. $SX_1 + SX_2 + SX_3$). It is not difficult to see that S^G/P^G is normal. Then we have $S^G/P^G = (S^N/P^N)^{G/N}$ and an exact sequence

$$0 \longrightarrow H^1(G/N, P^N) \longrightarrow H^1(G/N, S^N)$$

of cohomology groups. Put $h = e_{5\,4} + e_{6\,4} + e_{7\,4}$ and define A to be $1 + h + \dots + h^{p-1}$. Obviously $(1 - h)(P^H)_p = 0$ and $(1 - h)S^H \cap \operatorname{Ker} A \cap P^H = (1 - h)P^H$. Let $U_1 = X_5^{\,p} -$

$X_1^{p-1}X_5$, $U_2 = X_6^p - X_2^{p-1}X_6$ and $Y = U_1 - U_2$. Then Y is contained in $(S^H)_p$ and so $(1-h)Y \in (1-h)(S^H)_p \cap \operatorname{Ker} A \cap P^H$. We see that $(1-h)Y$ is non-zero, whion is a contradiction.

Proposition 2.5. Let W be a kG-submodule of V and let H denote the inertia group of (W) under the natural action of G. If R^G is a polynomial ring, then R^H and $R^G/(W)^G$ are polynomial rings and $(W)^H$ is generated by $(W)^G$ as an ideal of R^H.

Proof. By the definition of H we know that $(W)^H$ is unramified over $(W)^G$. On the other hand we easily see that $(\bar{k} \otimes_k R^H)_M \cong (\bar{k} \otimes_k R^H)_{M'}$, for any maximal ideals M, M' of $\bar{k} \otimes_k R^H$ which contain $(\bar{k} \otimes_k W)^H$, where $\bar{k}$ denotes the algebraic closure of k. $R^H_{(W)^H}$ is a regular local ring, and hence R^H is a polynomial ring. From Hironaka's lemma, R^H is a graded free R^G-module of rank $[G : H]$. Clearly H is a normal subgroup of G. Since G/H acts faithfully on $R^H/(W)^H$ and $R^G/(W)^G$ is contained in $(R^H/(W)^H)^{G/H}$, $R^H/(W)^H$ is also a graded free $R^G/(W)^G$-module of rank $[G : H]$. This implies that $(W)^H$ is generated by $(W)^G$ as an ideal of R^H. Because H acts trivially on V/W, $R^H/(W)^H$ is always a polynomial ring. Therefore $R^G/(W)^G$ is also a polynomial ring.

Suppose that G is a p-subgroup of $GL(V)$ and $\underline{X} = \{X_i : i \in I\}$ be a k-basis of V. The set $\underline{X}$ is said to be G-admissible, if there is a family $\{I_j : 1 \leq j \leq n\}$ of subsets of I with $I_j \subsetneq I_{j+1}$ such that

$$\oplus_{i\in I_j} kX_i \quad (1 \leq j \leq n)$$

are kG-submodules of V. We introduce order in the set of all G-admissible k-bases of V defined as ; $\underline{X} \leq \underline{Y}$ if

$$\prod_{i\in I} |GX_i| \leq \prod_{i\in I} |GY_i|.$$

Further we set

$$a_G(V) = -\min\left\{ \prod_{i\in I} |GX_i| \ : \ \underline{X} = \{X_i\} \text{ is a minimal G-admissible basis of } V \right\}.$$

Proposition 2.6. Let $V = V_n \supset \cdots \supset V_1 \supset V_o = (0)$ be a composition series consisting of kG-submodules. The following conditions are equivalent :

(1) R^G is a polynomial ring.

(2) There exists an n-dimensional graded polynomial subalgebra $S = k[f_1, \ldots, f_n]$ of R^G with

$$\prod_{1\leq i\leq n} \deg f_i \leq |G|$$

such that each $(V_i)\cap S$ is generated by $\{f_j : 1 \leq j \leq i\}$ as an ideal of S, where f_i are graded elements.

Using (2.5), we can prove this (cf. [13]).

The purpose of §2 is to state the next result, which is a generalization of (2.2).

Theorem 2.7. The following conditions on (a p-group) G are equivalent :

(1) R^G is a polynomial ring.

(2) There is a G-admissible k-basis $\{X_i : i\in I\}$ of V which satisfies

$$\prod_{i\in I} |GX_i| = |G|.$$

In the proof of (2.7), (2.2) plays an essential role (cf. [13]).

Corollary 2.8. Suppose that R^G is a Cohen-Macaulay ring. Then $a_G(V) \leq a(R^G)$ and the equality holds if and only if R^G is a polynomial ring.

This follows immediately from (2.7).

§3. Relative invariants. Let $R = \oplus_{i \geq 0} R_i$ be a noetherian factorial graded integral domain over a field $R_0 = k$ of characteristic p and let G be a finite subgroup of Aut R whose elements preserve the graduation of R. For a prime ideal P of R, $\underline{I}(P)$ stands for the inertia group of P under the natural action of G. Clearly $\underline{I}(P)$ is equal to $\underline{I}(\overline{P})$, where $\overline{P}$ is the maximal homogeneous ideal of R contained in P. If $ht(P) = 1$, $\underline{T}(P)$ denotes the maximal subgroup of $\underline{I}(P)$ which acts trivially on a generator of P, and let $\{P_i : 1 \leq i \leq n\}$ be the set consisting of all prime ideals such that $ht(P_i) = 1$ and $\underline{I}(P_i)/\underline{T}(P_i)$ is non-trivial. The integers $e_i = |\underline{I}(P_i)/\underline{T}(P_i)|$ $(1 \leq i \leq n)$ are said to be orders of generalized reflections in G. Exchanging indices of P_i, we may assume that

$$\bigcup_{1 \leq i \leq n} GP_i = \dot{\bigcup_{1 \leq i \leq m}} GP_i .$$

Since $\underline{I}(P_i) = \underline{I}(\overline{P}_i)$ and R is a noetherian factorial integral domain, we can choose irreducible homogeneous elements M_i from R such that $P_i = \overline{P}_i = RM_i$. Let $\overline{D}$ be the 1-cocycle of G defined by

$$g \longrightarrow g\Big(\prod_{1 \leq i \leq n} M_i\Big) / \prod_{1 \leq i \leq n} M_i$$

and for $\overline{X} \in Z^1(G, k^*)$ put

$$f_{\overline{X}} = \prod_{1 \leq i \leq n} M_i^{t_i(\overline{X})}$$

where $t_i(\overline{X}) = \inf \left\{ j \geq 0 : j \in Z, \overline{X}(g) = \overline{D}(g)^j \right.$ for all $\left. g \in \underline{I}(P_i) \right\}$. Furthermore we put

$$R_{\overline{X}} = \left\{ f \in R : g(f) = \overline{X}(g)f \quad \text{for all} \quad g \in G \right\},$$

whose elements are known as $\overline{X}$-invariants (invariants relative to $\overline{X}$ or relative invariants).

Proposition 3.1. The sequence

$$0 \longrightarrow H_o^1(G,k^*) \longrightarrow H^1(G,k^*) \xrightarrow[t]{} (Z/e_1Z, \ldots Z/e_mZ) \longrightarrow 0$$

is exact. Here $H_o^1(G, k^*)$ denotes the set $\left\{ \overline{X} \bmod B^1(G, k^*) : \overline{X} \in Z^1(G, k^*) \right.$ with $\left. t_i(\overline{X}) = 0 \ (1 \leq i \leq n) \right\}$ and the map t is defined by

$$t(\overline{X}) = (t_1(\overline{X}) \bmod e_1 Z, \ \ldots \ , t_m(\overline{X}) \bmod e_m Z).$$

Proof. For an integer $1 \leq i_o \leq m$ let F be the homogeneous element

$$\prod_{GP_i = GP_{i_o}} M_i \ .$$

Obviously F is a relative invariant of G and so it defines a 1-cocycle $\overline{Y}(i_o)$ of G. Then we can easily show that $g(F) = \overline{D}(g)F$ and

$$\left\{ i : 1 \leq i \leq m \quad \text{with} \quad t_i(\overline{Y}(i_o)) \neq 0 \right\} = \left\{ i_o \right\}.$$

The assertion follows from this.

By (3.1) we get a generalization of [4].

Theorem 3.2 (cf. [14]). The following conditions on a 1-cocycle $\overline{X}$ of G are equivalent :

(1) $R_{\overline{\chi}}$ is a graded free R^G-module.

(2) There is a unit $u_{\overline{\chi}}$ of R such that $u_{\overline{\chi}} f_{\overline{\chi}}$ is contained in $R_{\overline{\chi}}$.

If (1) and (2) hold, then we have $R_{\overline{\chi}} = R^G u_{\overline{\chi}} f_{\overline{\chi}}$.

Assume that G acts trivially on k and we denote by G_p a p-Sylow subgroup of G if $p>0$ and otherwise put $G_p = \{1\}$. The set $\underline{A}(G)$ is defined to be the union of $\underline{T}(P_i)$ $(1 \leq i \leq n)$ and $\underline{B}(G)$ denotes the set of all generalized reflections (for definition, see [11]) in G.

Corollary 3.3. Suppose that G_p is normal in G and R^{G_p} is a Cohen-Macaulay ring. Then we have

$$a(R^G) \leq a(R^{G_p}) - |\underline{B}(G)| + |\underline{A}(G)| .$$

Proof. By the Galois descent we see that R^{G_p} is a factorial domain, and hence R^{G_p} is a Gorenstein ring. Using [7, 10], we have an isomorphism $K_{R^G} \cong (R^{G_p})_{\overline{\chi}'}(a(R^{G_p}))$ of graded R^G-modules for some linear character $\overline{\chi}'$ of G/G_p (where K_{R^G} is the canonical module of R^G). The module $(R^{G_p})_{\overline{\chi}'}$ can be embedded in $R f_{\overline{\chi}}$ if $\overline{\chi}$ is the linear character of G which is associated with $\overline{\chi}'$. Therefore the inequality of (3.3) follows from (3.2).

Hereafter we will consider the following special case : Let R be the symmetric algebra $k(V)$ of a vector space V which is finite dimensional over k and let G be a finite subgroup of $GL(V)$.

Corollary 3.4. Suppose that $(|G|, p) = 1$ if p is positive. Then we have

$$a(R^G) \leqq -(\dim V + |\underline{B}(G)| - 1).$$

The equality holds if and only if R^G is a Gorenstein ring.

Proof. By the result of Watanabe [19] we see that $K_{R^G} \cong R_{\det^{-1}}(a(R))$ as graded R^G-modules. But there is a natural embedding $R_{\det^{-1}}(a(R)) \hookrightarrow R(a(R))f_{\det^{-1}}$ which preserves the graduations. From the definition of $a(R^G)$ and the equality $\deg f_{\det^{-1}} = |\underline{B}(G)| - 1$, we deduce that

$$a(R^G) \leqq a(R) - \deg f_{\det^{-1}}$$
$$= -(\dim V + |\underline{B}(G)| - 1).$$

By (3.2) R^G is a Gorenstein ring if and only if $R^G f_{\det^{-1}} = R_{\det^{-1}}$. Hence the remainder of this corollary is evident.

In [7] Goto has already obtained (3.4) with the additional hypothesis that the ground field k is the complex number field C, using some properties of Hilbert series of invariant subrings defined over C (his proof is interesting).

Proposition 3.5. Suppose that p is positive and G_p is normal in G. If G is realizable on the prime field of characteristic p and R^{G_p} is a polynomial ring, then the following conditions are equivalent :

(1) R^G is a Gorenstein ring.

(2) $a(R^G) = a_{G_p}(V) - |\underline{B}(G)| + |\underline{A}(G)|$.

Especially in the case where $\underline{A}(G) = \underline{B}(G)$, R^G is a Gorenstein ring if and only if G is contained in $SL(V)$.

Proof(outline). Clearly K_{R^G} is isomorphic to $(R^{G_p})_{\overline{X}}(a(R^{G_p}))$ for some linear character $\overline{X}$ of G/G_p in the category of graded R^G-modules. We regard $\overline{\chi}$ as a linear character of G and need only to show that $\overline{\chi} = \det^{-1}$. Exchanging a regular system of homogeneous parameters of R^{G_p}, we obtain a natural kG/G_p-isomorphism $R^{G_p} \longrightarrow k(U)$ for a kG/G_p-module U. Then it follows from $K_{R^G} \cong k(U)_{\det^{-1}}$ that $\overline{\chi}$ is equal to $\det^{-1}$. (We can also prove (3.5), extending the action of G/G_p to the fibre of the blowing-up of $\operatorname{Spec} R^{G_p}$ with center $\operatorname{Spec} k$.)

Example 3.6. Assume that $\dim V = 2$ and the order of G is divisible by $p>3$. Then we have

$$a(R^G) \leq \begin{cases} |\underline{A}(G)| - |\underline{B}(G)| - c(G) & \text{if } G \text{ is irreducible} \\ |\underline{A}(G)| - |\underline{B}(G)| - 1 - |G_p| & \text{otherwise} \end{cases}$$

where G_p is a p-Sylow subgroup of G and $c(G) = |SL(2, p^f)|/(p^{2f} - p^f) + p^{2f} - p^f$ for a maximal subgroup $SL(2,p^f)$ of G generated by transvections (if G is primitive irreducible, then maximal subgroups generated by transvections in G are conjugate to $SL(2,p^f)$). The equality holds if and only if R^G is a hypersurface. Further, in the case where $\underline{A}(G) = \underline{B}(G)$, R^G is a hypersurface if and only if G is contained in $SL(V)$.

Proposition 3.7. Suppose that p is positive and G is a p-group. Then R^G is a Cohen-Macaulay ring if and only if R^G is a Buchsbaum ring.

Proof. It suffices to prove the "if" part of this proposition. So we assume that R^G is not a Cohen-Macaulay ring. Then it is well known that $(\bar{k} \otimes_k R^G)_{\bar{k} \otimes_k R^G_+}$ is not a Cohen-Macaulay local ring, where R_+ is the unique homogeneous maximal ideal of R. Since G is unipotent, there is a non-trivial $G_a(\bar{k})$-action on the set of closed points of $\mathrm{Spec}(\bar{k} \otimes_k R)$ which commutes with the natural action of G. Therefore $G_a(\bar{k})$ acts transitively on the set of closed points of $\mathrm{Spec}(\bar{k} \otimes_k R^G)$ which are specializations of a homogeneous prime ideal P of $\bar{k} \otimes_k R^G$ with $\mathrm{ht}(P) = \dim R - 1$. Cohen-Macaulay loci of affine rings are open with respect to Zariski topology and hence we deduce that the localization of $\bar{k} \otimes_k R^G$ at P is not a Cohen-Macaulay ring. It follows from this that $(\bar{k} \otimes_k R^G)_{\bar{k} \otimes_k R^G_+}$ is not a Buchsbaum ring. By the faithfully flat descent of noetherian graded algebras, we see that R^G is not a Buchsbaum ring.

Now we suppose that G is a subgroup generated by pseudo-reflections and the order of G is a unit in k. Let N be a normal subgroup of G such that the quotient group G/N is abelian and let $\bar{t} : \mathrm{Hom}(G, \bar{k}^*) \longrightarrow GL_m(\bar{k})$ denote the homomorphism defined by

$$\bar{X} \longmapsto \mathrm{diag}(\bar{t}_1(\bar{X}), \ldots, \bar{t}_m(\bar{X}))$$

where $\bar{t}_i(\bar{X})$ is the image of $t_i(\bar{X})$ mod $e_i Z$ under the fixed embedding $Z/e_i Z \hookrightarrow \bar{k}^*$ as groups.

Lemma 3.8. There is an isomorphism $\bar{u} : G^{ab} \longrightarrow \mathrm{Hom}(G, \bar{k}^*)$ of groups such that for any element g of $\underline{I}(P_i)$ $(1 \leq i \leq n)$ if $GP_i = GP_j$ $(1 \leq j \leq m)$ $\bar{t}_j(\bar{u}(gG')) = \det g$ and otherwise $\bar{t}_j(\bar{u}(gG')) \doteq 1$ where G^{ab} is the commutator quotient of G and G' is the commutator subgroup of G.

Proof. Let g_i be a generator of $\underline{I}(P_i)$ and let

$\bar{d} : \underline{I}(P_1) \oplus \cdots \oplus \underline{I}(P_m) \longrightarrow G^{ab}$ be a homomorphism defined by

$$\bar{d}((\underbrace{1,\ldots,1}_{i-1 \text{ times}},g_i,1,\ldots,1)) = g_iG' \quad (1 \leq i \leq m).$$

Since G is generated by pseudo-reflections, $\bar{d}$ is an isomorphism. On the other hand we deduce from the definition of $\bar{t}$ that $\mathrm{Im}\,\bar{t}$ is generated by $\{\mathrm{diag}(\det g_1,1,\ldots,1), \ldots, \mathrm{diag}(1,\ldots,1,\det g_m)\}$. Let $\bar{q} : \underline{I}(P_1) \oplus \cdots \oplus \underline{I}(P_m) \longrightarrow \mathrm{Im}\,\bar{t}$ be an isomorphism such that

$$\bar{q}((\underbrace{1,\ldots,1}_{i-1 \text{ times}},g_i,1,\ldots,1)) = \mathrm{diag}(\underbrace{1,\ldots,1}_{i-1 \text{ times}},\det g_i,1,\ldots,1).$$

Then $\bar{u} = \bar{t}^{-1}\,\bar{q}\,\bar{d}^{-1}$ is an isomorphism as desired.

We define a homomorphism $\bar{v} : N \longrightarrow GL_m(\bar{k})$ such that the diagram

$$\begin{array}{ccc} \mathrm{Hom}(G, \bar{k}) & \xrightarrow{\ \bar{t}\ } & GL_m(\bar{k}) \\ \uparrow{\scriptstyle \bar{u}} & & \uparrow{\scriptstyle \bar{v}} \\ G^{ab} & \xleftarrow[\text{can.}]{} & N \end{array}$$

is commutative.

Theorem 3.9. R^N is a complete intersection if and only if $\bar{v}(N)$ coinsides with the group $G_D(\bar{k})$ in $GL_m(\bar{k})$ for some datum D (see [20], for definitions).

R_+^G is generated by an R^N-regular sequence (cf. [2, 3, 16]), and hence, to prove this theorem we consider the artinian ring $R^N/R_+^GR^N$. $R^N/R_+^GR^N$ is an epimorphic image of an affine normal semigroup ring. This viewpoint is important in the proof of (3.9) (cf. [14, 15]).

Corollary 3.10. If $\mathrm{Hom}(G, \bar{k}^*)$ is cyclic, then R^N

is a hypersurface.

Proof. By (3.1) we see that G acts transitively on $\underline{B}(G)$. Let X(N) be a subgroup of Hom(G, $\bar{k}^*$) which satisfies

$$\bigcap_{\bar{X}\in X(N)} \operatorname{Ker} \bar{X} = N$$

and let $\bar{Y}$ be an element of X(N) such that deg $f_{\bar{Y}}$ is minimal. Then it is easy to show that R^N is generated by $f_{\bar{Y}}$ as an algebra over R^G.

References

[1] G. Almkvist and R. Fossum: Decomposition of exterior and symmetric powers of indecomposable Z/pZ-modules in characteristic p and relations to invariants, Séminaire d'Algèbre Paul Dubreil, Lecture Notes in Math. 641, Springer-Verlag, Berlin, 1978, 1-111.

[2] N. Bourbaki: Groupes et algèbres de Lie, Chs. 4, 5 et 6, Herman, Paris, 1968.

[3] C. Chevalley: Invariants of finite groups generated by reflections, Amer. J. Math. 77 (1955), 778-782.

[4] A. Dress: On finite groups generated by pseudo-reflections, J. Algebra 11 (1969), 1-5.

[5] G. Ellingsrud and T. Skjelbred: Profondeur d'anneaux d'invariants en caracteristique p, Compositio Math. 41 (1980), 233-244.

[6] R. Fossum and P. Griffith: Complete local factorial rings which are not Cohen-Macaulay in characteristic p, Ann. scient. Ec. Norm Sup. 8 (1974), 189-200.

[7] S. Goto: On Gorenstein rings (in Japanese), Sûgaku 31 (1979), 349-364.

[8] S. Goto and K. Watanabe: On graded rings, I, J. Math.

Soc. Japan 30 (1978), 179-213.

[9] P. Griffith: Completions of rings of invariants and their divisor class groups, Nagoya Math. J. 72 (1978), 71-82.

[10] V. A. Hinic: On Gorenstein property of the ring of invariants of a Gorenstein ring, Math. USSR Izvestija 10 (1976), 47-53.

[11] M. Hochster and J. A. Eagon: Cohen-Macaulay rings, invariant theory and the generic perfection of determinantal loci, Amer. J. Math. 93 (1971), 1020-1058.

[12] H. Nakajima: Modular representations of abelian groups with regular rings of invariants, to appear in Nagoya Math. J.

[13] H. Nakajima: Regular rings of invariants of unipotent groups, to appear.

[14] H. Nakajima: On Stanley's theorem, to appear.

[15] H. Nakajima: Rings of invariants of finite groups which are hypersurfaces, to appear.

[16] J.-P. Serre: Groupes finis d'automorphismes d'anneaux locaux réguliers, Colloq. d'Alg. E. N. S., 1967.

[17] R. P. Stanley: Relative invariants of finite groups generated by pseudo-reflections, J. Algebra 49 (1977), 134-148.

[18] R. P. Stanley: Invariants of finite groups and their applications to combinatorics, Bull. Amer. Math. Soc. (New Series) 1 (1979), 475-511.

[19] K. Watanabe: Certain invariant subrings are Gorenstein, I, II, Osaka J. Math. 11 (1974), 1-8, 379-388.

[20] K. Watanabe: Invariant subrings which are complete intersections, I (Invariant subrings of Abelian groups), Nagoya Math. J. 77 (1980), 89-98.

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INJECTIVE DIMENSION OF GENERALIZED TRIANGULAR MATRIX RINGS

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This note is an abstract of the author's paper [7] and includes the modifications of some results in it.

Throughout this note, let R and S denote rings with identity, M an (S, R)-bimodule, and Λ a generalized triangular matrix ring defined by ${}_SM_R$, i.e.,

$$\Lambda = \begin{pmatrix} R & 0 \\ M & S \end{pmatrix}$$

with the addition by element-wise and the multiplication by

$$\begin{pmatrix} r & 0 \\ m & s \end{pmatrix}\begin{pmatrix} r' & 0 \\ m' & s' \end{pmatrix} = \begin{pmatrix} rr' & 0 \\ mr' + sm' & ss' \end{pmatrix}.$$

For an R-module U_R, $\mathrm{id}(U_R)$ ($\mathrm{fd}(U_R)$) denotes the injective (flat) dimension of U_R, respectively.

The estimation of $\mathrm{id}(\Lambda_\Lambda)$ in terms of $\mathrm{id}(R_R)$, $\mathrm{id}(M_R)$, and $\mathrm{id}(S_S)$ is determined by Reiten [6] when $\mathrm{id}(\Lambda_\Lambda) = 0$. Furthermore, in [8], Zaks shows that the injective dimension of an $n \times n$ lower triangular matrix ring over a semiprimary ring R is just equal to $\mathrm{id}(R_R) + 1$. An example is constructed to show that the condition on R being semi-primary is redundant in his theorem. In this note, we observe general cases.

Let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \Lambda$ and $e' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \Lambda$. Then $R \simeq e\Lambda e$, $M \simeq e'\Lambda e$, and $S \simeq e'\Lambda e'$.

Lemma 1. Let X be a right Λ-module with $X = Xe$.

(1) If X_R is projective, then X_Λ is projective.

(2) $\mathrm{Ext}^i_\Lambda(X_\Lambda, \Lambda_\Lambda) \cong \mathrm{Ext}^i_R(X_R, \Lambda e_R)$.

Lemma 2. Let Y be a right Λ-module.

(1) If Y_Λ is projective, then Ye'_S is projective.

(2) $\mathrm{Ext}^i_\Lambda(Y_\Lambda, e'\Lambda/e'\Lambda e_\Lambda) \cong \mathrm{Ext}^i_S(Ye'_S, S_S)$.

Lemma 3 [4, Proposition 4.1]. Every right ideal of Λ has the form of $\begin{pmatrix} X & \begin{matrix} 0 \\ K \end{matrix} \end{pmatrix}$, where K is a right ideal of S and $\begin{pmatrix} 0 \\ KM \end{pmatrix}_R \subseteq X_R \subseteq \begin{pmatrix} R \\ M \end{pmatrix}_R$.

Theorem 4. Assume that $\mathrm{fd}({}_SM)$ is finite. Then we have

$$\max(\mathrm{id}(R_R), \mathrm{id}(M_R), \mathrm{id}(S_S) - \mathrm{fd}({}_SM)) \leqq \mathrm{id}(\Lambda_\Lambda) \leqq \max(\max(\mathrm{id}(R_R), \mathrm{id}(M_R)) + \mathrm{fd}({}_SM), \mathrm{id}(S_S) - 1) + 1.$$

The following is essentially in [1, p. 346].

Lemma 5. Let A_S, ${}_SB_\Lambda$, and C_Λ be modules such that $\mathrm{Ext}^i_\Lambda(B, C) = 0$ $(i > 0)$ and $\mathrm{Tor}^S_i(A, B) = 0$ $(i > 0)$. Then there holds

$$\mathrm{Ext}^n_S(A, \mathrm{Hom}_\Lambda(B, C)) \cong \mathrm{Ext}^n_\Lambda(A \otimes_S B, C).$$

The proof of the following lemma is a short proof of [7, Lemma 6].

Lemma 6. Assume that ${}_SM$ is flat. Let
$$f^\#_i = \mathrm{Ext}^i_\Lambda(f, 1_\Lambda) : \mathrm{Ext}^i_\Lambda\left(\Lambda/\begin{pmatrix} R & 0 \\ KM & K \end{pmatrix}, \Lambda\right) \to \mathrm{Ext}^i_\Lambda\left(\begin{pmatrix} R & 0 \\ M & K \end{pmatrix}/\begin{pmatrix} R & 0 \\ KM & K \end{pmatrix}, \Lambda\right)$$
be the induced map by the inclusion map
$f : \begin{pmatrix} R & 0 \\ M & K \end{pmatrix}/\begin{pmatrix} R & 0 \\ KM & K \end{pmatrix} \longrightarrow \Lambda/\begin{pmatrix} R & 0 \\ KM & K \end{pmatrix}$, where K is a right ideal of

S. Then $\mathrm{Im}\, f_i^{\#}$ is contained in $\mathrm{Ext}_\Lambda^i(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix} / \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, e'\Lambda)$, a direct summand of $\mathrm{Ext}_\Lambda^i(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix} / \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, \Lambda)$.

Proof. Since $f_i^{\#} = f_{i1}^{\#} \oplus f_{i2}^{\#}$, where
$f_{i1}^{\#} : \mathrm{Ext}_\Lambda^i(\Lambda / \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, e\Lambda) \longrightarrow \mathrm{Ext}_\Lambda^i(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix} / \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, e\Lambda)$ and
$f_{i2}^{\#} : \mathrm{Ext}_\Lambda^i(\Lambda / \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, e'\Lambda) \longrightarrow \mathrm{Ext}_\Lambda^i(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix} / \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, e'\Lambda)$, and

$$\begin{aligned} \mathrm{Ext}_\Lambda^i(\Lambda / \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, e\Lambda) &\simeq \mathrm{Ext}_S^i(S/K, \mathrm{Hom}_\Lambda(e'\Lambda, e\Lambda)) \\ &\simeq \mathrm{Ext}_S^i(S/K, e\Lambda e') = 0, \end{aligned}$$

we have $\mathrm{Im}\, f_i^{\#} = \mathrm{Im}\, f_{i2}^{\#} \subseteq \mathrm{Ext}_\Lambda^i(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix} / \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, e'\Lambda)$.

Proposition 7. Assume that ${}_SM$ is flat and put $\max(\mathrm{id}(R_R), \mathrm{id}(M_R)) = i$.

(1) If $\mathrm{id}(S_S) > i$, then $\mathrm{id}(\Lambda_\Lambda) = \mathrm{id}(S_S)$.

(2) If $\mathrm{id}(S_S) < i \neq 0$, then $\mathrm{id}(\Lambda_\Lambda) = i$ if and only if $\mathrm{Ext}_R^i(M/KM, R \oplus M) = 0$ for every right ideal K of S.

(3) If $\mathrm{id}(S_S) = i \neq 0$ and if $\mathrm{Ext}_R^i(M/KM, R \oplus M) = 0$ for every right ideal K of S, then $\mathrm{id}(\Lambda_\Lambda) = i$.

(4) If $\mathrm{id}(S_S) = i \neq 0$ and if $\mathrm{Ext}_R^i(M/KM, R) \neq 0$ for some right ideal K of S, then $\mathrm{id}(\Lambda_\Lambda) = i + 1$.

Let K be a right ideal of S. Since

$$T(S/K, M) = \mathrm{Hom}_S(S/K, \mathrm{Hom}_R(M, M)) \simeq \mathrm{Hom}_R(S/K \otimes_S M, M),$$

we have the following spectral sequences

$$E_2^{p,q} = \mathrm{Ext}_S^p(S/K, \mathrm{Ext}_R^q(M, M)) \underset{p}{\Longrightarrow} R^nT(S/K, M)$$

and

$$\tilde{E}_2^{p,q} = \mathrm{Ext}_R^q(\mathrm{Tor}_p^S(S/K, M), M) \underset{q}{\Longrightarrow} R^nT(S/K, M),$$

where R^nT is the n-th derived functor of T. Then we have the edge homomorphisms $E_2^{n,0} \longrightarrow R^nT(S/K, M)$ and $R^nT(S/K, M) \longrightarrow \tilde{E}_2^{0,n}$ (cf. [1, Chapter XVI]). We define $\alpha : \mathrm{Ext}_S^n(S/K, \mathrm{Hom}_R(M, M)) \longrightarrow \mathrm{Ext}_R^n(S/K \otimes_S M, M)$ as the composition of the above edge homomorphisms. Let $\mu : S \longrightarrow \mathrm{End}(M_R)$ be the canonical map and $\rho : S/K \otimes_S M \simeq M/KM$ and put $g = \mathrm{Ext}_R^i(\rho, M)\cdot\alpha\cdot\mathrm{Ext}_S^i(S/K, \mu)$.

Proposition 8. Assume that ${}_SM$ is flat and let g be as above. If $\mathrm{id}(S_S) \leqq \max(\mathrm{id}(R_R), \mathrm{id}(M_R)) = i \neq 0$, then $\mathrm{id}(\Lambda_\Lambda) = i$ if and only if $\mathrm{Ext}_R^i(M/KM, R) = 0$ and $g : \mathrm{Ext}_S^i(S/K, S) \longrightarrow \mathrm{Ext}_R^i(M/KM, M)$ is an epimorphism for every right ideal K of S.

Proof. Let $\begin{pmatrix} X & 0 \\ & K \end{pmatrix}$ be a right ideal of Λ. Considering the following exact commutative diagrams

$$\mathrm{Ext}_\Lambda^{i+1}\left(\Lambda/\begin{pmatrix} R & 0 \\ M & K \end{pmatrix}, \Lambda\right) \to \mathrm{Ext}_\Lambda^{i+1}\left(\Lambda/\begin{pmatrix} X & 0 \\ & K \end{pmatrix}, \Lambda\right) \to \mathrm{Ext}_\Lambda^{i+1}\left(\begin{pmatrix} R & 0 \\ M & K \end{pmatrix}/\begin{pmatrix} X & 0 \\ & K \end{pmatrix}, \Lambda\right) \simeq \mathrm{Ext}_R^{i+1}((R \oplus M)/X, R \oplus M) = 0$$

and

$$\begin{array}{ccc} \mathrm{Ext}_S^i(S/K, S) & \xrightarrow{g} & \mathrm{Ext}_R^i(M/KM, M) \\ \uparrow\wr & & \downarrow\nu \\ & \mathrm{Ext}_R^i(M/KM, R) \oplus \mathrm{Ext}_R^i(M/KM, M) \simeq \mathrm{Ext}_R^i(M/KM, R \oplus M) & \\ & & \uparrow\wr \\ \mathrm{Ext}_\Lambda^i\left(\Lambda/\begin{pmatrix} R & 0 \\ KM & K \end{pmatrix}, \Lambda\right) & \longrightarrow & \mathrm{Ext}_\Lambda^i\left(\begin{pmatrix} R & 0 \\ M & K \end{pmatrix}/\begin{pmatrix} R & 0 \\ KM & K \end{pmatrix}, \Lambda\right) \to \end{array}$$

$$\to \mathrm{Ext}_\Lambda^{i+1}\left(\Lambda/\begin{pmatrix} R & 0 \\ M & K \end{pmatrix}, \Lambda\right) \to \mathrm{Ext}_\Lambda^{i+1}\left(\Lambda/\begin{pmatrix} R & 0 \\ KM & K \end{pmatrix}, \Lambda\right),$$

where ν is the injection map, we conclude that $\mathrm{id}(\Lambda_\Lambda) = i$ iff $\mathrm{Ext}^{i+1}_\Lambda(\Lambda/\begin{pmatrix} R & 0 \\ M & K \end{pmatrix}, \Lambda) = 0$ for every right ideal K of S iff $\mathrm{Ext}^i_R(M/KM, R) = 0$ for every right ideal K of S and g is an epimorphism, for $\mathrm{Ext}^{i+1}_\Lambda(\Lambda/\begin{pmatrix} R & 0 \\ KM & K \end{pmatrix}, \Lambda) \simeq \mathrm{Ext}^i_S(S/K, S) = 0$.

Proposition 9. Assume that ${}_SM$ is flat, $\mathrm{Ext}^n_R(M, M) = 0$ $(n > 0)$, and $S = \mathrm{End}(M_R)$. If $\mathrm{id}(S_S) \leqq \max(\mathrm{id}(R_R), \mathrm{id}(M_R)) = i \neq 0$, then $\mathrm{id}(\Lambda_\Lambda) = i$ if and only if $\mathrm{id}(M^*{}_S) \leqq i - 1$, where $M^* = \mathrm{Hom}_R(M, R)$.

Proof. Let $\begin{pmatrix} X & 0 \\ & K \end{pmatrix}$ be a right ideal of Λ. Considering the following exact commutative diagrams

$$\begin{array}{l}
\mathrm{Ext}^{i+1}_\Lambda(\Lambda/\begin{pmatrix} R & 0 \\ M & K \end{pmatrix}, \Lambda) \to \mathrm{Ext}^{i+1}_\Lambda(\Lambda/\begin{pmatrix} X & 0 \\ & K \end{pmatrix}, \Lambda) \\
\qquad\qquad\qquad \to \mathrm{Ext}^{i+1}_\Lambda(\begin{pmatrix} R & 0 \\ M & K \end{pmatrix}/\begin{pmatrix} X & 0 \\ & K \end{pmatrix}, \Lambda) \\
\qquad\qquad\qquad\qquad\qquad \downarrow \\
\qquad\qquad\qquad \mathrm{Ext}^{i+1}_R((R \oplus M)/X, R \oplus M) = 0
\end{array}$$

and

$$\begin{array}{ccc}
\mathrm{Ext}^i_S(S/K, S) & = & \mathrm{Ext}^i_S(S/K, \mathrm{End}(M_R)) \\
\wr\downarrow & & \wr\downarrow \\
\mathrm{Ext}^i_S(S/K, \mathrm{Hom}_\Lambda(\begin{pmatrix} 0 & 0 \\ M & S \end{pmatrix}, e'\Lambda)) & \to & \mathrm{Ext}^i_S(S/K, \mathrm{Hom}_\Lambda(\begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix}, e'\Lambda)) \\
\wr\downarrow\psi_1 & & \wr\downarrow\psi_2 \\
\mathrm{Ext}^i_\Lambda(S/K \otimes_S \begin{pmatrix} 0 & 0 \\ M & S \end{pmatrix}, e'\Lambda) & \longrightarrow & \mathrm{Ext}^i_\Lambda(S/K \otimes_S \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix}, e'\Lambda) \\
\wr\downarrow & & \wr\downarrow \\
\mathrm{Ext}^i_\Lambda(\Lambda/\begin{pmatrix} R & 0 \\ KM & K \end{pmatrix}, e'\Lambda) & \longrightarrow & \mathrm{Ext}^i_\Lambda(\begin{pmatrix} R & 0 \\ M & K \end{pmatrix}/\begin{pmatrix} R & 0 \\ KM & K \end{pmatrix}, e'\Lambda) \\
\wr\downarrow & & \downarrow\iota \\
\mathrm{Ext}^i_\Lambda(\Lambda/\begin{pmatrix} R & 0 \\ KM & K \end{pmatrix}, \Lambda) & \xrightarrow{f^\#_i} & \mathrm{Ext}^i_\Lambda(\begin{pmatrix} R & 0 \\ M & K \end{pmatrix}/\begin{pmatrix} R & 0 \\ KM & K \end{pmatrix}, \Lambda) \to
\end{array}$$

$$\to \mathrm{Ext}^{i+1}_\Lambda(\Lambda/\begin{pmatrix} R & 0 \\ M & K \end{pmatrix}, \Lambda) \longrightarrow \mathrm{Ext}^{i+1}_\Lambda(\Lambda/\begin{pmatrix} R & 0 \\ KM & K \end{pmatrix}, \Lambda) = 0,$$

where ψ_1 and ψ_2 are isomorphisms by Lemma 5, $f_i^{\#}$ as in Proposition 7, and ι the injection map. We conclude that $id(\Lambda_\Lambda) = i$ if and only if $\mathrm{Ext}_\Lambda^{i+1}(\Lambda/\begin{pmatrix} R & 0 \\ M & K \end{pmatrix}, \Lambda) = 0$ for every right ideal K of S iff for every right ideal K of S,

$$\mathrm{Ext}_\Lambda^i(\begin{pmatrix} R & 0 \\ M & K \end{pmatrix}/\begin{pmatrix} R & 0 \\ KM & K \end{pmatrix}, e\Lambda) \simeq \mathrm{Ext}_R^i(M/KM, R)$$
$$\simeq \mathrm{Ext}_S^i(S/K, \mathrm{Hom}_R(M, R)) = 0,$$

since $\mathrm{Im}\, f_i^{\#} \subseteq \mathrm{Ext}_\Lambda^i(\begin{pmatrix} R & 0 \\ M & K \end{pmatrix}/\begin{pmatrix} R & 0 \\ KM & K \end{pmatrix}, e'\Lambda)$ by Lemma 6.

Proposition 10 [6, Theorem 1.4.1]. Let $\mu : S \longrightarrow \mathrm{End}(M_R)$ be the canonical map. Then Λ_Λ is injective if and only if

(1) R_R, M_R, and $\ell_S(M) = \{s \in S;\ sm = 0$ for every $m \in M\}$ are all injective.

(2) μ is an epimorphism.

(3) $\mathrm{Hom}_R(M_R, R_R) = 0$.

Example 11. Let R be an infinite direct product of fields, I a maximal ideal containing their direct sum, and $M = R/I$. Let

$$\Lambda = \begin{pmatrix} R & 0 \\ M & \mathrm{End}(M_R) \end{pmatrix}.$$

Then R_R, M_R and Λ_Λ is injective.

Example 12. Let Λ be a 2×2 lower triangular matrix ring over a ring $R \neq 0$ with $id(R_R) = i < \infty$. Then $id(\Lambda_\Lambda) = i + 1$.

Example 13. Let

$$\Lambda = \begin{pmatrix} Z & 0 & \vdots & 0 \\ Q & Q & \vdots & 0 \\ \hdashline Q & Q & \vdots & Z \end{pmatrix}, \quad R = \begin{pmatrix} Z & 0 \\ Q & Q \end{pmatrix}.$$

Then $id(R_R) = 2$, $id((Q\ Q)_R) = 0$, $id(Z_Z) = 1$ and $id(\Lambda_\Lambda) = 2$.

Example 14. Let

$$\Lambda = \begin{pmatrix} Z & 0 & \vdots & 0 \\ Z & Z & \vdots & 0 \\ \hdashline 0 & Q & \vdots & Z \end{pmatrix}, \quad R = \begin{pmatrix} Z & 0 \\ Z & Z \end{pmatrix},$$

where $(0\ Q)_Z$ can be considered as a right R-module via $\sigma : R \longrightarrow Z$ $\left(\begin{pmatrix} z & 0 \\ z' & z'' \end{pmatrix} \longmapsto z''\right)$. Then $id(R_R) = 2$, $id(Z_Z) = 1$, $id((0\ Q)_R) = 0$ and $id(\Lambda_\Lambda) = 3$.

Example 15. Let $\Lambda_n (n > 2)$ be an $n \times n$ lower triangular matrix ring over a ring $R \neq 0$ with $id(R_R) = i < \infty$. Since Λ_n can be considered as

$$\begin{pmatrix} R & \vdots & 0 \cdots\cdots 0 \\ \hdashline R & \vdots & \\ \vdots & \vdots & \Lambda_{n-1} \\ R & \vdots & \end{pmatrix}$$

Then $id(\Lambda_{n\Lambda_n}) = id(\Lambda_{n-1\Lambda_{n-1}}) = \cdots = id(\Lambda_{2\Lambda_2}) = i + 1.$

References

[1] H. Cartan and S. Eilenberg: Homological Algebra, Princeton Univ. Press, Princeton, 1956.

[2] S. Chase: A generalization of the ring of triangular

matrices, Nagoya Math. J. 18 (1968), 13-25.

[3] R. Fossum, P. Griffith and I. Reiten: Trivial extensions of abelian categories, Lecture Notes in Math. 456, Springer-Verlag, Berlin, 1975.

[4] K. R. Goodearl: Ring Theory, Dekker, New York, 1976.

[5] I. Palmer and J.-E. Roos: Explicit formulae for the global homological dimensions of trivial extensions of rings, J. Algebra 27 (1973), 380-413.

[6] I. Reiten: Trivial extensions and Gorenstein rings, Thesis, University of Illinois, Urbana, 1971.

[7] K. Sakano: Injective dimension of generalized triangular matrix rings, to appear in Tsukuba J. Math.

[8] A. Zaks: Injective dimension of semi-primary rings, J. Algebra 13 (1969), 73-86.

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WEAKLY CODIVISIBLE MODULES AND STRONGLY η-PROJECTIVE MODULES

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In this note, ring R means a ring with unit and modules mean unital right R-modules.

P.E. Bland [3] has introduced the notion of the strongly M-projective modules as the generalization of well known notions of projective modules, quasi-projective modules and M-projective modules. (For the definitions, see [1] and [2].) But for the insight of these modules, we generalize this notion as follows. Let η be a subclass of R-modules. A module K_R is called a strongly η-projective module if $\mathrm{Hom}_R(K_R,-)$ preserves the exactness of any epimorphism $L_R \longrightarrow H_R$ such that $L_R = \Pi M_i$ for some $M_i \in \eta$. Also we call that N_R has a strongly η-projective cover if there is an epimorphism f: $K_R \longrightarrow N_R$ satisfying the following properties;

(1) K_R is a strongly η-projective module.

(2) Ker(f) is small in K_R.

(3) For $0 \neq x \in \mathrm{Ker}(f)$, there are R-module M_R in η and R-homomorphism f: $K_R \longrightarrow M_R$ such that $f(x) \neq 0$.

In the case that $\eta = \{M_R\}$, it coinsides with the original definitions of the strongly η-projective module and strongly η-projective cover due to Bland.

In [3], he has proved that if factor modules of modules cogenerated by M_R are also cogenerated by M_R, then a module N_R has the strongly M-projective cover iff $N/(N\cdot\mathrm{Ann}(M_R))$ has the projective cover as an $R/\mathrm{Ann}(M_R)$-module. In the half of this note, we show the above equivalent condition holds without the assumption for M_R. The proof for this result is due to the method of torsion theories. So we define fundamental notions of torsion theories. A subfunctor of the identity functor between the category of right R-modules is called a preradical. For a preradical t, we denote

$$\mathcal{T}_t = \{ L_R \mid t(L_R) = L_R \} \quad \text{and}$$
$$\mathcal{F}_t = \{ L_R \mid t(L_R) = 0 \}$$

whose elements are called torsion modules and torsion-free modules respectively. A preradical t is called a radical if $t(L_R/t(L_R)) = 0$ for any R-module L_R. $(\mathcal{T}_t, \mathcal{F}_t)$ is called a pre-torsion theory in the case t is a radical.

For a subclass η of R-modules, we define t_η by

$$t_\eta(L_R) = \bigcap \{ \mathrm{Ker}(f) \mid f \in \mathrm{Hom}_R(L_R, K_R),\ K_R \in \eta \}$$

for any R-module L_R. H. Katayama [4] has remarked that for a preradical t, t is a radical iff $t = t_\eta$ for some class η of R-modules. By this result, we can translate the notions of strongly M-projective modules and strongly M-projective covers into the notions of torsion theories. (See Lemma 1.)

For a preradical t, a module N_R is called a codivisible (resp. weakly codivisible) module with respect to $(\mathcal{T}_t, \mathcal{F}_t)$ if $\mathrm{Hom}_R(N_R, -)$ preserves the exactness of an epimorphism $h: L_R \longrightarrow H_R$ such that $\mathrm{Ker}(h) \in \mathcal{F}_t$ (resp. $L_R \in \mathcal{F}_t$). We call a module N_R has a weakly codivisible (resp. codivisible) cover with respect to $(\mathcal{T}_t, \mathcal{F}_t)$ if there exists an epimorphism $f: K_R \longrightarrow N_R$ such that

(1) K_R is weakly codivisible (resp. codivisible).

(2) $\mathrm{Ker}(f)_R$ is small in K_R.

(3) $t(K_R) \cap \mathrm{Ker}(f)_R = 0$
(resp. $\mathrm{Ker}(f)_R \in \mathcal{F}_t$).

In the last half of this note, we characterize the pre-torsion theory with respect to which every weakly codivisible module is codivisible. It will be shown that this pre-torsion theory is just coincided with the pseudo-hereditary pre-torsion theory (i.e. any submodule of $t(R_R)$ is torsion). By this theorem, we can determine, in Corollary 9, the pre-torsion theory with the condition that every module is codivisible. (This has been partially solved by K.M. Rangaswarmy [6. Corollary 15] in the case $(\mathcal{T}_t, \mathcal{F}_t)$ is pseudo-hereditary torsion theory.)

Last we characterize a module with the colocalization for a given torsion theory and we shall show that those modules are those M_R such that $t(M_R)\cdot t(R_R) = t(M_R)$.

1. ON STRONGLY η-PROJECTIVE MODULES.

Lemma 1. Let η be a subclass of R-modules and $t = t_\eta$. Then it holds that

(a) A module N_R is strongly η-projective iff N_R is weakly codivisible with respect to $(\mathcal{T}_t, \mathcal{F}_t)$.

(b) The property of (3) in the definition of a strongly η-projective cover is equivalent to $t(K_R) \cap \mathrm{Ker}(f)_R = 0$.

(c) $t(R_R) = \mathrm{Ann}(\eta) = \bigcap \{ \mathrm{Ann}(M_R) \mid M_R \in \eta \}$.

The next lemma plays an important role in this note.

Lemma 2. For a radical t, it holds that

(a) $N_R \cdot t(R_R) \subset t(N_R)$ for any R-module N_R.

(b) If N_R is weakly codivisible, then $t(N_R) = N_R \cdot t(R_R)$.

(c) Let $0 \longrightarrow A_R \longrightarrow B_R \longrightarrow C_R \longrightarrow 0$ be an exact sequence. If C_R is codivisible, then $t(A_R) = t(B_R) \cap A_R$.

Next is a slight generalization of [3. Proposition 3].

Theorem 3. For a radical t, the following statements are equivalent.

(1) A module N_R is weakly codivisible with respect to $(\mathcal{T}_t, \mathcal{F}_t)$.

(2) $N_R/(N_R \cdot t(R_R))$ is a projective $R/t(R_R)$-module.

The following theorem is a main theorem of this section.

Theorem 4. Let t be a radical such that $t = t_\eta$ for some class η of R-modules. Then the following statements are equivalent for a module N_R.

(1) N_R has a strongly η-projective cover.

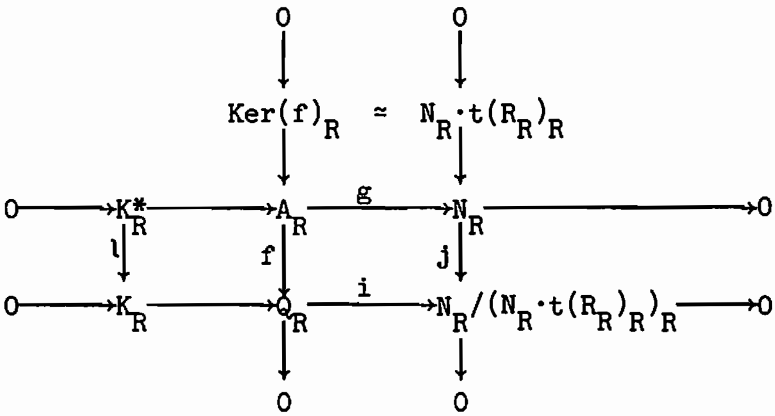
(2) N_R has a weakly codivisible cover.

(3) $N_R/(N_R \cdot t(R_R))$ has a projective cover as an $R/t(R_R)$-module.

The important part of the proof is (2) implies (1), so we give an outline of the proof.

Let $0 \longrightarrow K \longrightarrow Q \longrightarrow N_R/(N_R \cdot t(R_R)) \longrightarrow 0$ be a projective cover of $N_R/(N_R \cdot t(R_R))$ as an $R/t(R_R)$-module and

$j: N_R \longrightarrow N_R/(N_R \cdot t(R_R))_R$ a canonical map. We consider a fibre product (i. e. pull back) (A_R, f, g) of $(i, j, N_R/(N_R \cdot t(R_R)))$, then we have a commutative diagram with exact rows and columns;



We can show that

(a) $\mathrm{Ker}(f) = A_R \cdot t(R_R) = t(A_R)$.

(b) A_R is weakly codivisible by Theorem 3.

(c) $K^*_R \cap t(A_R)_R = K^*_R \cap \mathrm{Ker}(f)_R = 0$.

(d) K^*_R is small in A_R.

Here we only prove (d). Assume $L_R + K^*_R = A_R$ for $L_R \subset A_R$. Then $K^*_R \cdot t(R_R) + L_R \cdot t(R_R) = A_R \cdot t(R_R)$. Since $K_R \cdot t(R_R) = 0$, $K^*_R \cdot t(R_R) = 0$, hence $L_R \cdot t(R_R) = A_R \cdot t(R_R)$. Thus $\mathrm{Ker}(f) = A_R \cdot t(R_R) = L_R \cdot t(R_R) \subset L_R$.

On the other hand, $f(K^*_R) + f(L_R) = f(A_R) = Q_R$. That is $K_R + f(L_R) = Q_R$. Since K is small in Q as an $R/t(R_R)$-module, of course as an R-module, so $f(L_R) = Q_R$, that is $L_R + \mathrm{Ker}(f)_R = A_R$. Hence $L_R = A_R$.

2. ON PSEUDO-HEREDITARY PRE-TORSION THEORIES.

If $t = t_{\{E\}}$ for some injective module E_R, (in this case, $(\mathcal{T}_t, \mathcal{F}_t)$ is well known as a hereditary torsion theory), then it is clearly shown that weakly codivisible modules coincide with codivisible modules. So in this section, we determine a pre-torsion theory with respect to which weakly codivisible modules are codivisible modules. It turns out that this pre-torsion theory must be pseudo-hereditary defined as any submodule of $t(R_R)_R$ is torsion. Also we consider a following property (*) for a module N_R;

(*) If $K_R \subset N_R \cdot t(R_R)_R$, then $K_R \in \mathcal{T}_t$.

For a pseudo-hereditary pre-torsion theory, we have the following theorem.

Theorem 5. Assume t is a radical. The following statements are equivalent.

(1) $(\mathcal{T}_t, \mathcal{F}_t)$ is a pseudo-hereditary pre-torsion theory.

(2) Every module has the property (*).

(3) Every weakly codivisible module has the property (*).

(4) Every codivisible module has the property (*).

(5) A module N_R such that $t(N_R)_R = N_R \cdot t(R_R)_R$ has the property (*).

(6) If $t(N_R)_R = N_R \cdot t(R_R)_R$, then $t(K_R) = K_R \cap t(N_R)_R$ for any submodule K_R of N_R.

Next we consider the following property (**) for a module N_R.

(**) Any submodule of $N_R \cdot t(R_R)_R$ has no non-zero torsion-free submodules.

Theorem 6. The following assertions are equivalent for a radical t.

(1) Every weakly codivisible module is codivisible.

(2) Every module has the property (**).

(3) Every cyclic module has the property (**).

(4) Every weakly codivisible module has the property (**).

(5) For every codivisible module N_R, it holds that

(a) N_R has the property (**).

(b) $N_R/t(K_R)$ is codivisible for any $K_R \subset N_R$.

(6) Every module N_R such that $t(N_R)_R = N_R \cdot t(R_R)_R$ has the property (**).

(7) If N_R is weakly codivisible, then $t(K_R)_R = t(N_R)_R \cap K_R$ for any submodule K_R of N_R.

(8) If N_R is codivisible, then $t(N_R)_R \cap K_R = t(K_R)_R$ for any submodule K_R of N_R.

Next theorem is a main theorem of this section.

Theorem 7. The properties in Theorem 5 and Theorem 6 are equivalent.

Proof. We only show the properties in Theorem 6 implies those in Theorem 5 since the other part is clear.

Assume I_R is a right ideal contained in $t(R_R)_R$. Since $I_R/t(I_R)_R \subset (R_R/t(I_R)_R)_R \cdot t(R_R)_R$, $I_R/t(I_R)_R \in \mathcal{T}_t$ by Theorem 6 (2). Hence $I_R/t(I_R)_R \in \mathcal{T}_t \cap \mathcal{F}_t$, so $I_R/t(I_R)_R = 0$. This means $I_R = t(I_R)_R$.

From the above theorems, we have the next corollary. This gives a complete solution of the property [8. Theorem 15].

Corollary 8. Assume t is a radical. Then the following statements are equivalent.

(1) Every module is codivisible.

(2) It holds that

(a) $R/t(R_R)$ is a semi-simple artinian ring.

(b) $(\mathcal{T}_t, \mathcal{F}_t)$ is pseudo-hereditary.

An example of a pseudo-hereditary torsion theory is ones such that $t(R_R)_R = 0$. Next example gives a non-pseudo-hereditary torsion theory and also gives non-codivisible but weakly codivisible modules.

Example 9. Let Z be a ring of integers and p a prime number. We put $M_Z = Z_Z/(p\cdot Z)_Z$ and $t = t_{\{M\}}$. Then it holds that

(1) $(\mathcal{T}_t, \mathcal{F}_t)$ is not pseudo-hereditary.

(2) $Z/t(Z_Z)$ is a semi-simple artinian ring. (In fact $t(Z_Z) = p\cdot Z$.)

(3) Every Z-module is weakly codivisible.

(4) $Z/(p\cdot Z)_Z$ has not a codivisible cover.

3. THE MODULES WITH THE COLOCALIZATION.

For a radical t, we say that M_R has the colocalization with respect to $(\mathcal{T}_t, \mathcal{F}_t)$ if there exists an R-homomorphism $f: N_R \longrightarrow M_R$ such that

(1) $\mathrm{Ker}(f)_R \in \mathcal{F}_t$ and $\mathrm{Cok}(f)_R \in \mathcal{F}_t$.

(2) N_R is codivisible.

(3) N_R is torsion.

Theorem 10. Let t be a radical. Assume for a codivisible module A_R and its submodule B_R, $A_R/t(B_R)_R$ is codivisible. Then the following statements are equivalent.

(1) M_R has the colocalization.

(2) $Hom_R(t(M_R)_R, L_R/T_R) = 0$ for any torsion-free module L_R and its submodule T_R.

(3) $t(M_R)_R \varepsilon \mathcal{T}_t$ and $t(M_R)_R$ is weakly codivisible.

(4) $t(M_R)_R \cdot t(R_R)_R = t(M_R)_R$.

Remark: In the above theorem, the equivalences of (2), (3) and (4) is valid without the assumption for A_R. Also this assumption is equivalent to $t(t(B_R))_R = t(B_R)_R$ by Lemma 2 (c), so if t is idempotent, this assumption is naturally satisfied.

References

[1] G. Azumaya: M-projective and M-injective modules, unpublished.

[2] G. Azumaya, F. Mbuntun and K. Varadarajan: On M-projective and M-injective modules, Pacific J. Math., 59 (1975), 9 - 16.

[3] P.E. Bland: Remarks on strongly M-projective modules, Illinois J. Math., 23 (1979), 167 - 174.

[4] H. Katayama: A note on locally injective modules and QF-3' modules, preprint in Japanese.

[5] K. ohtake: Colocalization and localization, J. Pure and Appl. Algebra 11 (1977), 217 - 241.

[6] K.M. Rangaswarmy: Codivisible modules, Comm. in Algebra 2 (1974), 475 - 489.

[7] M. Sato: The concrete description of the colocalization, Proc. Japan Acad., 52 (1976), 501 - 504.

[8] Bo Stenström: Rings and Modules of Quotients, Lecture Notes in Math. 237, Springer Verlag, Berlin, 1971.

[9] M.L. Teply: Codivisible and projective covers, Comm. in Algebra 1 (1974), 23 - 38.

[10] K. Varadarajan: M-projective and strongly M-projective modules, Illinois J. Math. 20 (1976), 507 - 511.

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