PROCEEDINGS OF THE 14TH SYMPOSIUM ON RING THEORY

HELD AT SHINSHU UNIVERSITY, MATSUMOTO

JULY 30—AUGUST 1, 1981

EDITED BY

HISAO TOMINAGA

WITH THE COOPERATION OF

SHIZUO ENDO MANABU HARADA

KAZUO KISHIMOTO HIROYUKI TACHIKAWA

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OKAYAMA, JAPAN

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CONTENTS

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PAPERS

PREFACE

This volume contains the papers presented at the 14th Symposium on Ring Theory held at Shinshu University, July 30 - August 1, 1981.

The annual Symposium on Ring Theory was founded in 1968. The main aims of the Symposium are to provide a means for the dissemination of recent theories on rings and modules which are not yet widely known and to give algebraists an opportunity to report on recent progress in the ring theory.

The Symposium was organized by Professors Shizuo ENDO (Tokyo Metropolitan University), Manabu HARADA (Osaka City University), Hiroyuki TACHIKAWA (University of Tsukuba) and Hisao TOMINAGA (Okayama University); the 14th Symposium itself and these Proceedings were supported from the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (Subject No. 56340001) through the arrangements by Professor Hiroyuki TACHIKAWA. I would like to take this opportunity of thanking him for his arrangements.

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November 1981

H. TOMINAGA

PREFACE

This volume contains the papers presented at the [illegible] Symposium on Ring Theory held at Shinshu University, July 30 - August 1, 1991.

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[illegible] by [illegible] (Grant-in-Aid [illegible]), [illegible] (University of [illegible]) and [illegible] [illegible] The [illegible] Symposium [illegible] supported [illegible] Research from the Ministry of Education [illegible] (Project No. [illegible]) [illegible] Hiroyuki TACHIKAWA [illegible]

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SOME POLYNOMIAL IDENTITIES AND COMMUTATIVITY OF RINGS. I

Yuji KOBAYASHI

1. Introduction. There have been many results of the following type: A ring satisfying a cirtain polynomial identity is commutative. For example, a ring satisfying the identity $x^n=x$, $n(\geq 2)$ being a fixed integer, is commutative. This is a famous result by Jacobson. But the converse of this result is not true, that is, a commutative ring need not satisfy $x^n=x$. What types of polynomial identities both are satisfied by all commutative rings and induce ring-commutativity ?

Let $\mathbb{Z}\langle x_1,\ldots,x_r\rangle$ be the non-commutative polynomial ring in r variables $x_1,\ldots,x_r$ over the ring $\mathbb{Z}$ of integers. Let F be a polynomial in $\mathbb{Z}\langle x_1,\ldots,x_r\rangle$ and $n_1,\ldots,n_r$ be r integers. F is said to be homogeneous of degree $(n_1,\ldots,n_r)$, if all the non-zero terms in F are of degree n_i with respect to x_i for $i=1,\ldots,r$. $F_{n_1,\ldots,n_r}$ denotes the homogeneous component of F of degree $(n_1,\ldots,n_r)$. F is called balanced, if $F_{n_1,\ldots,n_r}(1,\ldots,1) = 0$ for any integers $n_1,\ldots,n_r$. The identity $F(x_1,\ldots,x_r) = 0$ is called balanced if F is balanced. Easily we can see

(*) An identity $F(x_1,\ldots,x_r) = 0$ is satisfied by all commutative rings if and only if it is balanced.

Now, our problem is to find a balanced identity which makes (or is apt to make) a ring commutative. However, there is a non-commutative ring which satisfies any homogeneous identity of total degree higher than 2. In fact, consider a non-commutative ring N satisfying $N^3=0$. To avoid this culdesac, we assume that every ring we consider has an identity element 1. Moreover, for simplicity, we consider only polynomials in two variables.

In §2, we discuss general balanced identities and give an answer to the question above. In §3, we concentrate upon the special identity $(xy)^n=x^n y^n$ and give some results, some conjectures and some problems on this identity.

2. General arguments. Hereafter, R represents a ring with 1. Let $M = x_1 \cdots x_d$ be a monic monomial in x and y of total degree d (x_i equals either x or y). We define the integers $\Phi(M)$ and $\Psi(M)$ as follows:

$$\Phi(M) = |\{(i,j) \mid 1 \leqq i<j \leqq d,\ x_i=x,\ x_j=y\}|,^{1)}$$

$$\Psi(M) = |\{(i,j) \mid 1 \leqq i<j \leqq d,\ x_i=y,\ x_j=x\}|.$$

Φ and Ψ are extended linearly to the functions of $\mathbb{Z}\langle x,y\rangle$ into $\mathbb{Z}$, which we denote by the same symbols Φ and Ψ respectively.

Lemma 1. If $F \in \mathbb{Z}\langle x,y\rangle$ is balanced, then $\Phi(F)+\Psi(F) = 0$.

Proof. For a monic monomial $M = x_1 \cdots x_d$ (x_i=x or y) of degree (m,n), we have by the definition of Φ and Ψ that $\Phi(M)+\Psi(M) = mn$. Therefore, for a balanced polynomial F we obtain

$$\Phi(F)+\Psi(F) = \sum_{m,n} (\Phi(F_{m,n})+\Psi(F_{m,n})) = \sum_{m,n} mn\cdot F_{m,n}(1,1) = 0.$$

Let n be a positive integer. R is called n-torsion free, if nx=0 implies x=0 for every $x \in R$.

---

1) For a set X, |X| denotes the cardinality of X.

Theorem 1 (Kobayashi [6]). *Let* $F \in \mathbb{Z}\langle x,y\rangle$ *be a homogeneous balanced polynomial of degree* (m,n). *Assume* $\Phi(F) \neq 0$. *If* R *satisfies* $F(x,y)=0$, *then* R *satisfies the identity*

$$(m-1)!(n-1)!\Phi(F)(xy-yx) = 0.$$

In particular, if R *is moreover* $(m-1)!(n-1)!\Phi(F)$-*torsion free, then* R *is commutative*.

In Theorem 1, the condition $\Phi(F) \neq 0$ (this is equivalent to $\Psi(F) \neq 0$ by Lemma 1) is indispensable (see [6, Example 3]). Moreover, in the conclusion, $(m-1)!(n-1)!\Phi(F)$ cannot be replaced by a proper divisor of $\Phi(F)$ (see [6, Example 4]). Thus $\Phi(F)$ is a very important index concerning commutativity of rings. For some F, $(m-1)!(n-1)!\Phi(F)$ can be replaced by $(\Phi(F))^k$ with k sufficiently large, but it is impossible in general. It depends on each individual identity. A homogeneous balanced polynomial $F \in \mathbb{Z}\langle x,y\rangle$ (or identity $F(x,y)=0$) is called *strong*, if $\Phi(F) \neq 0$ and any ring with 1 satisfying $F(x,y)=0$ satisfies the identity

$$(\Phi(F))^k(xy-yx) = 0$$

for some integer $k \geq 0$.

Lemma 2. *Let* $F \in \mathbb{Z}\langle x,y\rangle$ *be homogeneous and balanced. Assume* $\Phi(F) \neq 0$. *Then the following are equivalent*.

(1) F *is strong*.

(2) *Any* $\Phi(F)$-*torsion free ring with* 1 *satisfying* $F(x,y) = 0$ *is commutative*.

Proof. (1) ⇒(2): Easy.

(2) ⇒(1): Let us assume that (2) holds. Let R be a ring with 1. Let I be the ideal of R defined by

$$I = \{x \in R \mid (\Phi(F))^k x = 0 \text{ for some } k \geq 0\}.$$

Then the quotient ring R/I is $\Phi(F)$-torsion free and satisfies $F(x,y)=0$. Hence R/I is commutative by (2). Therefore, for any $x,y \in R$, there exists an integer $k(x,y) \geq 0$ such that $(\Phi(F))^{k(x,y)}(xy-yx) = 0$. On the other hand, by Theorem 1 we have $(m-1)!(n-1)!\Phi(F)(xy-yx) = 0$ for all $x,y \in R$, where we suppose that F is of degree (m,n). Hence we get

$$\Phi(F)\cdot((m-1)!(n-1)!,\ (\Phi(F))^{k(x,y)-1})(xy-yx) = 0$$

[2]

for all $x,y \in R$. It follows that R satisfies the identity $(\Phi(F))^{k}(xy-yx) = 0$ for a sufficiently large k.

Example. Let n be a positive integer.

(1) A homogeneous balanced polynomial of degree lower than 3 with respect to both x and y is strong by Theorem 1.

(2) $(xy)^n=(yx)^n$ is strong by Bell [3, Theorem 5].

(3) (c.f. [3, §2]). $x^ny^n=y^nx^n$ is not strong, when n has a divisor of the form $1+p^r+p^{2r}+\cdots+p^{sr}$, where p is a prime and r and s are positive integers. Therefore, $x^ny^n=y^nx^n$ is not strong for n=3,4,...,10. What about $x^{11}y^{11}=y^{11}x^{11}$?

Metatheorem. If R satisfies strong homogeneous balanced identities $F_1=0$, ..., $F_s=0$ such that $(\Phi(F_1), \ldots, \Phi(F_s))$[2] $= 1$, then R is commutative.

Proof. Since F_i is strong, there is an integer $k_i \geq 0$ such that $(\Phi(F_i))^{k_i}(xy-yx)=0$ for all $x,y \in R$ (i=1,...,s). Since $((\Phi(F_1))^{k_1}, \ldots, (\Phi(F_s))^{k_s}) = 1$, we have

$$xy-yx = ((\Phi(F_1))^{k_1}, \ldots,(\Phi(F_s))^{k_s})\cdot(xy-yx) = 0.$$

2) For integers $n_1,\ldots,n_s$, their greatest common divisor is denoted by $(n_1,\ldots,n_s)$.

Application. Since $(xy)^n=(yx)^n$ is strong as stated in Example (2) and $\Phi((xy)^n-(yx)^n) = n$, we have the following: _If_ R _satisfies the identities_ $(xy)^{n_1}=(yx)^{n_1}$, ..., $(xy)^{n_s}=(yx)^{n_s}$ _such that_ $(n_1,\ldots,n_s)=1$, _then_ R _is commutative_.

3. _The identity_ $(xy)^n=x^ny^n$. Let n be a positive integer. The first problem we wish to settle is

Problem 1. Is $(xy)^n=x^ny^n$ strong ? In other words, is an $\frac{n(n-1)}{2}$ - torsion free ring with 1 satisfying $(xy)^n=x^ny^n$ commutative ?

Though we know $(xy)^2=x^2y^2$ is strong by Theorem 1 (or by Johnsen, Outcalt and Yacub [4]), the problem is open for $n\geqq3$. The following result by Abu-Khuzam is interesting.

Theorem 2 (Abu-Khuzam [1]). _If_ R _is_ n(n-1)-_torsion free and satisfies_ $(xy)^n=x^ny^n$, _then_ R _is commutative_.

We define the subset E(R) of $\mathbb{Z}$ associated with R by

$$E(R) = \{n \in \mathbb{Z} \mid n>0 \text{ and } (xy)^n=x^ny^n \text{ for all } x,y \in R\}.$$

E(R) is a semigroup by multiplication and is called the _exponent semigroup_ of R (Tamura [10]). If Problem 1 is answered positively, then so is the following problem in virtue of Metatheorem.

Problem 2. Is R commutative if E(R) contains integers $n_1,\ldots,n_s$ such that $(n_1(n_1-1), \ldots, n_s(n_s-1)) = 2$?

The following is a partial answer to the problem.

Theorem 3 (Kobayashi [7]). _If_ E(R) _contains integers_ $n_1,\ldots,n_s$ _such that_ $(n_1(n_1-1), \ldots, n_s(n_s-1)) = 2$ _and some of_ n_i _is even_, _then_ R _is commutative_.

Theorem 3 contains the following well-known result: *If E(R) contains three consequtive integers, then R is commutative* (Ligh and Richoux [8]). More generally, it contains the following: *If E(R) contains m, m+1, n and n+1 such that (m,n) = 1 or 2, then R is commutative* (Bell [2] and Mogami [9]). Observing Theorem 3, we conjecture the following.

Conjecture 1. If E(R) contains integers $n_1,\ldots,n_s$ such that R is $\frac{1}{2}(n_1(n_1-1), \ldots, n_s(n_s-1))$-torsion free and some of n_i is even, then R is commutative.

The following result by Bell is a very special case of the conjecture.

Theorem 4 (Bell [2, Theorem 1]). *If E(R) contains integers n and n+1 and R is n-torsion free, then R is commutative.*

Considering the case s=1 in Conjecture 1, we have

Conjecture 1'. $(xy)^n=x^ny^n$ is strong for n even.

The author is very interested in the structure of E(R), on which we give

Conjecture 2. E(R) is either equal to {1} or expressible in the form

$$E(R) = \bigcap_{i=1}^{s} M(n_i) \bigcap N(n)$$

for some integers $n_1,\ldots,n_s \geqq 2$ and $n \geqq 1$, where

$$M(n) = \{kn+1, (k+1)n \mid k=0,1,2,..\}$$

and

$$N(n) = \{kn+1 \mid k=0,1,2,..\}$$

It is easily seen that Theorem 3 is a natural consequence of Conjecture 2. The reader may think that this conjecture is too bold. But we have the following conjecture on general exponent semigroups, which is true in many important cases. For a semigroup S, the exponent semigroup E(S) of S is defined in the same way as for a ring. For two subsets A and B of $\mathbb{Z}$, we write $A \doteqdot B$ if $(A \cup B) \setminus (A \cap B)$ is a finite set.

Fundamental Conjecture on exponent semigroups (Kobayashi [5]). For any semigroup S, it holds either $E(S) = \{1\}$ or

$$E(S) \doteqdot \bigcap_{i=1}^{s} M(n_i) \cap N(n)$$

for some integers $n_1, \ldots, n_s \geq 2$ and $n \geq 1$.

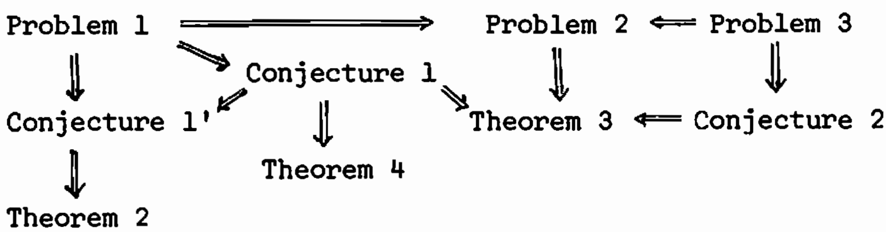
This conjecture is true, for example, for any finite semigroup S ([5, Theorem 5]). Concerning Problem 2 we give

Problem 3. Assume $E(R) \neq \{1\}$. Is E(R) expressible in the form

$$E(R) = \bigcap_{i=1}^{s} M(n_i)$$

for some integers $n_1, \ldots, n_s \geq 2$?

If Problem 3 has an affirmative answer, then so does Problem 2. The following diagram illustraites the relations among the theorems, the conjectures and the problems given in this section. In the diagram, $A \Rightarrow B$ means that B follows from A or that if A is affirmative, so is B.



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Faculty of Education
Tokushima University
Tokushima, Japan

SOME POLYNOMIAL IDENTITIES AND COMMUTATIVITY OF RINGS.II

Yasuyuki HIRANO

Throught this paper, R will denote an associative ring (with or without 1), and C the center of R. We denote by D(R) the commutator ideal of R. Given a, b $\in$ R, we set $[a,b] = ab - ba$ as usual, and formally write $a(1+b)$ (resp. $(1+b)a$) for $a+ab$ (resp. $a+ba$).

A ring R is called s-unital if for each x in R, $x \in Rx \cap xR$. If R is an s-unital ring, then for any finite subset F of R, there exists an element e in R such that $ex = xe = x$ for all x in F. Such an element e will be called a pseudo-identity of F.

We begin with reproving a theorem of Kezlan-Bell ([21], [4]).

Theorem 1 ([21], [4]). Let f be a polynomial in non-commuting indeterminates $x_1, \dots, x_k$ with integer coefficients. Then the following statements are equivalent:

1) For any ring R satisfying the polynomial identity $f = 0$, D(R) is a nil ideal.

2) Every semiprime ring satisfying $f = 0$ is commutative.

3) There is no prime number p such that $(GF(p))_2$ satisfies $f = 0$.

Proof. Obviously, 2) $\Rightarrow$ 1) $\Rightarrow$ 3). We prove that 3) implies 2). Note that the coefficients of f are relatively prime and $(\mathbb{Z})_2$ does not satisfy $f = 0$. It is enough to show that if R is a prime ring satisfying $f = 0$ then R is commutative. By [3, Theorem 7 (6)] (and Posner's theorem), the quotient ring Q of R is a simple ring satisfying $f=0$. Hence, by hypothesis, Q must be a central division algebra over Z of finite rank. Suppose Q is not commutative, and choose a maximal subfield K of Q. Then, again by [3, Theorem 7 (6)], $Q \otimes_Z K = (K)_m$ ($m^2 = [Q:Z] > 1$) satisfies $f=0$. But, by hypothesis, this is impossible.

Examples. (1) If a semiprime ring R satisfies one of the polynomial identities $[(xy)^{\ell},[(xy)^m,(yx)^n]]=0$, $(xy)^{\ell}\circ[(xy)^m,(yx)^n]=0$ and $[(xy)^{\ell},(xy)^m\circ(yx)^n]=0$, then R is commutative, where $x\circ y = xy+yx$. (Take $x=E_{11}+E_{12}$ and $y=E_{11}$.) This is [19, Theorem 3].

(2) Suppose that for each pair of elements x, y in R there exists an integer $n=n(x,y)$ such that $1<n\leq N$ and $[(xy)^n - x^ny^n,x]=0$. Then $D(R)$ is a nil ideal. In fact, R satisfies the polynomial identity

$f(x) = [(xy)^2 - x^2y^2,x]y[(xy)^3 - x^3y^3]y \cdots [(xy)^N - x^Ny^N,x] = 0$

but $f(E_{12},E_{21}) \neq 0$. This includes [8, Theorem 1] and [20, Theorem].

(3) If a semiprime ring R satisfies the polynomial identity $[[x^n,y]-[x,y^n],x]=0$ $(n>1)$, then R is commutative. (Take $x=E_{11}$ and $y=E_{12}$.) This includes [8, Theorems 2 and 3].

Remark 1. Let f be as in Theorem 1 3). As was stated in [14, Lemma 1], if R satisfies $f=0$ then there exists a positive integer m such that $[x,y]^m=0$ for all x, $y \in R$. However, this can be seen from [24, Lemma 1.6.37, p. 48].

Next, in connection with power maps, we consider the following ring-properties:

$P_1(n)$: R satisfies the identity $[x^n, y^n] = 0$.

$P_2(n)$: R satisfies the identities $(xy)^k = x^k y^k$ $(k = n, n+1)$.

$P_3(n)$: R satisfies the identities $(xy)^n = x^n y^n = y^n x^n$.

$P_4(n)$: R satisfies the identity $(xy)^n = (yx)^n$.

$P_5(n)$: R satisfies the identity $[x, (xy)^n] = 0$.

$P_6(n)$: R satisfies the identity $[x^n, y] = 0$.

$P_7(n)$: R satisfies the identity $[x^n, y] = [x, y^n]$.

$P_8(n)$: R satisfies the identity $[x^2\psi(x), y] = [x, y^n]$, where $\psi(x)$ is a polynomial with integer coefficients.

$P_9(n)$: R satisfies the identity $[x, (x+y)^n - y^n] = 0$.

Proposition 1 ([14]). If R is an s-unital ring, then $P_2(n) \Leftrightarrow P_3(n) \Rightarrow P_4(n) \Rightarrow P_5(n) \Leftrightarrow P_6(n) \Rightarrow P_1(n)$, and $P_6(n) \Rightarrow P_7(n)$.

Proof. Obviously, $P_3(n)$ implies $P_2(n)$ and $P_4(n)$, and $P_6(n)$ does $P_1(n)$, $P_5(n)$ and $P_7(n)$. Since $P_2(n)$ together with $P_6(n)$ implies $P_3(n)$, it is enough to show that $P_2(n) \Rightarrow P_5(n)$ and $P_4(n) \Rightarrow P_5(n) \Rightarrow P_6(n)$.

$P_2(n) \Rightarrow P_5(n)$. Since $xyx^n y^n = (xy)^{n+1} = x^{n+1} y^{n+1}$, we have $x[x^n, y]y^n = 0$, and therefore $x[x^n, y] = 0$. In particular, $x[x^n, y^n] = 0$. Hence, $[x, (xy)^n] = x\{(xy)^n - (yx)^n\} = x[x^n, y^n] = 0$.

$P_4(n) \Rightarrow P_5(n)$. It is immediate that $[x, (xy)^n] = x\{(xy)^n - (yx)^n\} = 0$.

$P_5(n) \Rightarrow P_6(n)$. As a consideration of $x = E_{12}$ and $y = E_{21}$ shows, D(R) is a nil ideal (Theorem 1). Let T be the (s-unital) subring of R generated by all n-th powers of elements of R. Let u be an arbitrary nilpotent element of R, and u' the quasi-inverse of u. If a is an arbitrary element of R, and e a pseudo-identity of $\{u, a\}$, then $[u,a] = [e+u, \{(e+u)(e+u')a\}^n] = 0$. In particular, every nilpotent element of T is in the center of T. Now, let s, t be in T. Since $s^n t^n - (st)^n$ is in the nil ideal $D(T)$, we get $s^n[s,t^n] = [s, s^n t^n] = [s,(st)^n] = 0$. Then, $[s,t^n] = 0$. This implies that $[x^n, y^{n^2}] = 0$ for all $x, y \in R$. So, according to [14, Lemma 3], we can find a positive integer k such that $kD(R) = 0$. This enables us to see that $x^{n^2k}[x,y^n] = [x, x^{n^2k} y^n] = [x, (x \cdot x^{nk-1} y)^n] = 0$. Hence, we obtain $[x,y^n] = 0$.

In [5, Theorem 3], it is shown that if a ring R with 1 has the property $P_2(n)$ and is generated by $\{a^{n^2} \mid a \in R\}$ or $\{a^{n(n+1)} \mid a \in R\}$, then it is commutative. However, the next is immediate.

Corollary 1. Let R be an s-unital ring having the property $P_2(n)$. If R is generated by $\{a^n \mid a \in R\}$, then R is commutative.

Acceding to Theorem 1, if R has any of the properties $P_2(n) - P_6(n)$, then $D(R)$ is contained in the prime radical of R.

Corollary 2 ([5, Theorem 1], [6, Theorems 2 and 5]). Let R be an s-unital ring having one of the properties $P_2(n) - P_6(n)$. Furthermore, if R has the property $Q(n)$: for each pair of elements x, y in R, $n[x,y] = 0$ implies $[x,y] = 0$, then R is commutative.

Proof. It is easy to see that an s-unital ring R satisfying $P_6(n)$ and $Q(n)$ is commutative.

The following exaples show that even if $(1\ \epsilon)R$ has the properties $P_1(n)$ and $Q(n)$, R is not necessary commutative.

Examples (Bell [6]). a) We consider the following subring of $(GF(4))_2$:

$$R = \left\{ \begin{bmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{bmatrix} \middle| \ a,\ b,\ c \in GF(4) \right\}.$$

It is easily verified that R has the properties $P_1(3)$ and $Q(3)$ and it is obvious that R is not commutative.

b) Let ϕ be an automorphism of $GF(p^k)$. We consider the ring $R = GF(p^k) \oplus GF(p^k)$ with multiplication given by $(a,b)(c,d) = (ac, ad + b\phi(c))$. R is called the Corbas (p,k,ϕ)-ring. As is easily seen, R is commutative if and only if ϕ is the identity automorphism. It is easily verified that R has the properties $P_1(p^k - 1)$ and $Q(p^k - 1)$.

We denote by N the set of all nilpotent elements of R. Then we have the following

Theorem 2 ([14]). If R is an s-unital ring having the properties $P_1(n)$ and $Q(n)$, then $DN = 0$, and in particular, $D^2 = 0$. Moreover, if every $u \in N$ with $u^2 = 0$ is central, then R is commutative.

According to Remark 1, if R has the property $P_7(n)$ then R satisfies the identity $[x,y]^h = 0$ for some positive integer h. By making use of this fact, we can prove the following

Proposition 2 ([14]). If $n > 1$, then $P_7(n) \Leftrightarrow P_8(n) \Leftrightarrow P_9(n) \Rightarrow P_6(n^\alpha)$ for some positive integer α.

Proof. Obviously, $P_7(n)$ implies $P_8(n)$. If R has $P_8(n)$ then
$$[x,(x+y)^n - y^n] = [x^2\psi(x),(x+y)-y] = [x^2\psi(x),x] = 0.$$
Next, if R has $P_9(n)$ then
$$[x,y^n] - [x^n,y] = [x,(x+y)^n] - [(x+y)^n,y] = [x+y,(x+y)^n] = 0.$$
We have seen the equivalence of $P_7(n)$, $P_8(n)$ and $P_9(n)$.

In order to show that $P_7(n) \Rightarrow P_6(n^\alpha)$ for some positive integer α, we require the following

Sublemma. If R satisfies the polynomial identity $[[x,y],z] = 0$, then $P_7(n)$ implies $P_6(n^6)$.

Proof. First, we claim that R satisfies the polynomial identity $(x^{(n-1)^2} - 1)[x,y^{n^3}] = 0$. Indeed,
$$\begin{aligned}0 &= [x^{n^2},y^n] - [x^n,y^{n^2}] = nx^{n(n-1)}[x^n,y^n] - nx^{n-1}[x,y^{n^2}] \\ &= n(x^{(n-1)^2} - 1)x^{n-1}[x,y^{n^2}] = (x^{(n-1)^2} - 1)[x^n,y^{n^2}] \\ &= (x^{(n-1)^2} - 1)[x,y^{n^3}]\end{aligned}$$
Since every ring is a subdirect sum of subdirectly irreducible ring with heart S $(\neq 0)$. Now, let a be an arbitrary element in the right annihilator $r(S)$ of S in R. If $[a,r^{n^3}]$ is non-zero for some $r \in R$ then, by the claim at the opening, the left ideal $I = \{x \in R \mid xa^{(n-1)^2} = x\}$ contains the non-zero central element $[a,r^{n^3}]$, so

that $I \supseteq S$. But then $s = sa^{(n-1)^2} = 0$ for all $s \in S$. This is a contradiction. We have thus seen that $[a, y^{n^3}] = 0$ for all $y \in R$. Next, we prove that R satisfies the identity $[x^{n^3}, y^{n^3}] = 0$. If $[x, y^{n^3}] = 0$ for all $x, y \in R$, there is nothing to prove. Now, assume that $[b, d^{n^3}] \neq 0$ for some $b, d \in R$. Then, again by the opening claim, the left annihilator $\ell(b^{(n-1)^2+1} - b)$ contains the non-zero element $[b, d^{n^3}]$, and so contains S. Then, since $b^{(n-1)^2+1} - b$ is in $r(S)$, it follows from what was just shown above that $[b^{(n-1)^2+1} - b, d^{n^3}] = 0$. Thus, at any rate, R satisfies the identity $[x^{(n-1)^2+1} - x, y^{n^3}] = 0$, and so the subring generated by all n^3-th powers of elements of R is commutative by a theorem of Herstein. Consequently, R satisfies the identity $[x^{n^3}, y^{n^3}] = 0$. Now, by $P_7(n)$, it is immediate that $[x^{n^6}, y] = [x^{n^3}, y^{n^3}] = 0$.

We now back to the proof of $P_7(n) \Rightarrow P_6(n^{\alpha})$. By Remark 1, there exists a positive integer h such that $[x,y]^h = 0$ for all $x, y \in R$. Choose a positive integer such that $n^{\kappa} \geq h$. Let T be the subring of R generated by all n^{κ}-th powers of elements of R. Since $[[x, y], z^{n^{\kappa}}] = [[x,y]^{n^{\kappa}}, z] = 0$ for all $x, y, z \in R$, we get $[s^{n^6}, t] = 0$ for all $s, t \in T$ (Sublemma). It therefore follows that $[x^{n^{2\kappa+6}}, y] = [x^{n^{\kappa+6}}, y^{n^{\kappa}}] = 0$ for all $x, y \in R$.

In [4], it is shown that if a ring R having the property $P_7(n)$ $(n > 1)$ is generated by $\{a^n \mid a \in R\}$, then R is commutative. This can be improved as follows:

Corollary 3 ([14]). Let R be a ring having the property $P_7(n)$ $(n > 1)$, and T the subring of R generated by $\{a^n \mid a \in R\}$. If the centralizer of T in R coincides with C, then R is commutative.

Corollary 4 ([5, Theorem 5] and [14]). If R is an s-unital ring having the properties $P_7(n)$ and $Q(n)$ $(n > 1)$, then R is commutative.

Theorem 3 ([17]). Let i, j be integers in the set $\{k \mid 1 \le k \le 9\}$, and $m, n > 1$. Suppose an s-unital ring R has the properties $P_i(m)$ and $P_j(n)$. If $(m,n) = 1$, then R is commutative.

Proof. According to Propositions 1 and 2, R has the properties $P_6(m^\alpha)$ and $P_6(n^\alpha)$ for some positive integer α. Hence, by [18, Theorem], R is commutative.

A ring-property P is called a C(n)-property if every ring with identity having the properties P and Q(n) is commutative. Corollaries 1 and 4 shows that if $n > 1$ then the properties $P_2(n)$ — $P_9(n)$ are C(n)-properties. The previous example b) shows that $P_1(p^k - 1)$ is not a $C(p^k - 1)$-property for every prime p and $k \ge 2$.

Theorem 4 ([17]). Let i, j be integers in the set $\{k \mid 2 \le k \le 9\}$, and $m, n > 1$. Suppose an s-unital ring R has the Properties $P_i(m)$ and $P_j(n)$. If R has the property $Q((m,n))$, then R is commutative.

Proof. Let e be a pseudo-identity of $\{a, b\} \subseteq R$, and e' a pseudo-identity of $\{a, b, e\}$. Let $S = \langle a, b, e, e' \rangle$ be the subring of R generated by $\{a, b, e, e'\}$, and $A = \ell_S(e) (= r_S(e))$. Then $e' + A$ is the identity element of S/A. Since $\langle a, b \rangle \cap A = 0$, we may regard $\langle a, b \rangle$ as a subring of S/A. Obviously, S/A has the properties $P_i(m)$ and $P_j(n)$. Moreover, we can easily see that S/A has the property $Q((m,n))$. Now, the rest of the proof is immediate by the lemma below.

Lemma 1. Let P_i be a $C(n_i)$-property which is inherited by every finitely generated subring ($i = 1, \ldots, t$), and $d = (n_1, \ldots, n_t)$. Suppose a ring R with 1 has the properties $P_1, \ldots, P_t$. If R has the property $Q(d)$ then R is commutative.

Proof. It suffices to prove the case $t = 2$. We show that R has the property $Q(n_1)$ (and therefore R is commutative). Suppose $n_1[a,b] = 0$ for some $a, b \in R$, and let R' be the subring of R generated by $\{1, a, b\}$. Then, we can easily see $n_1[x,y] = 0$ for all $x, y \in R'$. Since R' has the property $Q(d)$, the above implies that R' has the property $Q(n_2)$. Hence, R' is commutative, namely $[a,b] = 0$.

In [9], it is shown that if power map $f : x \to x^n$ $(n > 1)$ in R is a surjective ring homomorphism then R is commutative. Obviously, if the power map f is a ring ring endomorphism then R has the property $P_9(n)$, and hence by Proposition 2, $f^\alpha(R) \subseteq C$ for some α. Using this fact, we can characterize the class of finite rings in which a power map is a ring endomorphism. We need the following notation. If A is an algebra over a field K, we denote by (K,A) the ring whose additive group is the direct sum of K and A with multiplication given by

$$(k,a)(k',a') = (kk',ka'+k'a+aa').$$

Lemma 2. Let R be a ring with minimum condition for subrings. If the map $f : R \to R$ defined by $x \to x^n$ $(n > 1)$ is a ring endomorphism, then

$$R = (K_1,N_1) \oplus \dots \oplus (K_m,N_m) \oplus N,$$

where N_i is a nilpotent algebra over a field K_i and N is a nilpotent ring.

Proof. Without loss of generality, we may assume that R is indecomposable. There exists a positive integer γ such that $f^{\gamma}(R) = f^{2\gamma}(R)$. Obviously, $N_0 = \operatorname{Ker} f^{\gamma}$ is a nilpotent ideal. As can be easily seen, $K_0 = f^{\gamma}(R)$ is a semisimple (Artinian) ring, and therefore f induces an automorphism of K_0 and the additive group of R is the direct sum of K_0 and N_0. We consider the case that neither of K_0 and N_0 is zero. Since K_0 is in the center of R by Proposition 2, K_0 is a direct sum of fields in the center of R. Recalling here that R is indecomposable, we can easily see that K_0 is a field and N_0 is an algebra over K_0. We thus obtain $R = (K_0,N_0)$.

Theorem 5 ([15]). The following statements are equivalent:

1) R is a finite ring and there exists an integer $n > 1$ such that the map f defined by $x \to x^n$ is a ring endomorphism of R.

2) $R = (K_1, N_1) \oplus \dots \oplus (K_m, N_m) \oplus N$ with $(\prod_{j \in J} \text{char } K_j, \prod_{i=1}^{m} (|K_i| - 1)) = 1$, where N_i is a finite nilpotent algebra over a finite field K_i, N a finite nilpotent ring, and $J = \{j \mid N_j \neq 0\}$.

Proof. 1) $\Rightarrow$ 2). According to Lemma 2, we have

$$R = (K_1, N_1) \oplus \dots \oplus (K_m, N_m) \oplus N,$$

where N_i is a finite nilpotent algebra over a finite field K_i and N is a finite nilpotent ring. Let $p_i =$ ckar K_i. As can be easily seen, there exists a positive integer κ such that $f^{\kappa}(x) = x$ if x is in K_i and $f^{\kappa}(x) = 0$ if x is in N_i or N. Then, $n - 1$ is a multiple of $|K_i| - 1$. Moreover, for each $j \in J$, (K_j, N_j) contains a unit of order p_j, and therefore n is a multiple of p_j. Let λ be the least common multiple of $|K_i| - 1$ $(i = 1, \dots, m)$, and μ the least common multiple of p_j $(j \in J)$. Then $n = \xi\lambda = \eta\mu + 1$ for some positive integers ξ, η, which implies that $(\prod_{j \in J} p_j, \prod_{i=1}^{m} (|K_i| - 1)) = 1$.

2) $\Rightarrow$ 1). Let ν be the nilpotency index of N, and $p_i = \text{char } K_i$. There exists a positive integer τ such that p_j^{τ} exceeds the nilpotency index of N_j $(j \in J)$. Then, by hypothesis, we can select positive integers ξ, η such that

$$\nu \leq n = \xi \prod_{j \in J} p_j^{\tau} = \eta \prod_{i=1}^{m} (|K_i| - 1) + 1.$$

Now, it is easy to see that the map f defined by $x \to x^n$ is a ring endomorphism of R, in fact, $f(N) = 0$ and $f((k,u)) = k$ for all $(k,u) \in (K_i, N_i)$ $(i = 1, \dots, m)$.

In what follows, we consider rings with some particular variable identities.

Lemma 3 ([11, Lemma 2.1.1, p. 57]). If R contains a non-zero nil right ideal I satisfying a polynomial identity, then R is not semiprime. In particular, if R contains a non-zero nil right ideal of bounded index, then R is not semiprime.

Proof. By [22, Theorem 1.6. 36, p. 48] or [2, Corollary to Theorem 2], the upper nil radical of I coincides with the lower one (prime radical) of I. If $I^2 = 0$, there is nothing to prove. If $I^2 \neq 0$, we set $W = \{a \in I \mid aI = 0\}$. Then there is an ideal A of I such that $\bar{A}$ is a non-zero nilpotent ideal of $\bar{I} = I/W$. Now, it is immediate that AI is a non-zero right ideal of R.

Corollary 5. If a prime ring R contains a non-zero right ideal I satisfying a polynomial identity then R contains no non-zero right ideal.

Proof. Suppose R contains a non-zero nil right ideal S, and choose non-zero $s \in S$. Then, for any non-zero $t \in Is$, we see that tR is a non-zero nil right ideal satisfying a polynomial identity. But this is imposible by Lemma 3.

In [7], it is shown that if R is a semiprime ring with 1 and if for each pair of elements x, y in R there exists a positive integer $n = n(x,y)$ such that $(xy)^k - x^k y^k \in C$ $(k = n,\ n+1,\ n+2)$, then R is commutative. The next improves this result as well as [20, Theorem] (see also Example (2)).

Theorem 6 ([16]). Let ℓ, m be fixed positive integers. If R is an s-unital semiprime ring, then the following statements are equivalent:

1) R is commutative.

2) For each $x, y \in R$ there exists a positive integer $n = n(x,y)$ such that $[x^k y^k - (xy)^k, x] = 0$ and $[x^k y^k - (xy)^k, y] = 0$ $(k = n, n+1, n+2)$.

3) For each $x, y \in R$ there exists a positive integer $n = n(x,y)$ such that $[y^k x^k - (xy)^k, y] = 0$ $(k = n, n+1, n+2)$.

4) For each $x, y \in R$ there exists a positive integer $n = n(x,y)$ such that $[(xy)^k, (yx)^m] = 0$ $(k = n, n+\ell)$.

Proof. We prove only that 4) implies 1). Without loss of generality, we may assume that R is a prime ring. Since $[(xy)^n, (yx)^m] = 0 = [(xy)^{n+\ell}, (yx)^m]$, we can easily see

(†) $\qquad (xy)^n[(xy)^\ell, (yx)^m] = 0.$

Now, let $a^2 = 0$. Then it is known that aR is a nil right ideal (see the proof of [19, Theorem 1]). Since aR is quasi-regular and R is s-unital, we see that aR satisfies the polynomial identity $[\{(1+x)(1+y)\}^\ell, \{(1+y)(1+x)\}^m] = 0$. Hence, by Lemma 3, it follows that $a = 0$, namely, R is a reduced ring. Since the reduced prime ring R contains no non-zero zero-divisors, (†) proves that R satisfies the polynomial identity $[(xy)^\ell, (yx)^m] = 0$. Hence, by Theorem 1 (or by Example (1)), R is commutative.

We conclude this paper with refering to two recent results due to I.N. Herstein [12] and A.A. Klein, I. Nada and H.E. Bell [22].

Theorem 7 ([12]). Let R be a ring in which, given $x, y, z \in R$ there exist positive integers $m = m(x,y,z)$, $n = n(x,y,z)$, and $q = q(x,y,z)$ such that $[[x^m, y^n], z^q] = 0$. Then the commutator ideal of R is nil. Equivalently, the nilpotent elements of R form an ideal N such that R/N is commutative.

Conjecture. Let k be a positive integer and suppose that for each $x, y \in R$, there exist positive integers m, n such that $[x^m, y^n]_k = [\ldots[x^m, \underbrace{y^n], y^n]\ldots y^n}_{k}] = 0$. Then the commutator ideal of R is nil.

For rings with 1, the answer is yes.

Theorem 8 ([22]). Let R be a ring with 1, and k a positive integer. If for each pair of elements x, y in R there exist positive integers $m = m(x,y)$ and $n = n(x,y)$ such that $[x^m, y^n]_k = 0$, then D(R) is a nil ideal.

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Department of Mathematics,
Faculty of Science,
Okayama University

A GENERALIZATION OF KRULL ORDERS

Hisaaki FUJITA

In [10-13], H. Marubayashi has introduced a notion of noncommutative Krull rings (called Krull orders in [13]), which is a class of maximal orders in simple Artinian rings, and many interesting results have been obtained (see [5,14] too). Krull orders are a Krull domain type generalization of Dedekind prime rings. A Dedekind prime ring is an HNP (Hereditary Noetherian Prime) ring such that it is a maximal order in its maximal quotient ring. In noncommutative ring theory it is also important to consider HNP rings which are not maximal orders. In this paper we present a class of prime Goldie rings which includes both Krull orders and HNP rings with enough invertible ideals, and we study its ideal theory and give some inheritance properties and examples.

Throughout this paper, all rings are associative with identity. Conditions on rings or ideals are assumed to hold on right and left sides unless otherwise stated. R is a prime Goldie ring which is not Artinian and Q is a maximal quotient ring of R. M_R (resp. $_RM$) signifies that M is regarded as a right (resp. left) R-module.

If X and Y are submodules of Q, then we put $(X:Y)_\ell = \{q \in Q ; qY \subset X\}$, $(X:Y)_r = \{q \in Q ; Yq \subset X\}$, $X^{-1} = \{q \in Q ; XqX \subset X\}$, $O_\ell(X) = \{q \in Q ; qX \subset X\}$ and $O_r(X) = \{q \in Q ; Xq \subset X\}$. A right (resp. left) R-submodule I of Q is a right (resp. left) (fractional) R-ideal if $aR \subset I \subset bR$ (resp. $Ra \subset I \subset Rb$) for some units a,b in Q. A right (or left) R-ideal is integral if it is contained in R.

1. Reflexive R-ideals. For a prime Goldie ring R, let $F(R)$, $F^*(R)$ and $D(R)$ denote the set of R-ideals, the set of reflexive R-ideals and the set of reflexive R-ideals A such that $O_\ell(A) = R = O_r(A)$, respectively. For a right (resp. left) R-ideal I, we put $I_v = (R:(R:I)_\ell)_r$ (resp. ${}_vI = (R:(R:I)_r)_\ell$). For a right ideal I, since $I_v \cong \mathrm{Hom}_R(\mathrm{Hom}_R(I,R),R)$ canonically, I_R is reflexive if and only if $I = I_v$. An R-ideal A is invertible if $A(R:A)_r = R = (R:A)_\ell A$. First, we generalize invertible ideals.

Lemma 1.1. Let $A \in F(R)$, and let ${}_RA$ be reflexive. Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) where

(1) $O_\ell(A) = R$,

(2) $(A(R:A)_r)_v = R$, and

(3) $A^{-1} = (R:A)_r$.

Moreover, if ${}_v(A(R:A)_\ell) = O_\ell(A)$, then (3) implies (1).

Proposition 1.2. Let $A, B \in F(R)$. Then

(1) $(AB)_v = (AB_v)_v$.

(2) If $B \in D(R)$, then $(AB)_v = (AB_v)_v = ((A_v)B)_v$.

(3) If $A \in D(R)$, then $A^{-1} \in D(R)$.

(4) If A_R is reflexive and if $B \in D(R)$, then $O_\ell(A) = O_\ell(AB) = O_r((R:AB)_\ell) = O_\ell((AB)_v)$.

(5) $D(R)$ forms a group under the multiplication "$\circ$" defined by $A \circ B = (AB)_v$.

We call an element of $D(R)$ a quasi-invertible R-ideal. R has enough quasi-invertible ideals provided that, for any nonzero ideal A of R, there exists $B \in D(R)$ such that $A_v \supset B$ and ${}_vA \supset B$.

Next, we generalize idempotent ideals.

Lemma 1.3. Let $A \in F^*(R)$ and integral. Then (1) $\Rightarrow$(2) $\Leftrightarrow$(3) where

(1) $(A^2)_v = A$,

(2) $O_\ell(A) = (R:A)_\ell$, and

(3) $(R:A)_\ell A = A$.

Moreover, if ${}_v(A(R:A)_\ell) = O_\ell(A)$, then (2) implies (1)* ${}_v(A^2) = A$.

Proposition 1.4. Suppose that ${}_v(A(R:A)_\ell) = O_\ell(A)$ and $((R:A)_rA)_v = O_r(A)$ hold for $A \in F^*(R)$ with $A \subset R$. Then the following conditions are equivalent.

(1) $(A^2)_v = A$.

(1)* ${}_v(A^2) = A$.

(2) $O_\ell(A) = (R:A)_\ell$.

(2)* $O_r(A) = (R:A)_r$.

(3) $(R:A)_\ell A = A$.

(3)* $A(R:A)_r = A$.

A reflexive ideal A of R is said to be quasi-idempotent if $(A^2)_v = A = {}_v(A^2)$.

2. Ideal theory of generalized Krull orders. A subring of Q which contains R is called an overring of R. Following Marubayashi[11], an overring R' of R is right (resp. left) essential over R if the following two conditions hold:

(1) There is a perfect right (resp. left) additive topology F (resp. G) on R such that $R' = R_F$ (resp. $R' = {}_GR$).

(2) If $I \in F$ (resp. $J \in G$), then $R'I = R'$ (resp. $JR' = R'$).

(1) is equivalent to the condition that the inclusion mapping $R \hookrightarrow R'$ is a right (resp. left) flat epimorphism. A subset C of R is a regular Ore set if each element of C is regular and if R satisfies the right and left Ore condition with respect to C. If C is a regular Ore set of R, $R_C = \{rc^{-1} \in Q \; ; \; r \in R$ and $c \in C\}$ is an essential overring of R. We set $S(R) = \{q \in Q \; ; \; qA \subset R$ and $A'q \subset R$ for some nonzero ideals A, A' of $R\}$, and we call it the Asano overring of R. $S(R) = Q$ if and only if R is bounded (i.e., any essential one-sided ideal contains a nonzero ideal). Concerning the basic properties of HNP rings we refer to [3].

Definition. A prime Goldie ring R is a generalized Krull order if there is a family $\{R_i\}_{i \in I}$ of overrings of R which satisfies the following conditions:

(GK1) $R = \bigcap_{i \in I} R_i \cap S(R)$.

(GK3) For each $i \in I$, R_i is a semilocal HNP, essential overring of R whose Jacobson radical is a maximal invertibleideal; and $S(R)$ is a Noetherian simple, essential overring of R.

(GK3) Each regular element of R is a unit of R_i for almost all $i \in I$.

Remark. We may remove the condition that the Jacobson radical of R_i is a maximal invertible ideal.

Given a generalized Krull order $R = \bigcap_{i \in I} R_i \cap S(R)$, we shall use the following notations:

For each $i \in I$, $A_i' = \mathrm{rad}(R_i)$ and $A_i = A_i' \cap R$, $\{M_{i1}', \ldots, M_{in(i)}'\}$ is the cycle of (idempotent) maximal ideals of R_i and $M_{ij} = M_{ij}' \cap R$ $(1 \le j \le n(i))$.

For an ideal I of R, we set $C(I) = \{c \in R \,;\, c+I$ is regular in $R/I\}$. If $C(I)$ is a regular Ore set of R, $R_{C(A)}$ is denoted by R_A.

First of all we investigate the relation between a generalized Krull order and its overrings.

Proposition 2.1. (1) Let R' be a semilocal HNP, essential overring of R, and put $A' = \mathrm{rad}(R')$ and $A = A' \cap R$.

(a) A is a semiprime ideal.

(b) $R/A \cong R+A'/A'$ is an order in a semisimple ring R'/A'.

(c) $C(A)$ is a regular Ore set of R and $R' = R_A$.

(2) If S is a simple essential overring of R, $AS = S = SA$ for any nonzero ideal A of R. Conversely if $S = S(R)$ and if $AS = S = SA$ for any nonzero ideal A of R, S is a simple essential overring of R.

Lemma 2.2. Let C be a regular Ore set of R, and suppose that R satisfies the ascending chain condition for integral reflexive one-sided R-ideals. Then

(1) For any nonzero reflexive ideal A of R, $AR_C = R_CA$. Furthermore, suppose that R_C is hereditary. Then for any nonzero ideal A of R, $AR_C = R_CA$.

(2) If $A \in D(R)$ and integral, then $AR_C \in D(R)$.

Let $R = \bigcap_{i \in I} R_i \cap S(R)$ be a generalized Krull order, and let I be a reflexive right R-ideal. Then as in [10,

§4], we have $I = \bigcap_{i\in I} IR_i \cap IS(R)$. Thus by (GK3), we conclude that R satisfies the ascending chain condition for integral reflexive one-sided R-ideals. If R is a maximal order, for any nonzero ideal A of R, A_R is reflexive if and only if $_RA$ is reflexive. Form this fact, the ideal theory of Krull orders is studied by using reflexive ideals. The following proposition enables us to make use of reflexive ideals to study our ideal theory of generalized Krull orders, though they are not necessarily maximal orders.

Proposition 2.3. Let R be a generalized Krull order.

(1) For any nonzero ideal A of R, A_R is reflexive if and only if $_RA$ is reflexive.

(2) For any nonzero ideals A, B, C of R, $((AB)_v \cdot C)_v = (ABC)_v = (A(BC)_v)_v$.

(3) For any nonzero reflexive ideal A of R, $_v(A(R:A)_\ell) = O_\ell(A)$ and $((R:A)_rA)_v = O_r(A)$.

Proposition 2.4. Let $R = \bigcap_{i\in I} R_i \cap S(R)$ be a generalized Krull order. Then

(1) If R is a maximal order, R is a Krull order.

(2) $M_{ij}\in F^*(R)$ and $A_i\in D(R)$ for any $i\in I$, $j = 1,\dots,n(i)$.

(3) R has enough quasi-invertible ideals.

(4) $S(R) = \cup\{A^{-1}$; A is a quasi-invertible ideal$\}$.

Lemma 2.5. Let R' be an overring of R such that the inclusion mapping $R \hookrightarrow R'$ is a right flat epimorphism and let A' be an ideal of R' such that $A = A' \cap R \in D(R)$. Then if I is a right ideal of R with $I \supset A$ and $IR' = R'$, then $I_v = R$.

Next, we study the reflexive prime ideals.

Proposition 2.6. Let $R = \bigcap_{i \in I} R_i \cap S(R)$ be a generalized Krull order. Then

(1) $\{M_{ij} ;\ i \in I,\ 1 \le j \le n(i)\}$ is the set of reflexive prime ideals of R.

(2) M is a reflexive prime ideal if and only if M is a maximal reflexive ideal.

(3) Each reflexive prime ideal is either quasi-invertible or quasi-idempotent.

For each $i \in I$, $M_{i1}, \ldots, M_{in(i)}$ are linked by the following relations.

Lemma 2.7. Let $R = \bigcap_{i \in I} R_i \cap S(R)$ be a generalized Krull order. Then

$M_{i2} = (A_i M_{i1} A_i^{-1})_v, \ldots, M_{i1} = (A_i M_{in(i)} A_i^{-1})_v$.

Proposition 2.8. Let $R = \bigcap_{i \in I} R_i \cap S(R)$ be a generalized Krull order. Then for any $i_1, \ldots, i_k \in I$, $X = A_{i_1} \cap \ldots \cap A_{i_k} \in D(R)$.

Lemma 2.9. Let A be a quasi-invertible ideal of R and let $M_1, \ldots, M_n$ be reflexive prime ideals of R such that $A \not\subset M_i$ for any $i = 1, \ldots, n$. Put $I = M_1 \cap \ldots \cap M_n$. Then ${}_v(AI) = A \cap I = (IA)_v$.

Proposition 2.10. Let $R = \bigcap_{i \in I} R_i \cap S(R)$ be a generalized Krull order. If $i, k \in I$ and $i \ne k$, $\{M_{i1}, \ldots, M_{in(i)}\}$ and $\{M_{k1}, \ldots, M_{kn(k)}\}$ are disjoint.

We are now in a position to prove the following theorem that is the first aim of this section.

Theorem 2.11. Let $R = \bigcap_{i \in I} R_i \cap S(R)$ be a generalized Krull order. Then $D(R)$ forms a free abelian group with generators $\{A_i\}_{i \in I}$.

Proof. Omitted.

For the second theorem we prepare the following lemma.

Lemma 2.12. Let R be a generalized Krull order, and let $M_1, \ldots, M_n$ be reflexive prime ideals of R such that $I = M_1 \cap \ldots \cap M_n$ is not contained in any quasi-invertible ideals. Then $(I^n)_v$ is quasi-idempotent.

A reflexive ideal I is said to be eventually quasi-idempotent if $(I^n)_v = {}_v(I^n)$ is quasi-idempotent for some $n \geq 1$.

Theorem 2.13. Let R be a generalized Krull order. Then for each nonzero ideal A of R, there are a quasi-invertible ideal X and an eventually quasi-idempotent ideal I such that $A_v = X \circ I$, and such X and I are uniquely determined by A_v.

Remark. The uniqueness of the theorem holds for any nonzero ideal of any HNP ring.

3. Inheritance properties and examples. In this section we shall give some examples and inheritance porperties.

R is a Krull order if and only if R is a maximal, generalized Krull order by Proposition 2.4 and [13, Proposition 2.1].

Proposition 3.1. R is an HNP ring with enough invertible ideals if and only if R is a hereditary generalized Krull order.

Let A be a Krull domain with field of quotients K, let Σ be a finite dimensional central simple K-algebra, and let Λ be an A-order in Σ (i.e., $\Lambda \supset A$, $\Lambda K = \Sigma$ and each element of Λ is integral over A). Then Λ is a tame A-order if Λ_A is reflexive and Λ_p is a hereditary A_p-order for each minimal prime ideal p of A (see [4]).

Proposition 3.2. Any tame A-order Λ over a Krull domain A is a bounded generalized Krull order.

Proposition 3.3. If R is a generalized Krull order, then so is eR_ne where $e = e^2 \in R_n$, the full n×n matrix ring over R.

For the latter use, we note the following lemma.

Lemma 3.4. Let R be a semilocal Noetherian prime ring. Then R is hereditary if and only if $\mathrm{rad}(R)$ is invertible.

Let $R[x]$ denote the ring of polynomials over R in an indeterminante x with $rx = xr$ for all $r \in R$.

Lemma 3.5. Let R be a semilocal HNP ring whose Jacobson radical A is a maximal invertible ideal. Then

(1) $C(A[x])$ is a regular Ore set of $R[x]$.

(2) $R[x]_{A[x]}$ is a semilocal HNP ring with the maximal invertible Jacobson radical $A[x]R[x]_{A[x]}$.

(3) $R[x]_{A[x]} \cap Q[x] = R[x]$.

Theorem 3.6. If R is a generalized Krull order, so is $R[x]$. Moreover $D(R[x]) \simeq D(R) \oplus D(S(R)[x])$.

The proof of the next theorem will give a short proof to [14, Theorem 1.10].

Theorem 3.7. Let $R = \bigcap_{i \in I} R_i \cap S(R)$ be a generalized Krull order, and let I' be a subset of I. Then $T = \bigcap_{i \in I'} R_i \cap S(R)$ is also a generalized Krull order.

Proof. Omitted.

Theorem 3.8. Let $R = \bigcap_{i \in I} R_i \cap S(R)$ be a generalized Krull order, and let C be a regular Ore set of R. Then $T = R_C$ is also a generalized Krull order. In this case, $T = \bigcap_{i \in I'} R_i \cap S(T)$ for some subset I' of I and $S(T) = TS(R) = S(R)T$.

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Institute of Mathematics
University of Tsukuba

A GENERALIZATION OF SEMIPERFECT MODULES

Shigeki NAKAHARA

The main purpose of this note is to generalize the notion of semiperfect modules in terms of preradicals. In particular the results allows several new characterizations of semiperfect rings and modules (Corollaries 3 and 5 below).

Throughout this note, unless otherwise specified, $\underline{R}$ will always denote a ring with identity and all modules will be understood to be unital right R-modules.

Let M be a module. We denote its Jacobson radical by $\underline{J(M)}$. Given a submodule N of M, we say N is <u>small</u> in M if $T + N = M$ implies $T = M$ for any submodule T of M. Other definitions correspond to the ones found in Bass [1], Mares [2] and Stenström [5].

1. Preliminaries

In this section we shall introduce the notions of "semicover" and "normality" for preradicals, and, by using them, generalize semiperfect modules of Mares, as follows:

Let M be a module and let ρ be any preradical on modules (see [5] for details concerning preradicals). Then an epimorphism $P \xrightarrow{\pi} M \longrightarrow 0$ is called a <u>ρ-semicover</u> of M if P is a projective module and $\mathrm{Ker}\ \pi \subset \rho(P)$. A projective module is said to be <u>ρ-semiperfect</u> (resp. <u>ρ-perfect</u>) if every factor module of it (resp. of every direct sum of its copies) has a ρ-semicover.

Of course, semiperfect (resp. perfect) modules are

J-semiperfect (resp. J-perfect), where "J" denotes the Jacobson radical. Furthermore, the converse to this is also true, as we shall see later (Corollary 3 below).

The following definition will play a central role throughout this note: A preradical ρ is said to be normal if whenever P is a nonzero projective module then $\rho(P) \neq P$.

For example, the Jacobson radical "J" is normal by [1, Proposition 2.7]. The singular preradical "Z" over any ring (see [5] for definition) is also normal by [6, Example 1]. In particular, the latter example tells us that normality does not imply smallness in general (see also Example 1 below). For further information concerning normality, the reader is referred to [6, §2].

Lemma 1. Let ρ be a normal preradical on modules. Then:

(1) Let M be a module and $P \xrightarrow{\pi} M \longrightarrow 0$ a ρ-semicover. Suppose that M has also a projective cover $Q \xrightarrow{\xi} M \longrightarrow 0$. Then there exists an isomorphism $f: P \longrightarrow Q$ such that $\pi = \xi f$.

(2) If a module M has a ρ-semicover and if $M\rho(R) = M$, then $M = 0$.

Proof. See [6, Proof of Lemma 1.2].

2. The main results

For the structure of ρ-semiperfect modules, we have the following:

Theorem 2. Let ρ be any normal preradical and P any projective module. Then P is ρ-semiperfect (resp. ρ-perfect) if and only if P is semiperfect (resp. perfect) and $\rho(P) = J(P)$.

Proof. See [6, proof of Theorem 1.3].

Corollary 3. A projective module is semiperfect (resp. perfect) if and only if it is J-semiperfect (resp. J-perfect).

If we restrict ourselves to the ring R, we can get further conditions for R to be ρ-semiperfect:

Theorem 4. Let ρ be a normal preradical. Then the following conditions are equivalent:

(1) R is a ρ-semiperfect ring (that is, the right module R_R is ρ-semiperfect).

(2) Every simple module has a ρ-semicover.

(3) Every semisimple module has a ρ-semicover.

(4) Every right $R/\rho(R)$-module has a ρ-semicover as an R-module and $\rho(R)$ is small in R as a right ideal.

Proof. See [6, proof of Theorem 1.7].

Combining this theorem and Corollary 3 one obtains:

Corollary 5. The following conditions are equivalent for any ring R:

(1) R is a semiperfect ring.

(2) Every simple module has a J-semicover.

(3) Every semisimple module has a J-semicover.

(4) Every right $R/J(R)$-module has a J-semicover as an R-module.

Remark. Condition (2) of this corollary is a generalization of a result proved by Sandomierski [3, Theorem 4] and Mueller [4, p.465].

Example 1. The smallness assumption of $\rho(R)$ in Theorem 4(4) cannot be dropped in general even when ρ is a normal "radical" (recall that a preradical σ is called a radical if $\sigma(M/\sigma(M)) = 0$ for any module M): Let $R = \mathbb{Z}$ (the ring of integers) and p any prime number. Set $\rho(M) = Mp$ for every module M. Then ρ is a normal radical and every right $R/\rho(R)$-module has a ρ-semicover as an R-module, but $\rho(R)$ is not small in R.

Example 2. There exists a module which has a J-semicover, but has no projective covers: Let R be any ring such that J(R) is not right T-nilpotent, and let $M = F/J(F)$, where F is a free module with countably infinite rank. Then M has a J-semicover F, but has no projective covers, as is easily seen by Lemma 1(1) above and the standard argument as in the proof of [3, Theorem 5].

3. Appendices

This section consists of three different topics related to the above, namely, the liftability of idempotents; a characterization of the Jacobson radical over a perfect ring; and torsion-free rings for normal preradicals. We shall consider these topics in turn:

((I)) Firstly, we deal with the problem of the possibility of lifting idempotents modulo the Jacobson radical of a ring, as an application of semicovers (we utilized a more general version of the following proposition in the proofs of [6, Theorems 1.3 and 1.7]).

Proposition 6. Let R be any ring and $\bar{R} = R/J(R)$. Then idempotents can be lifted modulo $J(R)$ if and only if any direct summand of the R-module $\bar{R}_R$ has a J-semicover.

Proof. "NECESSITY": Let I be any direct summand of $\bar{R}_R$. Then we may write $I = \varepsilon R$ for some idempotent ε in $\bar{R}$ and, by assumption, ε is induced by an idempotent e in R (i.e. $\varepsilon = e + J(R)$). Clearly it follows that the natural epimorphism $eR \longrightarrow I \longrightarrow 0$ is a J-semicover.

"SUFFICIENCY": To show this, it is enough to show that any direct decomposition $\bar{R} = \bigoplus_\alpha I_\alpha$ can be lifted to R. In fact, let $P_\alpha \xrightarrow{\pi_\alpha} I_\alpha \rightarrow 0$ be a J-semicover for each α (it exists by hypothesis). Then $\bigoplus_\alpha P_\alpha \xrightarrow{\oplus_\alpha \pi_\alpha} \bar{R} \longrightarrow 0$ is also a J-semicover. Since R is a projective cover of $\bar{R}$, it follows immediately from Lemma 1(1) that $\bigoplus_\alpha P_\alpha$ yields a desired direct decomposition of R.

((II)) The second topic is as follows: If R is a ρ-semiperfect ring for a normal preradical ρ, then we have $\rho(R) = J(R)$ by Theorem 2 and hence $\rho(P) = J(P)$ for every projective module P, since $\rho(P) = P\rho(R)$. But Example 3 below shows that in general $\rho \neq J$ even if R is a ρ-perfect ring. This discussion motivates the study of the relation between the Jacobson radical and (normal) preradicals. The following proposition gives a condition under which the above equality holds, by characterizing the Jacobson radical among radicals over a perfect ring.

Proposition 7. Let R be a right perfect ring and ρ any preradical. Then $\rho = J$ if and only if ρ is a radical such that $\rho(R) = J(R)$.

Proof. Clearly we need only to prove the "if" part. Assume that ρ is a radical and $\rho(R) = J(R)$. Let M be any module and $P \xrightarrow{\pi} M \longrightarrow 0$ a projective cover of M. Then since $\rho(P) = J(P)$, P becomes a ρ-semicover of M, so we have $\rho(M) = \pi(\rho(P)) = \pi(J(P)) = J(M)$ by [5, Lemma 1.2], as was to be shown.

In this proposition, the requirement that ρ is a radical is needed, as the next example shows:

Example 3. Let R be a quasi-Frobenius ring which is not semisimple Artinian, and let ρ = the singular preradical. Then we have $\rho(R) = J(R)$ by the right self-injectivity of R (so $\rho(P) = J(P)$ for every projective module P). But it is readily seen that $\rho \neq J$. (Note that this ρ is left exact and normal.)

((III)) Finally, we shall give a couple of results for rings R satisfying the following property:

(*) ... "$\rho(R) = 0$ for any normal preradical ρ."

For example, von Neumann regular rings have this property (see the second remark after Proposition 9). Note that any ring with this property is in particular right nonsingular.

Now, it is well known that R is a simple ring (i.e. there exist no nontrivial two-sided ideals) if and only if any nonzero module is faithful. In this connection we get:

Proposition 8. The following assertions are equivalent for any ring R:

(1) R satisfies the above property (*).

(2) $\rho(P) = 0$ for any projective module P and any normal preradical ρ.

(3) Any module M such that $\mathrm{Hom}_R(P,M) \neq 0$ for every nonzero projective module P is faithful.

Proof. See [6, Corollary 2.5].

Remark. As is easily seen [7, Proposition 1.2], the condition that $\mathrm{Hom}_R(P,M) \neq 0$ for a projecive module $P \neq 0$ is equivalent to that $M\tau(P) \neq 0$, where $\tau(P)$ is the trace ideal of P, i.e., the image of the natural pairing $\mathrm{Hom}_R(P,R) \otimes P \longrightarrow R$. In particular, this shows that the converse to (3) above is always true: If M is any faithful module over any ring R, then $\mathrm{Hom}_R(P,M) \neq 0$ for every projective module $P \neq 0$. (More generally, it can be easily verified that M is faithful if and only if every projective module is isomorphically embedded in a direct product of copies of M. As a special case of this, we observe that an injective module is faithful if and only if the torsion-free class cogenerated by it contains all projective modules.)

Proposition 9. If R is a commutative ring, then the following assertions are equivalent:

(1) R is a semisimple Artinian ring.

(2) R is a Noetherian ring satisfying (*).

Proof. (1) implies (2) is obvious. Conversely, assume (2) holds. Let $I \neq 0$ be an arbitrary ideal and define a preradical ρ_I such that $\rho_I(M) = MI$ for any module M. Then since $\rho_I(R) = I$, ρ_I is not normal by assumption. Hence we have $\rho_I(P) = P$ for some projective module $P \neq 0$. It then follows immediately that $\rho_I(R) \supset \tau(P)$. But, since $\tau(P)$ is a nonzero idempotent ideal and further is finitely generated by assumption, we see easily that $\tau(P)$ is generated by an idempotent element $e_1 \neq 0$; so that we have a direct decomposition $I = e_1R \oplus I_1$ for some ideal I_1. If $I_1 \neq 0$, then we have a similar decomposition $I_1 = e_2R \oplus I_2$ by the same argument as above. Repeating this procedure, we get a strictly ascending chain

$$e_1R \subsetneqq e_1R \oplus e_2R \subsetneqq \cdots\cdots .$$

Since this chain stops by assumption and the e_i are orthogonal, it follows clearly that I is generated by an idempotent element. Hence any ideal is a direct summand of R; therefore R is semisimple Artinian, as was to be shown.

Remarks. 1. Since the above ρ_I is in fact an (epi-preserving) radical, Assertion (1) is also equivalent to:

(2') R is a Noetherian ring such that $\rho(R) = 0$ for any normal "radical" ρ.

2. In the above proof of (2) $\Rightarrow$ (1), we have shown substantially that if R is a commutative Noetherian ring, then (*) implies regularity. However, this implication does not hold in general unless the Noetherian condition is assumed. In fact, let R be the ring of sequences $(\alpha_1,\alpha_2,\cdots,\alpha_n,\beta,\beta,\cdots)$ where $n > 0$, the α_i are in the rational function field $K(X)$ over a field K and β is in the polynomial ring $K[X]$. Then one checks easily that R satisfies (*) but is not regular (cf. Jacobson [8, p.211]).

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Department of Mathematics
Faculty of Science
Osaka City University

CONTINUOUS MODULES AND SEMIPERFECT MODULES

Kiyoichi OSHIRO

Recent Harada's work [3]$\sim$[7] motivated the author's study of the extending and lifting property on modules. In this note we shall announce some results studied in [11], [12].

Throughout this paper R is an associative ring with identity and all modules considered are unitary right R-modules. For a module M and its submodules N_1 and N_2 with $N_1 \subseteq N_2$ we use the following symboles:

$\mathcal{L}(M)$: The set of all submodules of M.

J(M): The Jacobson radical of M.

$N_1 \subseteq_e N_2$: N_1 is essential in N_2.

$N_1 \subseteq_c N_2$: N_1 is co-essential in N_2, i.e., N_2/N_1 is small in M/N_1.

Definition. A module M is extending module (resp. lifting module) provided for any A in $\mathcal{L}(M)$ there exists a direct summand $A^* \mathrel{\langle\!\oplus} M$ such that $A \subseteq_e A^*$ (resp. $A^* \subseteq_c A$).

Definition ([8], [11]). A module M is continuous if M is an extending module and satisfies the condition: for any direct summand A of M, every monomorphic image of A to M is a direct summand. M is quasi-continuous if M is an extending module and satisfies the condition: for direct summands A_1, A_2 of M, the condition $M \supseteq_e A_1 \oplus A_2$ implies $M = A_1 \oplus A_2$.

Definition ([12]). M is semiperfect if M is a lifting module and satisfies the condition: for any direct summand

A of M, every sequence $M \to A \to 0$ splits. M is quasi-semiperfect if M is a lifting module and satisfies the condition: for direct summands A_1, A_2 of M, if $M = A_1 + A_2$ and $A_1 \cap A_2$ is small in M (then) $A_1 \cap A_2 = 0$.

Remark. 1) (quasi-) semiperfect modules and (quasi-) continuous modules are mutually dual notions and we know that quasi-injective module $\Rightarrow$ continuous module $\Rightarrow$ quasi-continuous module; semiperfect module $\Rightarrow$ quasi-semiperfect module. In the case every homomorphic image of M has a projective cover, quasi-projective module $\Rightarrow$ semiperfect module.

2) A projective module P is semiperfect if and only if it is semiperfect in the sense of Mares [9], i.e., every homomorphic image of P has a projective cover (cf. [1]).

Theorem 1 ([8], [11]). The following conditions are equivalent for a given module M.

1) M is quasi-injective.

2) $M \oplus M$ is continuous.

3) $M \oplus M$ is quasi-continuous.

Theorem 1^*([12]). Let M be a module such that every homomorphic image of $M \oplus M$ has a projective cover. Then the following are equivalent.

1) M is quasi-projective.

2) $M \oplus M$ is semiperfect.

3) $M \oplus M$ is quadi-semiperfect.

Natation. Let M be a module and $\mathcal{N} = \{N_\alpha\}_I$ a subfamily of $\mathcal{L}(M)$. By $M(\mathcal{N})$ we denote the set of all x in M such that x lies in allmost all N_α but finite. Then $M(\mathcal{N})$ is a submodule of M, and the map $\eta_n: M(n) \to \sum_I \oplus (M/N_\alpha)$ given by $x \to \sum_I (x + N_\alpha)$ is well defined. In general η_n is not epimorphic. As is easily seen it is epimorphic if and only if for any finite subset F of I

$$M = (\bigcap_F N_\alpha) + (\bigcap_{I-F} N_\beta).$$

Definition. Let $n = \{N_\alpha\}_I$ be a subfamily of $\mathcal{L}(M)$. We say n is co-independent if $N_\alpha \neq M$ for all $\alpha \in I$, $M = M(n)$ and η_n is epimorphic.

Proposition 1 ([12]). Let $n = \{N_\alpha\}_I$ be a co-independent subfamily of $\mathcal{L}(M)$ and put $T_\alpha = \bigcap_{I-\{\alpha\}} N_\beta$ for all $\alpha \in I$ and $X = \bigcap_I N_\alpha$. Then

$$M/X = \sum_I \oplus (T_\alpha/X),$$

$$N_\alpha = \sum_{I-\{\alpha\}} T_\beta \text{ for all } \alpha \in I.$$

Proposition 2([12]). Let X be a submodule of M and $M/X = \sum_I \oplus (T_\alpha/X)$ a decomposition with $T_\alpha \supsetneqq X$ for each $\alpha \in I$. Put $N_\alpha = \sum_{I-\{\alpha\}} T_\beta$ for all $\alpha \in I$. Then

$\{N_\alpha\}_I$ is co-independent,

$X = \bigcap_I N_\alpha$,

$T_\alpha = \bigcap_{I-\{\alpha\}} N_\beta$ for all $\alpha \in I$.

By the propositions above we see that there exists a one to one map between the family of all co-independent subfamilies of $\mathcal{L}(M)$ and the family of all decompositions of all homomorphic images of M.

Definition ([12]). Let X be a submodule of M and $M/X = \sum_I \oplus (T_\alpha/X)$ a decomposition of M/X. We say $M/X = \sum_I \oplus (T_\alpha/X)$ is co-essentially lifted to a decomposition of M if there exists a decomposition $M = X^* \oplus \sum_I \oplus T_\alpha$ such that $X^* \subseteq X$, $T_\alpha = X + T_\alpha$ and $0 \subseteq_c (T_\alpha \cap X)$ in T_α for all $\alpha \in I$.

Definition ([7]). Let $n = \{N_\alpha\}_I$ be an independent subfamily of $\mathcal{L}(M)$. We say $\sum_I \oplus N_\alpha$ is essentially extended to a decomposition of M if there exists a decomposition $M = X \oplus \sum_I \oplus N_\alpha^*$ with $N_\alpha \subseteq_e N_\alpha^*$ for all $\alpha \in I$.

The above two concepts are mutually dual by the following shows:

Proposition 3 ([12]). Let $n = \{N_\alpha\}_I$ be a co-independent subfamily of $\mathcal{L}(M)$, and let $M/X = \sum_I \oplus (T_\alpha/X)$ be its corresponding decomposition. Then $M/X = \sum_I \oplus (T_\alpha/X)$ is co-essentially lifted to a decomposition of M if and only if there exists a (co-independent) subfamily $\{N_\alpha^*\}_I$ such that $N_\alpha^* \subseteq_c N_\alpha$ for all $\alpha \in I$ and $\bigcap_I N_\alpha^* \leq_\oplus M$.

Definition ([7], [12]). A module M has the extending property of direct sum if for any independent subfamily $\mathcal{N} = \{N_\alpha\}_I$ $\sum_I \oplus N_\alpha$ is essentially extended to a decomposition of M. Dually M has the lifting property of direct sum if for any co-independent family $\mathcal{N} = \{N_\alpha\}_I$, its corresponding decomposition is co-essentially lifted to a decomposition of M.

Theorem 2 ([11]). The following conditions are equivalent for a given module M.

1) M has the extending property of direct sum.

2) M is quasi-continuous and every internal direct sum of submodules of M which is a locally direct summand of M ([2]) is a direct summand of M.

3) M is written as $M = \sum_I \oplus M_\alpha$ with the following conditions:

i) Each M_α is uniform.

ii) $\{M_\alpha\}_I$ is locally semi-T-nilpotent ([2]).

iii) For any partition $I = I_1 \cup I_2$ and any submodule A of $\sum_{I_1} \oplus M_\alpha$, every homomorphism from A to $\sum_{I_2} \oplus M_\alpha$ is extended to one from $\sum_{I_1} \oplus M_\alpha$ to $\sum_{I_2} \oplus M_\beta$.

Theorem 2^*([12]). The following conditions are equivalent for a given module M:

1) M has the lifting property of direct sum.

2) M is quasi-semiperfect.

3) M is written as $M = \sum_I \oplus M_\alpha$ with the following conditions:

i) Each M_α is hollow.

ii) For any submodule N of M there exists a subset $J \subseteq I$ such that $M = N + \sum_J \oplus M_\alpha$ and $N \cap \sum_J \oplus M_\alpha$ is small in M.

iii) For any partition $I = I_1 \cup I_2$ and any (small) submodule A of $\sum_{I_2} \oplus M_\alpha$, every homomorphism from $\sum_{I_1} \oplus M_\alpha$ to $(\sum_{I_2} \oplus M_\beta)/A$ is induced from $\sum_{I_1} \oplus M_\alpha$ to $\sum_{I_2} \oplus M_\beta$.

Theorem 3 ([11],[12]). Let M be a module with an indecomposable decomposition $M = \sum_I \oplus M_\alpha$. Then M is continuous (resp. semiperfect) if and only if it is quasi-continuous (resp. quasi-semiperfect) and for any $\alpha \in I$ every monomorphism (resp. epimorphism) from M_α to M_α is isomorphic.

Remark. 1) Theorem 2 give a generalization of the following Mares's result ([9]): A projective module P is semiperfect if and only if it satisfies the conditions:

i) J(P) is small,

ii) P/J(P) is completely reducible,

iii) Every direct decomposition of P/J(P) is induced from a decomposition of P.

2) In S. Mohamed [10] dual-continuous modules are introduced. This concept just coincides with that of our sem-perfect modules. He then asked what is the structure of a dual-continuous module M with J(M) = M. Theorem 2* is the complete solusion of this problem.

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Department of Mathematics
University of Yamaguchi.

V-RINGS RELATIVE TO HEREDITARY TORSION THEORIES

Yasuhiko TAKEHANA

Villamayor has considered V-rings with the property that every simple module is injective. The main purpose of this paper is to give a torsion theoretical generalization of V-rings. Theorem 2 generalizes Theorem 2.1 of [4], stating that any torsion simple right R-module is divisible if and only if each right ideal I of R with R/I torsion is an intersection of maximal right ideals of R. Applying Theorem 2 for the Goldie or Lambek torsion theories, we obtained Corollaries 4 and 5. We consider in Corollary 4 a ring (called a $\sigma(G)$-V-ring) for which every singular simple right module is injective, and in Corollary 5 a $\sigma(L)$--V-ring for which every dense right ideal is an intersection of maximal right ideals. We characterize V-rings in terms of $\sigma(G)$-V-rings or $\sigma(L)$-V-rings in Proposition 7 which is closely related to Theorem 8 in [5]. In Theorem 8 it is proved that commutative $\sigma(G)$-V-rings turn out to be V-rings. Finally two examples are given to show that $\sigma(L)$-V-rings are not necessary $\sigma(G)$-V-rings and $\sigma(G)$-V-rings are not always V-rings.

We assume a knowledge of torsion theory. For example, see [2]. Throughout of this paper, R is a ring with a unit, every right R-module is unital and Mod-R is the category of right R-modules. A subfunctor of the identity functor of Mod-R is called a preradical. A preradical t is called a left exact radical if $t(N) = N \cap t(M)$ and $t(M/t(M)) = 0$ hold for any right R-module M and any submodule N of M.

We put $T_t = \{M \in \text{Mod-}R;\ t(M) = M\}$ and $F_t = \{M \in \text{Mod-}R;\ t(M) = 0\}$, whose elements are said to be torsion and torsionfree modules, respectively. A right R-module M is called singular if any element of M is annihilated by a large right ideal of R. For a right R-module M, Z(M), E(M) and J(M) denote the singular submodule of M, the injective hull of M and the intersection of all maximal submodules of M. A right R-module M is called divisible if $\text{Hom}_R(-, M)$ preserves the exactness for every exact sequence $0 \to A \to B \to C \to 0$ with $C \in T_t$. A right ideal of R is called dense if $\text{Hom}_R(R/I, E(R)) = 0$.

At first we consider a right R-module which cogenerates all torsion right R-modules.

Lemma 1. Let σ be a left exact radical. Then the following conditions on a right R-module C are equivalent.

(1) C cogenerates each torsion right R-module.

(2) For each torsion simple right R-module S, C contains a copy of $E_\sigma(S)$, where $E_\sigma(S)$ denote the divisible hull of S(i.e. $E_\sigma(S)/S = \sigma(E(S)/S)$.).

Proof. (1) → (2): Let S be a torsion simple right R-module. Since T_σ is closed under taking extensions, $E_\sigma(S)$ is torsion. Thus $E_\sigma(S) \subset \Pi(C)$, and so C contains $E_\sigma(S)$, for S is the smallest nonzero submodule of $E_\sigma(S)$.

(2) → (1): Let N be a torsion right R-module and $o \neq n \in N$. Then nR has a torsion simple homomorphic image S. Since $E_\sigma(S)$ is divisible, there exists an $f \in \text{Hom}_R(N, E_\sigma(S))$ with $f(n) \neq o$. But then $E_\sigma(S) \subset C$ by (2), and so C cogenerates N.

Now we consider V-rings by using hereditary torsion theories.

Theorem 2. Let σ be a left exact radical. Then the following conditions are equivalent.

(1) Any torsion simple right R-module is divisible.

(2) For any torsion right R-module M, $J(M) = 0$ holds.

(3) If I is a right ideal of R with R/I torsion, then I is an intersection of maximal right ideals of R.

Proof. (1) $\rightarrow$ (2): Let C denote the product of all torsion simple right R-modules. By the assumption, C is divisible, and so for any $M \in T_\sigma$, $M \subset \Pi C$ holds by Lemma 1. Thus $J(M) = 0$ holds.

(2) $\rightarrow$ (3): Obvious.

(3) $\rightarrow$ (1): Let I be a right ideal of R with R/I torsion, S a torsion simple right R-module and $f \in Hom_R(I, S)$. We need to extend f to a mapping of R into S. We may assume f is an epimorphism. We put Ker(f) = K. It is sufficient to show that the following exact sequence $0 \rightarrow I/K \rightarrow R/K \rightarrow R/I \rightarrow 0$ splits. Since I/K and R/I is torsion, R/K is torsion. Thus by the assumption there exists a maximal right ideal L of R such that L contains K and does not contain I. Then $L + I = R$ and $L \cap I = K$, and so $R/K = (L/K) \oplus (I/K)$. Thus the above sequence splits, as desired.

We call a ring satisfying the equivalent conditions of the preceding theorem a σ-V-ring.

Corollary 3. Let R be a σ-V-ring for a left exact radical σ and L a right ideal of R with R/L torsion. Then $L^2 = L$ holds.

Proof. Since $L/(L^2)$ is a homomorphic image of a direct sum of some copies of R/L, $L/(L^2)$ is torsion. As T_σ is closed under taking extensions, $R/(L^2)$ is torsion. Thus by (3) of Theorem 2, L^2 is an intersection of maximal right ideals of R. Then it follows from the same argument as in the proof of Corollary 2.2 in [4] that $L^2 = L$ holds.

Let G(M)(L(M)) denote the Goldie(Lambek) torsion submodule of a right R-module M, respectively. Note that G and L are left exact radicals, $G(M)/Z(M) = Z(M/Z(M))$, $G(M) = M$ if and only if Z(M) is large in M, $L(M) = M$ if and only if $\mathrm{Hom}_R(M,E(R)) = 0$, $G(M) \supset Z(M) \supset L(M)$, if $Z(R) = 0$ then $G(M) = Z(M) = L(M)$ and if M is divisible with respect to (T_G, F_G) then M is injective, for a right R-module M.

Now we apply Theorem 2 for the Goldie or the Lambek torsion theory.

Corollary 4. The following conditions are equivalent.

(1) Any singular simple right R-module is injective.

(2) For each right R-module M with Z(M) large in M, $J(M) = 0$ holds.

(3) If I is a right ideal of R with Z(R/I) large in R/I, then I is an intersection of maximal right ideals of R.

Corollary 5. The following conditions are equivalent.

(1) If S is a simple right R-module with $\mathrm{Hom}_R(S, R) = 0$ and I a dense right ideal of R, then for any $f \in \mathrm{Hom}_R(I,S)$, f is extended to a mapping of R into S.

(2) For each right R-module M with $\mathrm{Hom}_R(M,E(R)) = 0$, $J(M) = 0$ holds.

(3) Any dense right ideal of R is an intersection of

maximal right ideals of R.

Corollary 6. Suppose that Z(R) = 0. Then the following conditions are equivalent.

(1) If S is a simple right R-module with $Hom_R(S,R) = 0$, then S is injective.

(2) For any singular right R-module M, J(M) = 0 holds.

(3) Any large right ideal of R is an intersection of maximal right ideals of R.

Proof. If Z(R) = 0, then the Lambek torsion theory coincides with the Goldie torsion theory. Thus this is clear by Corollaries 4 and 5.

We call a ring satisfying the equivalent conditions of Corollary 4 or Corollary 5 a σ(G)-V-ring or a σ(L)-V-ring, respectively. It is clear that each σ(G)-V-ring is a σ(L)-‑V-ring.

The following proposition is closely related to Theorem 8 of [5].

Proposition 7. The following conditions are equivalent.

(1) R is a V-ring.

(2) R is a σ(L)-V-ring and every minimal right ideal of R is injective and Z(R) = 0.

(3) R is a σ(G)-V-ring and every minimal right ideal of R is injective.

Proof. (1) → (2): If R is a V-ring, then Z(R) = 0 holds by (b) of Lemma 2.3 in [4], as desired.

(2) → (3): It follows from the fact that the Goldie

torsion theory coincides with the Lambek torsion theory.

(3) → (1): By (3) of Theorem 8 in [5], it is sufficient to prove that for every cyclic singular right R-module M, $J(M) = 0$ holds. This is clear by (2) of Corollary 4.

Next we consider commutative σ(G)-V-rings.

Theorem 8. Each commutative σ(G)-V-ring is a V-ring.

Proof. It is well known that R is commutative, then R is a V-ring if and only if R is a Von-Neumann regular ring. It is sufficient to prove that for every right ideal I of R $I^2 = I$ holds. If I is a large right ideal of R, then $I^2 = I$ holds by an application of Proposition 3 for the Goldie torsion theory. Now let L be a right ideal of R and J a complement of L in R(i.e. J is maximal in $\{J \subset R;\ J \cap L = 0\}$). Then it is well known that L + J is large in R. Thus $L + J = (L + J)^2 = L^2 + LJ + LJ + J^2 = L^2 + J^2$, and so $L^2 = L$ as desired.

The following example is given to show that σ(G)-V--rings are not necessary V-rings.

Example 1. Let k be a field, $R = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$, $M = \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix}$ and $K = \begin{pmatrix} 0 & 0 \\ k & k \end{pmatrix}$. Then it is easily verified that M is a unique proper large right ideal of a ring R and $Z(R) = 0$. Since M is a maximal right ideal of R, R is a σ(G)-V-ring by (3) of Corollary 6. But $J(R) = M \cap K \neq 0$, and so R is not a V-ring.

A commutative σ(L)-V-ring is not always a V-ring.

Example 2. Let k be a field, $R = \{\begin{pmatrix} a & b \\ o & a \end{pmatrix};\ a, b \in k\}$ and $M = \begin{pmatrix} o & k \\ o & o \end{pmatrix}$. Then R is a commutative ring and has only one non-trivial right ideal M. Since $J(R) = M \neq 0$, R is not a V-ring. It is clear that R is a $\sigma(L)$-V-ring.

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Institute of Mathematics
University of Tsukuba

TRIVIAL EXTENSIONS OF TILTED ALGEBRAS

Mitsuo Hoshino

Let A be a connected artin algebra over the center C and T(A) the trivial extension of A by an A-bimodule $\mathrm{Hom}_C(A,I)$, where I is the injective envelope of C/rad C over C. In this situation we shall ask the following.

Problem 1. When is T(A) of finite representation type ?

This problem has been already considered and partially answered by Müller [12], Green-Reiten [6] and so on. Especially in [11] Iwanaga-Wakamatsu proved that if A has a square-zero radical, T(A) is of finite representation type if and only if the graph of A is a Dynkin diagram. Moreover in [15] Tachikawa proved that if A is hereditary, T(A) is of finite representation type if and only if A is of finite representation type, i.e. the graph of A is a Dynkin diagram. These results suggest to us that the Dynkin diagrams play an essential role in our problem.

In the section 1, we shall recall some definitions we need.

In the section 2, we shall give a sufficient condition for self-injective artin algebras to be of finite representation type, that is, we shall prove the following theorem.

Theorem 1. Let R be a connected self-injective artin algebra and assume that there exists a component of the stable Auslander-Reiten quiver of R whose Cartan class is a Dynkin diagram. Then R is of finite representaton type.

The converse of this theorem is due to Riedtmann [13] for the case R is an algebra over an algebraically closed field, and to Todorov [16] for the case R is an arbitrary artin algebra. It seems, however, that our Theorem 1 has not been yet announced.

Now Problem 1 can be reduced to the following form.

Problem 2. When does a Dynkin diagram appear as the Cartan class of a component of the stable Auslander-Reiten quiver of T(A) ?

In the section 3, we shall study, in the general situation, the relations between two kinds of DTr, one of which is defined on mod A, the other on mod T(A) and both of which act on mod A. The results of this section, especially Proposition 3.4 and Theorem 3.7, will play an essential role in the proof of Theorem 2.

In the section 4, we shall study tilted algebras and their complete slices, which was introduced by Happel-Ringel [8], and also the stable Auslander-Reiten quiver of T(A). The purpose of this section is to give a partial answer to Problem 2, that is, to prove the following theorem.

Theorem 2. Let B be a connected hereditary artin algebra and T_B a tilting module with $\mathrm{End}(T_B) = A$. Then the dual quiver of B appears as the complete τ-section of a component of the stable Auslander-Reiten quiver of T(A). In particular if B is of finite representation type, there exists a component of the stable Auslander-Reiten quiver of T(A) whose Cartan class is a Dynkin diagram.

This is the main result of this paper.

As an immediate consequence of Theorems 1 and 2, we obtain the following theorem.

Theorem 3. Let B be a connected hereditary artin algebra and T_B a tilting module with $\mathrm{End}(T_B) = A$. Then T(A) is of finite representation type if and only if B is of finite representation type.

After completing this paper, the author learned that the similar result was obtained by D.Hughes and J.Washbüsch (cf. [10]). However their proof is entirely different from ours. It should be noted that in our Theorem 2 we do not exclude the case the graph of B contains a circle.

Throughout this paper; All modules are finitely generated, most modules are right modules and homomorphisms operate by the opposite side of the scalars. Given a artin algebra A, mod A denotes the category of finitely generated right A-modules, τ_A (resp. τ_A^{-1}) denotes DTr (resp. TrD) defined on mod A. For an A-module X, $\Omega_A^i X$ denotes the i-th syzygy module of X. D always denotes the duality.

1. Some definitions.

In this section, we recall some definitions we need in this paper, but refer to [5] for trivial extensions and to [2] for DTr and almost split sequences.

Given an artin algebra A, a finitely generated A-module T_A is said to be a tilting module if it satisfies the following three properties;

(1) $\mathrm{projdim}\, T_A \leq 1$,

(2) $\mathrm{Ext}_A^1(T,T) = 0$,

(3) there is an exact sequence $0 \to A_A \to T'_A \to T''_A \to 0$ with T'_A and T''_A direct sums of direct summands of T_A.

In the case A is hereditary, $\mathrm{End}(T_A)$ is said to be a <u>tilted algebra</u> (see [8] for details).

The <u>Auslander-Reiten quiver</u> of an artin algebra A has as vertices the isomorphism classes of indecomposable A-modules, and there is an arrow from $[X]$ to $[Y]$ if $\mathrm{Irr}(X,Y) \neq 0$, which is endowed with the valuation (d_{XY}, d'_{XY}) such that $d_{XY} = \dim \mathrm{Irr}(X,Y)_{K(X)}$ and $d'_{XY} = \dim_{K(Y)} \mathrm{Irr}(X,Y)$, where $\mathrm{Irr}(X,Y) = \mathrm{rad}(X,Y)/\mathrm{rad}^2(X,Y)$ and $K(X) = \mathrm{End}(X)/\mathrm{rad}\,\mathrm{End}(X)$ (see [14] for details), and simply written by $[X] \longrightarrow [Y]$ if $d_{XY} = 1 = d'_{XY}$.

An indecomposable module X is said to be <u>stable</u> if $\tau^n X \neq 0$ for any integer n. The <u>stable Auslander-Reiten quiver</u> of A is the full subquiver of the Auslander-Reiten quiver of A consisting of the isomorphism classes of the stable indecomposable modules.

Given an artin algebra A, let $\{S(1),\dots,S(n)\}$, $\{P(1),\dots,P(n)\}$ and $\{I(1),\dots,I(n)\}$ be the complete sets of simple, indecomposable projective and indecomposable injective modules respectively such that

$$\mathrm{top}\,P(i) \simeq S(i) \simeq \mathrm{soc}\,I(i),\ 1 \leq i \leq n.$$

The <u>quiver</u> of A has $\{1,\dots,n\}$ as vertices and there is an arrow from i to j if $\mathrm{Ext}_A^1(S(i),S(j)) \neq 0$, which is endowed with the valuation (d_{ij}, d'_{ij}) such that

$$d_{ij} = \dim \mathrm{Hom}_A(P(j), \mathrm{rad}\,P(i)/\mathrm{rad}^2 P(i))_{K(P(j))}$$

$$d'_{ij} = \dim_{K(I(i))} \mathrm{Hom}_A(\mathrm{soc}^2 I(j)/\mathrm{soc}\,I(j), I(i)),$$

and as above simply written by $i \longrightarrow j$ if $d_{ij} = 1 = d'_{ij}$.

The <u>graph</u> of A is obtained from the quiver of A by forgetting the orientations of the arrows (see [4] for details).

The <u>dual quiver</u> of A means the quiver obtained from the quiver of A by transposing the orientations and the valuations of the arrows.

In what follows, given a valued quiver we denote by (d_{xy}, d'_{xy}) the image of an arrow from x to y under the given valuation.

A <u>stable valued Riedtmann quiver</u> is a valued quiver without loops nor multiple arrows and with a bijective transformation τ of vertices such that $(\tau x)^+ = x^-$ for all x, where x^+ (resp. x^-) denotes the set of end-points (resp. start-points) of arrows starting from (resp. ending in) x, and $d_{\tau y,x} = d'_{xy}$, $d'_{\tau y,x} = d_{xy}$ for any valued arrow from x to y.

Given a stable valued Riedtmann quiver Δ, a full subquiver Γ of Δ is said to be a <u>complete τ-section</u> if it satisfies the following two properties;

(1) Γ contains exactly one vertex from each τ-orbit of Δ,

(2) if there is a path $x_0 \to \cdots\cdots \to x_n$ with x_0, x_n in Γ and without a subpath of the form $\tau y \to z \to y$, then all x_i belong to Γ.

Let Γ be a valued oriented tree, i.e. a valued quiver with the underlying graph a tree. We define a stable valued Riedtmann quiver $\mathbb{Z}\Gamma$ as follows; its vertices are the ordered pairs (n,x) with n an integer and x a vertex of Γ, there are valued arrows from (n,x) to (n,y) and from $(n+1,y)$ to (n,x) for all integer n and all valued arrow from x to y, where the valuations are such that

$$d_{(n,x),(n,y)} = d_{xy} = d'_{(n+1,y),(n,x)},$$
$$d'_{(n,x),(n,y)} = d'_{xy} = d_{(n+1,y),(n,x)},$$

and the bijective transformation τ is defined as follows; $\tau(n,x) = (n+1,x)$ for all (n,x).

Given a valued quiver Γ, we define the Cartan matrix $C(\Gamma)$ as follows; $c_{xx} = 2$ and $c_{xy} = -d_{xy} - d'_{yx}$ $(x \neq y)$ for all vertices x,y, where $d_{xy} = 0 = d'_{xy}$ if there is no arrow from x to y.

Structure Theorem (Riedtmann [13]). Let Γ, Γ' be valued oriented trees. Then $\mathbb{Z}\Gamma$ and $\mathbb{Z}\Gamma'$ are isomorphic if and only if $C(\Gamma)$ and $C(\Gamma')$ are similar. Given a stable valued Riedtmann quiver Δ, there is a valued oriented tree Γ and an admissible automorphism group G of $\mathbb{Z}\Gamma$ such that $\Delta \simeq \mathbb{Z}\Gamma/G$.

In the above, $C(\Gamma)$ is said to be the Cartan class of Δ (see [7] for details). Note that if Δ has as the complete τ-section a valued oriented tree Γ, then $C(\Gamma)$ is the Crtan class of Δ.

2. Proof of Theorem 1.

Theorem 1 is an immediate consequence of the following two propositions.

Proposition 2.1(Auslander [1]). Let R be a connected artin algebra and assume that there exists a bounded length component of the Auslander-Reiten quiver of R. Then these are all isomorphism classes of indecomposable modules, and R is of finite representation type.

Proof. See [1, Theorem 6.5].

Proposition 2.2. Let R be a self-injective artin algebra and C a component of the stable Auslander-Reiten quiver of R whose Cartan class is a Dynkin diagram. Then C is finite.

Proof. See [0]. This proposition is essentially due to the fact that for a Dynkin diagram, the coxter transformation has a finite period. Also we can directly prove this by calculating the composition lengths as Todorov [16] did.

3. Relations between two kinds of DTr.

Throughout this section; A is an artin algebra over the center C, $Q = \mathrm{Hom}_C(A,I)$ is an A-bimodule, where I is the injective envelope of $C/\mathrm{rad}\,C$ over C. R is the trivial extension of A by Q. $L = \mathrm{Hom}_A({}_AQ_A,-)$ and $G = -\otimes_A Q_A$ are endofunctors of $\mathrm{mod}\,A$.

The results of this section will be used in the next section to prove Theorem 2.

Lemma 3.1. Let P and I be the indecomposable projective and injective modules respectively such that $\mathrm{top}\,P \simeq \mathrm{soc}\,I$. Then followings hold;

(1) $GP \simeq I$ and $LI \simeq P$,

(2) $DHom_A(P,X) \simeq Hom_A(X,I)$ as End(X)-modules for any A-module X_A.

Proof. Clear.

Lemma 3.2. Let $P_1 \xrightarrow{\rho} P_0 \longrightarrow X \longrightarrow 0$ be the minimal projective resolution. Then $\tau X \simeq \operatorname{Ker} G\rho$. In particular, if A is symmetric, $\tau X \simeq \operatorname{Ker} \rho$.

Proof. See [15, Lemma 2.3].

Lemma 3.3. R is symmetric.

Proof. See [11, Proposition 1].

Proposition 3.4. For any A-module X_A, there is a natural exact sequence of the form

$$0 \longrightarrow \tau_A X \longrightarrow \tau_R X \longrightarrow \Omega_A X \oplus \Omega_R GX \longrightarrow 0.$$

In particular, if $\operatorname{projdim} X_A \leq 1$ and $\operatorname{injdim} {}_A X \leq 2$, there is a natural exact sequence of the form

$$0 \longrightarrow \tau_A X \longrightarrow \tau_R X \longrightarrow \tau_R LGX \longrightarrow 0.$$

Proof. See [0]. In the above proposition "natural" means that given a homomorphism $X_A \xrightarrow{f} Y_A$, there exists the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau_A X & \longrightarrow & \tau_R X & \longrightarrow & \Omega_A X \oplus \Omega_R X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau_A f} & & \downarrow{\scriptstyle \tau_R f} & & \downarrow{\scriptstyle \begin{pmatrix} \Omega_A f & 0 \\ 0 & \Omega_R G f \end{pmatrix}} & & \\ 0 & \longrightarrow & \tau_A Y & \longrightarrow & \tau_R Y & \longrightarrow & \Omega_A Y \oplus \Omega_R Y & \longrightarrow & 0, \end{array}$$

where $\tau_A f$, $\Omega_A f$ are determined modulo projectives over A and $\tau_R f, \Omega_R G f$ are determined modulo projectives over R.

Lemma 3.5. For any A-module X_A, followings hold;

(1) $\operatorname{projdim} X_A \leq 1$ iff $L\tau_A X = 0$,

(2) $\operatorname{injdim} X_A \leq 1$ iff $G\tau_A^{-1} X = 0$.

Proof. See [0].

Lemma 3.6. If $\operatorname{projdim} X_A \leq 1$, for any Z_A we have

$$\operatorname{Ext}_R^1(Z, \tau_A X) \simeq \operatorname{Ext}_A^1(Z, \tau_A X).$$

Proof. See [0].

Finally, as an immediate consequence of Proposition 3.4 and Lemmas 3.5 and 3.6, we obtain the following theorem.

Theorem 3.7. For any indecomposable A-module X_A, followings are equivalent;

(1) $\operatorname{projdim} X_A \leq 1$ and $\operatorname{injdim} \tau_A X \leq 1$,

(2) $\tau_A X \simeq \tau_R X$,

(3) the almost split sequence ending in X in mod A is also the almost split sequence in mod R.

4. Proof of Theorem 2.

Throughout this section; B is a connected hereditary artin algebra over a field k. $\{S(1),\ldots,S(n)\}$, $\{P(1),\ldots,P(n)\}$ and $\{I(1),\ldots,I(n)\}$ are the complete sets of simple, indecomposable projective and indecomposable injective B-modules respectively such that

$$\operatorname{top} P(i) \simeq S(i) \simeq \operatorname{soc} I(i), \ 1 \leq i \leq n.$$

$T_B = \bigoplus_{i=1}^{n} T_i$ is a tilting module with $\mathrm{End}(T_B) = A$ and pairwise non-isomorphic T_i's, thus A is basic. ${}_AQ_A = D({}_AA_A)$ and ${}_BQ_B = D({}_BB_B)$ are A- and B-bimodules respectively. R is the trivial extension of A by ${}_AQ_A$.

$F = \mathrm{Hom}_B({}_AT_B, -)$ and $F' = \mathrm{Ext}^1_B({}_AT_B, -)$ are functors from $\mathrm{mod}\,B$ to $\mathrm{mod}\,A$. $H = \mathrm{Hom}_B(-, {}_AT_B)$ and $H' = \mathrm{Ext}^1_B(-, {}_AT_B)$ are functors from $\mathrm{mod}\,B$ to $\mathrm{mod}\,A^{op}$, where $\mathrm{mod}\,A^{op}$ denotes the category of finitely generated left A-modules. $L = \mathrm{Hom}_A({}_AQ_A, -)$ and $G = -\otimes_A Q_A$ are endofunctors of $\mathrm{mod}\,A$.

$$d_{ij} = \dim \mathrm{Irr}(I(i), I(j))_{\mathrm{End}(I(i))},$$

$$d'_{ij} = \dim_{\mathrm{End}(I(j))} \mathrm{Irr}(I(i), I(j)).$$

Theorem of Brenner-Butlar. ${}_AT$ is also a tilting module with $\mathrm{End}({}_AT) = B$. The full subcategories $\{X_B \mid F'X = 0\}$ and $\{Y_A \mid \mathrm{Tor}^A_1(Y, T) = 0\}$ are equivalent under the restrictions of the functors F and $-\otimes_A T_B$ which are mutually inverse to each other, and similarly the full subcategories $\{X_B \mid FX = 0\}$ and $\{Y_A \mid Y\otimes T = 0\}$ are equivalent under the restrictions of the functors F' and $\mathrm{Tor}^A_1(-, {}_AT_B)$ which are mutually inverse to each other.

Proof. See [8, Theorem 2.1].

Lemma 4.1. For any $1 \le i \le n$, followings hold;

(1) if $I(i)$ is not a direct summand of T_B, then

$$DHI(i) \simeq DH(I(i)/S(i)),$$

(2) if $I(i)$ is a direct summand of T_B, then $DHI(i)$ is an indecomposable injective A-module such that

$$DHI(i)/\mathrm{soc} \simeq DH(I(i)/S(i)),$$

where the isomorphism is induced by a canonical surjection $I(i) \longrightarrow I(i)/S(i)$ in either cases.

Proof. See [0].

Proposition 4.2. For any X_B, $\operatorname{projdim} FX_A \leq 1$.

Proof. See [8, Lemma 5.1].

Proposition 4.3. For any indecomposable B-module X_B, followings hold;

(1) if $F'X = 0$, then

$$\tau_A FX \simeq DH'X \quad \text{and} \quad GFX \simeq DHX,$$

(2) if X_B is not projective, then

$$DH'X \simeq F\tau_B X \quad \text{and} \quad DHX \simeq F'\tau_B X.$$

Proof. See [0].

Corollary 4.4. For any non-projective indecomposable B-module X_B with $F'X = 0$, we have

$$\tau_A FX \simeq F\tau_B X \quad \text{and} \quad GFX \simeq F'\tau_B X.$$

Remark. Put $\mathcal{X} = \{X_A \mid X \otimes T = 0\}$, $\mathcal{Y} = \{Y_A \mid \operatorname{Tor}_1^A(Y,T) = 0\}$. In [8] Happel-Ringel showed that the pair $(\mathcal{X}, \mathcal{Y})$ forms a splitting torsion theory (see [8, Theorem 6.3]). The above corollary gives another proof of this, because we obtain

$$\begin{aligned} \operatorname{Ext}_A^1(Y,X) &\simeq D\overline{\operatorname{Hom}}_A(X,\tau_A Y) \\ &\simeq D\overline{\operatorname{Hom}}_A(X,F\tau_B(Y\otimes T)) \\ &= 0 \end{aligned}$$

for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ (see [9] for details).

Proposition 4.5. The minimal right almost split map ending in $FI(i)$ in $\operatorname{mod} A$ is of the form

$$(\bigoplus_j FI(j)^{d_{ji}}) \oplus \tau_A F(I(i)/S(i)) \longrightarrow FI(i).$$

Proof. See [0].

Lemma 4.6. $\operatorname{Hom}_A(GFI(i), FI(j)) \neq 0$ iff $I(i) \simeq P(j)$.

Proof. See [0].

Lemma 4.7. If $\operatorname{Hom}_B(I(j), I(i)) \neq 0$ with $i \neq j$, then we have

$$\operatorname{Ext}^1_R(FI(i), FI(j)) = 0.$$

Proof. See [0].

Proposition 4.8. If $HI(i) = 0$, then the almost split sequence ending in $FI(i)$ in $\operatorname{mod} R$ is of the form

$$0 \longrightarrow \tau_R FI(i) \longrightarrow (\bigoplus_j FI(j)^{d_{ji}}) \oplus \tau_R F(I(i)/S(i)) \longrightarrow FI(i) \longrightarrow 0.$$

Proof. See [0].

Proposition 4.9. If $HI(i) \neq 0$ and $H(I(i)/S(i)) = 0$, then the almost split sequence ending in $FI(i)$ in $\operatorname{mod} R$ is of the form

$$0 \to \tau_R FI(i) \to (\bigoplus_j FI(j)^{d_{ji}}) \oplus \tau_R F(I(i)/S(i)) \oplus P \to FI(i) \to 0,$$

where P is the projective cover of $FI(i)$ over R.

Proof. See [0].

Lemma 4.10. Assume that there is an irreducible map in $\mathrm{mod}\, R$ of the form

$$FI(i)^{d_{ij}} \longrightarrow FI(j)$$

for all j with $\mathrm{Irr}(I(i),I(j)) \neq 0$. Then there is an irreducible map in $\mathrm{mod}\, R$ of the form

$$FI(i) \longrightarrow F(I(i)/S(i)).$$

Proof. See [0].

Proposition 4.11. Assume that $H(I(i)/S(i)) \neq 0$ and there is an irreducible map in $\mathrm{mod}\, R$ of the form

$$FI(i) \longrightarrow F(I(i)/S(i))$$

Then the almost split sequence ending in $FI(i)$ in $\mathrm{mod}\, R$ is of the form

$$0 \longrightarrow \tau_R FI(i) \longrightarrow (\bigoplus_j FI(j)^{d_{ji}}) \oplus \tau_R F(I(i)/S(i)) \longrightarrow FI(i) \longrightarrow 0.$$

Proof. See [0].

Now we can prove Theorem 2. Since B is hereditary, the dual quiver of B is isomorphic to the full subquiver of the Auslander-Reiten quiver of B consisting of the isomorphism classes of indecomposable injective modules. Moreover, since R is self-injective, the stable Auslander-Reiten quiver of R is obtained from the Auslander-Reiten quiver of R by deleting only the isomorphism classes of indecomposable projective modules. Therefore, it is sufficient to prove the following proposition.

Proposition 4.12. For any i, the middle term of the almost split sequence ending in $FI(i)$ in $\mathrm{mod}\,R$ is, up to a projective direct summand, of the form

$$(\bigoplus_j FI(j)^{d_{ji}}) \oplus \tau_R F(I(i)/S(i)).$$

Proof. By induction along the sink sequences in the full subquiver $\{I(1),\ldots,I(n)\}$ of the Auslander-Reiten quiver of B.

If $I(i)$ is a sink, this is the case Proposition 4.8 or Proposition 4.9.

Consider now some $I(i)$, and assume that it is already shown for all $I(j)$ with $\mathrm{Irr}(I(i),I(j)) \neq 0$.

If $H(I(i)/S(i)) = 0$, this is also the case Proposition 4.8 or Proposition 4.9.

In the case $H(I(i)/S(i)) \neq 0$, by Lemma 4.10 the assumption of Proposition 4.11 is satisfied.

This finishes the proof.

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Institute of Mathematics
University of Tsukuba

ON THE INDECOMPOSABILITY OF AMALGAMATED SUMS

Hideto ASASHIBA and Takeshi SUMIOKA

Introduction. We introduce a consept of "independent map" and study the relationship between the "independence" of a map and the indecomposability of its cokernel in certain situations.

Let R be a right artinian ring with 1 and consider the exact sequence

$$0 \longrightarrow K \xrightarrow{f} \oplus_{i=1}^{n} L_i \xrightarrow{g} M \longrightarrow 0$$

of finitely generated right R-modules which does not split. We will see later that if M is indecomposable then f is "independent" but the converse is not true in general. In Tachikawa [4] we see the converse holds if the following conditions are satisfied: i) R is right serial; ii) n = 2 or 3; iii) each L_i is local (therefore uniserial) and its composition length is 2; iv) K is simple; v) each f_i ,the i-th coordinate map of f , is a monomorphism. Our aim is to prove under each of the following two weaker conditions that this converse assertion is still true:

1) i) n is an arbitrary natural number $\geqslant 2$;
  ii) each L_i is local but not simple for $i = 1,\dots,n$;
 iii) K is simple,

2) i) n is an arbitrary natural number > 2;
  ii) each L_i is colocal for $i = 1,\dots,n$;
 iii) f_1 is monomorphic and L_1/f_1K is simple.

Unless otherwise stated we assume throughout this note that R is a right artinian ring with 1, J stands for the Jacobson radical of R and all modules are unital finitely generated right R-modules. For maps $f : K \longrightarrow \oplus_I L_i$ and $g : \oplus_I L_i \longrightarrow M$, we denote $f = (f_i)_I^T$, $g = (g_i)_I$ where for each $i \in I$ the projection $p_i : \oplus_I L_i \longrightarrow L_i$ and the injection $k_i : L_i \longrightarrow \oplus_I L_i$ are canonical and $f_i = p_i f$, $g_i = g k_i$ which are called the i-th coordinate maps of f and g respectively. In case $I = \{a, a+1, \ldots, b\}$, $(f_i)_I^T$, $(g_i)_I$ are written as $(f_i)_{i=a}^{T\ b}$, $(g_i)_{i=a}^{b}$ respectively. The notation $I_1 \amalg I_2 \amalg \ldots \amalg I_n = I$ means the union $I_1 \cup I_2 \cup \ldots \cup I_n = I$ is disjoint.

1. Independent map

(1.1) Definition. A homomorphism $f = (f_i)_{i=1}^{T\ n} : K \longrightarrow \oplus_{i=1}^{n} L_i$ $(= L)$ is called independent (with respect to the decomposition $L = \oplus_{i=1}^{n} L_i$) in case for each $i = 1, \ldots, n$ and each $h = (h_i)_{i=1}^{n} : L \longrightarrow L_i$, $hf = \sum_{i=1}^{n} h_i f_i = 0$ implies h_i is not isomorphic for each $i = i, \ldots, n$. And f is called dependent in case f is not independent.

(1.2) Proposition. For every homomorphism $f : K \longrightarrow \oplus_{i=1}^{n} L_i$ $(= L)$ putting $g : \oplus_{i=1}^{n} L_i \longrightarrow \mathrm{Cok}\, f$ to be the canonical epimorphism, the following statements are equivalent:

a) f is dependent;

b) There is a homomorphism $h : \oplus_{i \neq j} L_i \longrightarrow L_j$ for some $j = 1, \ldots, n$ such that $f_j = h (f_i)_{i \neq j}^{T}$;

c) g_i is a split monomorphism for some $i = 1, \ldots, n$;

d) There is a split epimorphism $p : M \longrightarrow M_1$ and a subset $I \subseteq \{1, \ldots, n\}$ such that $p(g_i)_I : \oplus_I L_i \longrightarrow M_1$ is isomorphic.

Further if each L_i is indecomposable then these four conditions are equivalent to

e) $\{h \in End(L) \mid hf = 0\} \not\subset$ rad End(L) where rad(-) denotes the Jacobson radical of (-).

Proof. a) $\Rightarrow$ b). If f is dependent then there exist $i = 1,\ldots,n$ and $h : L \longrightarrow L_i$, $h = (h_i)_{i=1}^{n}$ such that $hf = 0$ and h_j is an isomorphism for some $j = 1, \ldots,n$. Since h_j is isomorphic we may assume $i = j$ and $h_j = 1_{L_j}$. Then $hf = 0$ implies that $f_j = (-h_i)_{i\neq j}(f_i)^T_{i\neq j}$.

b) $\Rightarrow$ c). Suppose that $f_j = (-h_i)_{i\neq j}(f_i)^T_{i\neq j}$. Taking $h = (h_1,\ldots,h_{j-1},1_{L_j},h_{j+1},\ldots,h_n)$ we have $hf = 0$. Therefore there is a homomorphism $p : M \longrightarrow L_j$ such that $h = pg$ where $g : L \longrightarrow M$ (= L/fK) is canonical. Let $k_j : L_j \longrightarrow L$ be the inclusion map then $pg_j = pgk_j = hk_j = 1_{L_j}$. Thus g_j is a split monomorphism.

c) $\Rightarrow$ d). Trivial.

d) $\Rightarrow$ a). Suppose d) holds. Taking $h = p_i[p(g_i)_I]^{-1}pg$ where $p_i : \oplus_I L_i \longrightarrow L_i$ $(i \in I)$ is a canonical projection, we have $hf = 0$ and $h_i = 1_{L_i}$ thus f is dependent.

If each L_i is indecomposable then rad End(L) = $\{(f_{ij}) \in End(\oplus_{i=1}^{n} L_i) \mid f_{ij}$ is not isomorphic for each $i,j = 1,\ldots,n\}$. From this fact the equivalence of a) and e) is immediate. OK.

(1.3) Corollary. If f is independent with respect to a decomposition $L = \oplus_{i=1}^{n} L_i$ with each L_i indecomposable then independence of f does not depend on the decomposition of L.

(1.4) Corollary. If a monomorphism f is not split and $\mathrm{Cok}\, f$ is indecomposable then f is independent (with respect to any decomposition of L).

Proof. If f is not split and f is dependent then from (1.2) L_i is a proper nonzero direct summand of $\mathrm{Cok}\, f$ for some $i = 1,\ldots,n$. OK.

(1.5) Corollary. Let $K_i \lneqq L_i$ for each $i = 1,2$ and $h : K_1 \longrightarrow K_2$ be an isomorphism. Define $f_1 = k_1$, $f_2 = k_2 h$ where $k_i : K_i \longrightarrow L_i$ is the inclusions for each $i = 1,2$. Then h or h^{-1} can be extendable to a homomorphism $L_1 \longrightarrow L_2$ or $L_2 \longrightarrow L_1$ respectively if and only if $(f_i)^T_{i=1,2} : K_1 \longrightarrow L_1 \oplus L_2$ is dependent.

(1.6) Example. There is a monomorphism $f : K \longrightarrow L$ such that f is independent (with respect to any decomposition of L) but $\mathrm{Cok}\, f$ is decomposable.

Let the exact sequence

$$0 \longrightarrow K \xrightarrow{f} L \xrightarrow{g} M \longrightarrow 0$$

be the projective cover of $M = M_1 \oplus M_2$ where M_1 and M_2 are nonprojective indecomposables. Then f is not split and by (1.2) f is independent but $\mathrm{Cok}\, f = M$ is decomposable.

2. Proof of the theorems

(2.1) Lemma. Let R be an arbitrary ring with 1 and suppose a right R-module M is decomposed as follows

$$M = \oplus_{i=1}^{n} L_i = M_1 \oplus M_2$$

where L_i is a simple right R-module for each $i = 1,\dots,n$. Then there is a partition of the set $\{1,\dots,n\}$: $I_1 \amalg I_2 = \{1,\dots,n\}$ such that $M = M_1 \oplus (\oplus_{I_2} L_i) = (\oplus_{I_1} L_i) \oplus M_2$.

proof. We prove the assertion by induction on n. Let L_1' and L_1'' be the images of L_1 under the projections $M \longrightarrow M_1$ and $M \longrightarrow M_2$, respectively. Then $M = L_1' \oplus L_2 \oplus \dots \oplus L_n$ or $M = L_1'' \oplus L_2 \oplus \dots \oplus L_n$. We may assume that the former holds. We denote the image of every submodule $N \leqslant M$ by $\overline{N}$ under the canonical projection $M \longrightarrow M/L_1'$. Then we have $\overline{M} = \overline{L}_2 \oplus \dots \oplus \overline{L}_n = \overline{M}_1 \oplus \overline{M}_2$. By hypothesis of induction there is a partition $I_1' \amalg I_2' = \{2,\dots,n\}$ such that $\overline{M} = \overline{M}_1 \oplus (\oplus_{I_2'} \overline{L}_i) = (\oplus_{I_1'} \overline{L}_i) \oplus \overline{M}_2$. Therefore $M = M_1 + (\oplus_{I_2'} L_i) = L_1' + (\oplus_{I_1'} L_i) + M_2^1$ for $L_1' \leqslant M_1$. Comparing the composition lengths, we see the sums are direct and it follows $M = L_1 \oplus (\oplus_{I_1'} L_i) \oplus M_2$. This completes the proof taking $I_1 = \{1\} \cup I_1'$, $I_2 = I_2'$. OK.

(2.2) <u>Remark</u>. This lemma holds more generally. Let $M = \oplus_I L_i = \oplus_J M_j$ be completely indecomposable decomposition of an R-module M where the ring R is not nècessarily an artinian ring. Then for any finite subset $J' = \{j_1,\dots,j_n\}$ of J there exists a subset $I' = \{i_1,\dots,i_n\}$ of I such that $L_{i_k} \cong M_{j_k}$ for each $k = 1,\dots,n$ and

$$M = (\oplus_{I'} L_i) \oplus (\oplus_{J-J'} M_j) = (\oplus_{I-I'} L_i) \oplus (\oplus_{J'} M_j).$$

(See [1,Theorem 1.7])

We are now in the position to prove the propositions in the introduction.

(2.3) Theorem. Let $0 \longrightarrow K \xrightarrow{f} L \xrightarrow{g} M \longrightarrow 0$ be an exact sequence of R-modules such that $L = \oplus_{i=1}^{n} L_i$ with L_i local but not simple for each $i = 1,\ldots,n$ and K is simple (n is an arbitrary natural number $\geqslant 2$). Then the following conditions are equivalent:

a) M is indecomposable;

b) M has no direct summand which is isomorphic to L_i for some $i = 1,\ldots,n$;

c) f is independent.

Proof. Since $\operatorname{Im} f$ is contained in the radical of L, the exact sequence dose not split so by (1.2) the implications a) $\Rightarrow$ b) $\Rightarrow$ c) are clear.

c) $\Rightarrow$ a). Now assume that M is decomposable, say $M = M_1 \oplus M_2$, $M_1, M_2 \neq 0$, and let $\pi_i : M \longrightarrow M_i$ be canonical projections for $i = 1,2$. We denote the image of every submodule $N \leqslant M$ by $\overline{N}$ under the canonical projection $M \longrightarrow M/MJ$ where J is the Jacobson radical of R. Then since $\operatorname{Ker} g \leqslant LJ$ we have

$$\overline{M} = \oplus_{i=1}^{n} \overline{gL_i} = \overline{M}_1 \oplus \overline{M}_2$$

where $\overline{gL_i}$ is simple for every $i = 1,\ldots,n$. By (2.1) there is a partition $I_1 \amalg I_2 = \{1,\ldots,n\}$ such that

$$\overline{M} = \overline{M}_1 \oplus (\oplus_{I_2} \overline{gL_i}) = (\oplus_{I_1} \overline{gL_i}) \oplus \overline{M}_2$$

but since MJ is small in M, it follows that

$$M = M_1 + \Sigma_{I_2} gL_i = \Sigma_{I_1} gL_i + M_2$$

which means that both $\pi_1(g_i)_{I_1}$ and $\pi_2(g_i)_{I_2}$ are epimorphisms. It is easily verified that $|\operatorname{Ker}\pi_1(g_i)_{I_1}| + |\operatorname{Ker}\pi_2(g_i)_{I_2}| = |K|$ where $|N|$ denotes the composition length of N for each R-module N. Then since K is simple i.e. $|K| = 1$, either $\pi_1(g_i)_{I_1}$ or $\pi_2(g_i)_{I_2}$ must be an isomorphism. Hence f is dependent by (1.2). OK.

(2.4) Theorem. Let $0 \longrightarrow K \xrightarrow{f} L \xrightarrow{g} M \longrightarrow 0$ be an exact sequence of R-modules such that $L = \oplus_{i=1}^{n} L_i$ (n is a natural number $\geqslant 2$) with L_1 indecomposable and $L_2, \ldots, L_n$ colocal, the first coordinate map f_1 is monomorphic and $\mathrm{Cok}\, f_1$ is simple. Then if f is independent, M is indecomposable.

Proof. Since f_1 is monomorphic, $(g_i)_{i\neq 1}$ is monomorphic for $\mathrm{Im}\, f \geqslant \mathrm{Ker}\, g$. Then we have $\oplus_{i\neq 1} gL_i \leqslant M$ where $gL_i \cong L_i$ for all $i \neq 1$ and there is a submodule N of $\mathrm{soc}\, M$, the right socle of M, such that $N \oplus (\oplus_{i\neq 1} \mathrm{soc}\, gL_i) = \mathrm{soc}\, M$ where $\mathrm{soc}\, gL_i \cong \mathrm{soc}\, L_i$ is simple for each $i = 2, \ldots, n$. Now suppose M is decomposable, say $M = M_1 \oplus M_2$, $M_1, M_2 \neq 0$, and let $\pi_i : M \longrightarrow M_i$ be the canonical projections for $i = 1,2$. Then it holds that

$$\mathrm{soc}\, M = N \oplus (\oplus_{i\neq 1} \mathrm{soc}\, gL_i) = \mathrm{soc}\, M_1 \oplus \mathrm{soc}\, M_2.$$

Hence by (2.1) there exists a partition $I_1 \amalg I_2 = \{2, \ldots, n\}$ and a direct sum decomposition $N = N_1 \oplus N_2$ such that

$$\mathrm{soc}\, M = \mathrm{soc}\, M_1 \oplus N_2 \oplus (\oplus_{I_2} \mathrm{soc}\, gL_i)$$

$$\mathrm{soc}\, M = N_1 \oplus (\oplus_{I_1} \mathrm{soc}\, gL_i) \oplus \mathrm{soc}\, M_2.$$

Since $\mathrm{soc}\, T$ is essential in T for any module T, we have $M_1 \cap \oplus_{I_2} gL_i = 0$ and $\oplus_{I_1} gL_i \cap M_2 = 0$ which means that the composite maps $\pi_1 (g_i)_{I_1}$ and $\pi_2 (g_i)_{I_2}$ are monomorphisms. On the other hand $\mathrm{Cok}\, (g_i)_{i\neq 1}$ is simple or zero for $\mathrm{Cok}\, (g_i)_{i\neq 1} = M/\oplus_{i\neq 1} gL_i = (\oplus_{i\neq 1} gL_i + gL_1)/\oplus_{i\neq 1} gL_i \cong gL_1/[(\oplus_{i\neq 1} gL_i) \cap gL_1] = gL_1/gf_1K$ and L_1/f_1K is simple. As easily seen $|\mathrm{Cok}\, \pi_1 (g_i)_{I_1}| + |\mathrm{Cok}\, \pi_2 (g_i)_{I_2}| = |\mathrm{Cok}\, (g_i)_{i\neq 1}| = 1$ or 0, hence either $\pi_1 (g_i)_{I_1}$ or $\pi_2 (g_i)_{I_2}$ is isomorphic. Thus f is dependent by (1.2). OK.

(2.3) is also a generalization of [3, Theorem 3.7] in right artinian (semiprimary) case. From (1.2) when each L_i is indecomposable we obtain that f is independent iff $\{h \in \mathrm{End}(L) \mid hf = 0\} \subseteq \mathrm{rad}\ \mathrm{End}(L)$. This latter condition is already considered in [2, p.339 Condition V].

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Department of Mathematics
Osaka City University
Osaka 558, Japan.

INTEGRAL REPRESENTATIONS OF CYCLIC GROUPS OF ORDER pq

Tadashi MITSUDA

In a module-theoretic way, considering the representations of a finite group G over Z is reduced to investigating ZG-lattices which are finitely generated Z-free ZG-modules. So the classification of indecomposable ZG-lattices is an important problem. In general, however, the number of isomorphism classes of indecomposable ZG-lattices is not necessarily to be finite. Relating to the finiteness, the following result is well-known.

(Heller-Reiner[2], Jones[4])

For a finite group G, the number of isomorphism classes of indecomposable ZG-lattices is finite if and only if for each prime p dividing $|G|$, Sylow p-subgroup of G is cyclic of order p or p^2.

But there are few results about the classifications of ZG-lattices. For example, all the indecomposable lattices are classified in the case when G is cyclic of order p (Diederichsen[1], Reiner[6]), cyclic of order p^2 (Reiner[8]) or non-abelian of order pq (Pu[5]), where p and q are distinct primes. For further references on integral representations, see Reiner[7].

Now, in this paper we will treat the case when G is an abelian group of order pq. Throughout this paper, we fix the following notation.

$G = \langle\sigma\rangle \times \langle\tau\rangle \quad \sigma^q = \tau^p = 1 \qquad p \neq q$: primes

ζ_n : primitive n-th root of unity

$\zeta := \zeta_p \ , \quad \theta := \zeta_q$

$R := \mathbb{Z}[\zeta]$, $T := \mathbb{Z}[\theta]$, $S := \mathbb{Z}[\zeta\theta]$

$r := |(\mathbb{Z}/q\mathbb{Z})^{\times} : \langle p(\mathrm{mod}.q)\rangle|$, $s := |(\mathbb{Z}/p\mathbb{Z})^{\times} : \langle q(\mathrm{mod}.p)\rangle|$

$rf = q - 1$

h_A : class number of A

$u(A)$: unit group of A

$|B : u^{*}(A)|$: index of $\mathrm{Im}(u(A) \longrightarrow B)$ in B

$K := K_1 \oplus \ldots\ldots \oplus K_r$, $K_i \simeq F_{p^f}$

$K^{(1)} := 0 \oplus K_2^{\times} \oplus \ldots\ldots \oplus K_r^{\times}$, ..., $K^{(2^r-1)} := 0 \oplus \ldots\ldots \oplus 0 \oplus K_r^{\times}$

$\tilde{K} := F_p \oplus K_1 \oplus \ldots\ldots \oplus K_r$

$\tilde{K}^{(1)} := 0 \oplus K_1^{\times} \oplus \ldots\ldots \oplus K_r^{\times}$, ..., $K^{(2^{r+1}-1)} := 0 \oplus \ldots\ldots \oplus 0 \oplus K_r^{\times}$

$\bar{K}^{(i)} := F_p^{\times} \oplus K^{(i)}$

§ 1. Extensions

For a ZG-lattice M, put $M_0 = \{m \in M \mid \Phi_p(\tau)m = 0\}$ and $M_1 = M/M_0$, where $\Phi_p(X)$ denotes the p-th cyclotomic polynomial. Then we have an exact sequence of ZG-lattices,

$$0 \longrightarrow M_0 \longrightarrow M \longrightarrow M_1 \longrightarrow 0$$

Since $ZG/(\Phi_p(\tau)) \simeq R\langle\sigma\rangle$ and $ZG/(\tau - 1) \simeq Z\langle\sigma\rangle$, M_0 and M_1 are regarded as lattices over $R\langle\sigma\rangle$ and $Z\langle\sigma\rangle$ respectively. So every ZG-lattice can be obtained as an extension of a $Z\langle\sigma\rangle$-lattice by an $R\langle\sigma\rangle$-lattice.

By the results of Reiner, the only indecomposable $Z\langle\sigma\rangle$-lattices are of the following three types.

(I) Z

(II) $\mathfrak{C}$: non-zero ideal of T

(III) $0 \longrightarrow Z \longrightarrow P \longrightarrow \mathfrak{C} \longrightarrow 0$

($\mathfrak{C}$ is of type II)

In a similar way, we can show that every indecomposable $R\langle\sigma\rangle$-lattice is one of the following three types.

(I) $\mathfrak{a}$: non-zero ideal of R

(II) $\mathfrak{b}$: non-zero ideal of S

(III) $0 \longrightarrow \mathfrak{a} \longrightarrow X \longrightarrow \mathfrak{b} \longrightarrow 0$

($\mathfrak{a}$ and $\mathfrak{b}$ are of types of (I) and (II) respectively.)

Therefore we can get every ZG-lattice as an extension of the form,

$$(*) \quad 0 \longrightarrow \sum_i \mathfrak{a}_i^{(e_i)} \oplus \sum_j \mathfrak{b}_j^{(f_j)} \oplus \sum_k X_k^{(u_k)} \longrightarrow M \longrightarrow Z^{(c)} \oplus \sum_l \mathfrak{c}_l^{(d_l)} \oplus \sum_m P_m^{(e_m)} \longrightarrow 0$$

Now the problem is to determine when such M is indecomposable.

§ 2. Indecomposable genera

Let Π be a finite group. For $Z\Pi$-lattices X and Y, we say that they have the same genus and denote $X \sim Y$, if they are locally isomorphic. In fact, X and Y have the same genus when $X_{(p)} \cong Y_{(p)}$ only for each prime p dividing $|\Pi|$. Let $X \sim Y$, then it is also well-known that X is indecomposable if and only if Y is so. Hence the indecomposability is a genus property. Now recall the following result of Jacobinski, which is obtained in much more general situation but we state only in our case here. For other quoted results we will do in the same manner.

Proposition 1 (Jacobinski [3]). For a $Z\Pi$-lattice M, the following two conditions are equivalent.

(i) M is decomposable.

(ii) There exists a $Z\Pi$-lattice L such that $L_{(p)}$ is a direct summand of $M_{(p)}$ for any p dividing $|\Pi|$.

So examining the indecomposability of M of the form (*) is reduced to investigating the non-existence of such a local direct summand as in Proposition 1. For that purpose we use the next simple remark.

Lemma. Let G be defined as before, and N, L be ZG-lattices. For given exact sequences;

$0 \longrightarrow N_{(p)} \longrightarrow X \longrightarrow L_{(p)} \longrightarrow 0$ as $Z_{(p)}G$-lattices,

$0 \longrightarrow N_{(q)} \longrightarrow Y \longrightarrow L_{(q)} \longrightarrow 0$ as $Z_{(q)}G$-lattices,

there exists a ZG-lattice M such that

$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$, $M_{(p)} \simeq X$ and $M_{(q)} \simeq Y$.

Now we must determine the indecomposable lattices in local case. We can again get every $Z_{(p)}G$-lattice by an extension as before. And in this case, we use the following result of Jones for examining the indecomposability.

Proposition 2 (Jones [4]). Let G be as before, and M be a $Z_{(p)}G$-lattice. Then the following two conditions are equivalent.

(i) M is decomposable.

(ii) There exists a $Z_{(p)}G$-lattice N such that $\hat{N}$ is a direct summand of $\hat{M}$, where ^ means the p-adic completion.

Then we need the results about the complete local case. But since in the complete local case Krull-Schmidt theorem holds, we can rather easily determine the isomorphism classes of indecomposable lattices. Using this result, we get the theorem in local case.

Theorem 1. With the notation defined as before, there are $2^r + 4$ indecomposable $Z_{(p)}G$-lattices up to isomorphism. They are the following ones.

$Z_{(p)}$, $R_{(p)}$, $T_{(p)}$, $S_{(p)}$, $Z_{(p)}\langle\tau\rangle$ and

$0 \longrightarrow S_{(p)} \longrightarrow V \longrightarrow T_{(p)} \longrightarrow 0$ (non-split extension)

Now using Lemma and Theorem 1. repeatedly, we have

Theorem 2. With the notation as before, there are $2^{r+s+2} - 2^{r} - 2^{s} + 5$ indecomposable genera of ZG-lattices. And every indecomposable ZG-lattice can be obtained as one of the following extension forms.

(1) Z

(2) $\mathfrak{a}$: non-zero ideal of R

(3) $\mathfrak{b}$: non-zero ideal of S

(4) $\mathfrak{c}$: non-zero ideal of T

(5) $0 \longrightarrow \mathfrak{a} \longrightarrow X \longrightarrow \mathfrak{b} \longrightarrow 0$ (non-split extension)

(6) $0 \longrightarrow Z \longrightarrow P \longrightarrow \mathfrak{c} \longrightarrow 0$ (non-split extension)

(7) $0 \longrightarrow \mathfrak{a} \longrightarrow M \longrightarrow Z \longrightarrow 0$ (non-split extension)

(8) $0 \longrightarrow \mathfrak{a} \longrightarrow M \longrightarrow P \longrightarrow 0$ (non-split extension)

(9) $0 \longrightarrow \mathfrak{b} \longrightarrow M \longrightarrow \mathfrak{c} \longrightarrow 0$ (non-split extension)

(10) $0 \longrightarrow \mathfrak{b} \longrightarrow M \longrightarrow P \longrightarrow 0$ (non-split extension)

(11) $0 \longrightarrow X \longrightarrow M \longrightarrow Z \longrightarrow 0$ (non-split extension)

(12) $0 \longrightarrow X \longrightarrow M \longrightarrow \mathfrak{c} \longrightarrow 0$ (non-split extension)

(13) $0 \longrightarrow X \longrightarrow M \longrightarrow P \longrightarrow 0$ (non-split extension)

(14) $0 \longrightarrow \mathfrak{a} \oplus \mathfrak{b} \longrightarrow M \longrightarrow P \longrightarrow 0$

with $M_{(p)} \simeq Z_{(p)}\langle\tau\rangle \oplus V_i$, $M_{(q)} \simeq R_{(q)} \oplus S_{(q)} \oplus Z_{(q)}\langle\sigma\rangle$

(15) $0 \longrightarrow X \longrightarrow M \longrightarrow Z \oplus \mathfrak{c} \longrightarrow 0$

with $M_{(p)} \simeq Z_{(p)}\langle\tau\rangle \oplus V_i$, $M_{(q)} \simeq W_j \oplus Z_{(q)} \oplus T_{(q)}$, where V_i and W_j are indecomposable $Z_{(p)}G$-lattice and indecomposable $Z_{(q)}G$-lattice obtained as in Theorem 1. respectively.

By this theorem we know all the indecomposable genera and the extension forms of indecomposable lattices. So the last problem is to determine the isomorphism classes in each extension.

§ 3. Isomorphism classes

Recall the following result of Reiner. By that Proposition, the computation of the cardinarity of isomorphism classes of indecomposable lattices is reduced to counting the number of orbits under the action of Aut( ) on Ext( ) given in [8].

Proposition 3 (Reiner [8]). Let Π be a finite group, and L, N be $Z\Pi$-modules with $\mathrm{Hom}_{Z\Pi}(L,N) = 0$. And let X_i (i=1,2) be a $Z\Pi$-module which is determined by $\xi_i \in \mathrm{Ext}^1_{Z\Pi}(N,L)$ (i =1,2). Then the following two conditions are equivalent.

(i) $X_1 \simeq X_2$

(ii) $\xi_2 = \gamma \cdot \xi_1 \cdot \delta$ for some $\gamma \in \mathrm{Aut}_{Z\Pi}(L)$, $\delta \in \mathrm{Aut}_{Z\Pi}(N)$.

In general, it is very difficult to express the number of orbits. Though, in the case of s = 1, which means $(Z/pZ)^{\times} = \langle q(\mathrm{mod}.p)\rangle$, we give the formula. But it is also considerably complicated. Note that the above condition is always satisfied when p = 2.

Theorem 3. With the notation defined as before, assume that s = 1, i.e. $(Z/pZ)^{\times} = \langle q(\mathrm{mod}.p)\rangle$. Then the number of isomorphism classes of indecomposable ZG-lattices is given by

$$1 + 2h_R + 2h_T + h_S + h_R h_T + 2h_R h_S \cdot |F_q(\zeta)^{\times} : u^*(S)|$$
$$+ (2h_T h_S + h_R h_T h_S) \cdot \sum_{i=1}^{f-1} |K^{(i)} : u^*(S)|$$
$$+ h_R h_T h_S |F_q(\zeta)^{\times} : u^*(S)| \cdot \sum_{i=1}^{f-1} |K^{(i)} : u^*(R\langle\sigma\rangle) \cdot u^*(T)|$$

$$+ h_R h_T h_S \, |F_q(\zeta)^{\times} : u^*(S)| \sum_{i=1}^{2^{r+1}-1} |\tilde{K}^{(i)} : u^*(R\langle\sigma\rangle)|$$
$$+ h_R h_T h_S \, |F_q(\zeta)^{\times} : u^*(S)| \sum_{i=1}^{2^r-1} |\bar{K}^{(i)} : u^*(R\langle\sigma\rangle)\cdot u^*(Z \oplus T)| .$$

Corollary. Assume further that $p = 2$ and $(Z/qZ)^{\times} = \langle 2(\mathrm{mod}.\,q)\rangle$. Then the number of isomorphism classes of indecomposable ZG-lattices is
$$h_T^2 + 6h_T + 3 + 5h_T^2 \cdot |F_2(\theta)^{\times} : u^*(T)| + 2h_T^2 |F_2(\theta)^{\times} : u^*(Z\langle\sigma\rangle)| .$$

Now we have the formula, but it is quite difficult to compute each term explicitly. For G with $|G| < 30$, a straightforward calculation yields the following table.

$\lvert G\rvert$	p	q	r	s	number of indecomposable ZG-lattices
6	2	3	1	1	17
10	2	5	1	1	17
14	2	7	2	1	31
15	3	5	1	1	21
21	7	3	2	1	33
22	2	11	1	1	31
26	2	13	1	1	45

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Department of Mathematics
Tokyo Metropolitan University

ON MULTIPLICATIVE INDUCTIONS

Yoshito OGAWA

Throughout of this paper, let G be a finite group and let H be a subgroup of index ℓ. We denote the symmetric group of ℓ letters by $\mathfrak{S}_\ell$. Setting $G = \cup_{i=1}^{\ell} t_i H$, we have

$$gt_i = t_{\sigma(i)}h_i,\ g \in G,\ \sigma \in \mathfrak{S}_\ell,\ h_i \in H,\ 1 \leqq i \leqq \ell.$$

We state some non-vanishing theorems in the cohomolgy of finite groups. Evens' norm mapping [3] plays a crutial rolè. The norm is a multiplicative analogue of the usual transfer. First we explain the norm from the view-point of 'multiplicative inductions' (§1 - §4), and then describe our theorems (§5). The proof will be published elsewhere.

1. Algebras (cf. Taylor [7, §1]). We begin with multiplicative induction of algebras. This is partly a generalization of Hasse's Galois algebra (see Example below). Let k be a field. Let A be a k-algebra on which H acts (i.e. an H-algebra over k). We define a multiplicative induction of A by

$$\mathrm{Map}_H(G,A) = \{\phi: G \to A \mid \phi(hg) = h\phi(g),\ g \in G,\ h \in H\}.$$

Here $\mathrm{Map}_H(G,A)$ is a G-algebra with k-algebra structure induced by one of A and with the G-action given by $(x\phi)(g) = \phi(gx)$, $x \in G$. Then we have a covariant functor

$$\mathrm{Map}_H(G,\cdot): \underline{\text{H-algebras}} \to \underline{\text{G-algebras}}.$$

Note that $\mathrm{Map}_H(G,\cdot)$ is the right adjoint functor of the restriction: $\underline{\text{G-algebras}} \to \underline{\text{H-algebras}}$. Specially $\mathrm{Map}_H(G,\cdot)$ preserves 'direct products'.

Since $G = \cup_{i=1}^{\ell} Ht_i^{-1}$, we have $\mathrm{Map}_H(G,A) \simeq \Pi_{i=1}^{\ell} A_i$: $\phi \to$

$\Pi_{i=1}^{\ell}\phi(t_i^{-1})$, $A_{i_\ell} = A$. Thus $\Pi_{i=1}^{\ell}A_i$ becomes a G-algebra by

$$g(\Pi_{i=1}^{\ell}a_i) = \Pi_{i=1}^{\ell}h_{\sigma^{-1}(i)}\, a_{\sigma^{-1}(i)},\ a_i \in A_i.$$

This gives another definition of multiplicative induction of algebras.

Example. Let F be a finite Galois extention field of k with Galois group G. Then $Map_{\diamond}(G,F)$ is a Galois algebra over F with Galois group G. That is, $Map_{\diamond}(G,F)$ is a G-algebra over F, which is isomorphic to FG as an FG-module. Observe that $Map_{\diamond}(G,F) \underset{\to}{\simeq} FG$: $\phi \mapsto \Sigma_{g\in G}\phi(g^{-1})g$.

cf. (Riehm [4, §4] and Evens [3, §5]). If A is a commutative H-algebra, then $\otimes_{i=1}^{\ell}A_i$ $(A_i = A)$ is a G-algebra by $g(\otimes_{i=1}^{\ell}a_i) = \otimes_{i=1}^{\ell}h_{\sigma^{-1}(i)}\, a_{\sigma^{-1}(i)}$, $a_i \in A_i$, where $\otimes$ is defined over k. $\otimes_{i=1}^{\ell}A_i$ is called the Corestriction. $\otimes_{i=1}^{\ell}\cdot$ is the left adjoint functor of the restriction: Commutative G-algebras → Commutative H-algebras, and so preserves 'direct sums'.

2. Modules ([3, §3], Dress [1,§5]). From now on $\otimes$ will be defined over $\mathbb{Z}$, which is the ring of rational integers. The restriction: $\mathbb{Z}G$-modules → $\mathbb{Z}H$-modules has the left (resp. right) adjoint functor $\mathbb{Z}G\otimes_{\mathbb{Z}H}\cdot$ (resp. $Hom_{\mathbb{Z}H}(\mathbb{Z}G,\cdot) = Map_H(G,\cdot)$). Two adjoint functors are isomorphic and preserve finite direct sums. Shapiro's lemma in the cohomology of finite groups arises from this isomorphism.

Multiplicative induction of a module is called the monomial module: for a $\mathbb{Z}H$-module M, $\otimes_{i=1}^{\ell}M_i$ $(M_i = M)$ is a $\mathbb{Z}G$-module by $g(\otimes_{i=1}^{\ell}m_i) = \otimes_{i=1}^{\ell}h_{\sigma^{-1}(i)}\, a_{\sigma^{-1}(i)}$, $m_i \in M_i$. Then

$$\otimes_{i=1}^{\ell}\cdot: \mathbb{Z}H\text{-modules} \to \mathbb{Z}G\text{-modules}$$

preserves 'tensor products given the diagonal action'.

3. Projective resolutions ([2, §5]). A $\mathbb{Z}G$-projective resolution can be constructed from a $\mathbb{Z}H$-projective resolution $X = (X_n)_{n \geq 0}$. Now $\otimes^{\ell} X$ is an acyclic $\mathbb{Z}G$-complex by

$$g(\otimes_{i=1}^{\ell} x_{p_i}) = (-1)^s \otimes_{i=1}^{\ell} h_{\sigma(i)}^{\to}\, x_{p_{\sigma^{-1}(i)}},$$
$$g \in G,\ x_i \in X_i,\ s = \Sigma_{\substack{1 \leq i < j \leq \ell \\ \sigma(i) > \sigma(j)}} p_i p_j.$$

Note that $\otimes^{\ell} X$ is not always $\mathbb{Z}G$-projective. Let W be a $\mathbb{Z}\mathfrak{S}_{\ell}$-projective resolution of $\mathbb{Z}$. Then W is an acyclic $\mathbb{Z}G$-chain complex via $G \to \mathfrak{S}_{\ell}$: $g \mapsto \sigma$. Given the diagonal G-action, $W \otimes (\otimes^{\ell} X)$ is a required $\mathbb{Z}G$-projective resolution of $\mathbb{Z}$.

cf. A fundamental theorem of Serre [5, Theorem 1] states that, if Γ is a torsion free group and if Γ' is a subgroup of finite index ℓ, then $cd(\Gamma) = cd(\Gamma')$. The key of the proof is to show that, if X is a $\mathbb{Z}\Gamma'$-projective resolution of $\mathbb{Z}$, then $\otimes^{\ell} X$ is itself a $\mathbb{Z}\Gamma$-projective resolution of $\mathbb{Z}$.

4. Cohomology rings ([3, §5 and §6]). Let k be the prime field of characteristic $p \geq 0$. We set

$$a(H) = \oplus_{u=0}^{\infty} H^{2u}(H,k),\quad a(G) = \oplus_{v=0}^{\infty} H^{2v}(G,k).$$

(If $p = 2$, the restriction to even degrees is unnecessary.) Using the above resolutions (§3), Evens defined norm: $a(H) \to a(G)$ as follows. Let $\alpha = \oplus_{u=0}^{\infty} \alpha_{2u} \in a(H)$, where $\alpha_{2u} \neq 0$ for only finitely many $u \geq 0$. We choose a cocycle f_{2u} which represents α_{2u} for each $u \geq 0$. Then there is a cochain g_{2v} for G such that

$$g_{2v}(\Sigma_{i+i_1+\dots+i_{\ell}=2v} w_i \otimes x_{i_1} \otimes \dots \otimes x_{i_{\ell}})$$
$$= \omega(w_0)\Sigma_{j_1+\dots+j_{\ell}=v} f_{2j_1}(x_{2j_1}) \dots f_{2j_{\ell}}(x_{2j_{\ell}}),$$

where $v \geq 0$, $w_i \in W_i$, $x_i \in X_i$ and ω is the augmentation of W (ω: $W_0 \to \mathbb{Z}$). Clearly g_{2v} is a cocycle, and we denote the cohomology class containing g_{2v} by γ_{2v}. Then norm(α) is defined by $\oplus_{v=0}^{\infty}\gamma_{2v}$.

Next we describe the properties of a. First, a is a contravariant functor

<u>Finite groups</u> $\to$ <u>Commutative graded k-algebras</u>.

Let $G \geqq L \geqq H$, $G \geqq K$, $g \in G$, $\alpha,\beta \in a(H)$, $\gamma \in a(G)$. For brevity, we write

$$\text{norm}: a(H) \to a(G): \alpha \mapsto {}^{G}\alpha,$$
$$\iota^* : a(G) \to a(H): \alpha \mapsto \alpha_H,$$
$$\theta^* : a(H^g) \to a(H): \alpha \mapsto \alpha^g,$$

where $\iota: H \to G$, $\theta: H \to H^g: h \to g^{-1}hg$ and $H^g = gHg^{-1}$. If $h \in H$, then $\alpha^h = \alpha$. Though ${}^{G}(\gamma_H) = \gamma^{\ell}$ does not always hold, the following conditions are satisfied:

$$\text{(N.1)} \quad {}^{G}(\alpha\cdot\beta) = {}^{G}\alpha\cdot{}^{G}\beta$$
$$\text{(N.2)} \quad {}^{H}\alpha = \alpha, \quad {}^{G}({}^{L}\alpha) = {}^{G}\alpha$$
$$\text{(N.3)} \quad {}^{L^g}(\alpha^g) = ({}^{L}\alpha)^g$$
$$\text{(N.4)} \quad ({}^{G}\alpha)_K = \Pi_{g\in K\backslash G/H}\, {}^{K}(\alpha^g{}_{H^g \cap K}),$$

where $K\backslash G/H$ is a transversal of the (K,H) double coset in G.

5. Application. Evens introduced the norm in order to give an algebraic proof of the nontriviality of the integral cohomology of finite groups [3, Theorem 3], which was geometrically proved by Swan [6]. We are interisted in the case where the coefficients are simple modules. Now we describe our results with the following notations:

k: the prime field of characteristic $p \geqq 0$

P: a finite p-group ($\neq 1$)

H: a p'-subgroup of Aut(P)

G: the semi-direct product of P by H

Z: the center of P

$q = |P:Z|$

$E = \langle x \in Z \mid x^p = 1 \rangle$

$i \geqq 0$

$S^i(E)$: the i-th symmetric power of the kG-module E

M: any simple kG-module.

Theorem 1. If M is a composition factor of $S^i(E)$ as a kG-module, then $H^{2qi}(G,M) \neq 0$.

To prove this, it is important to construct a polynomial ring contained in $a(P)$ from one in $a(E)$ by using the norm.

Theorem 2. Furthermore, if each element of $H-\{1\}$ has a fixed-point-free action on P, then $\oplus_{i=1}^{|H|} H^{2qi}(G,M) \neq 0$.

Remark. Let G' be an arbitrary Frobenius group such that the order of the Frobenius kernel is divisible by p. Note that G' is the extension of its maximai normal p'-subgroup by some G in Theorem 2. If we regard the blocks of kG' as the sets of isomorphic classes of simple kG'-modules, then the nontriviality of cohomology groups in Theorem 2 distinguishes the principal block of kG' from other blocks.

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Department of Mathmatics
Tokyo Metropolitan University

On $\aleph_0$-continuous regular rings

Jiro KADO

In 1936, von Neumann and Murrary discovered a regular ring which coordinates the lattice of projections of certain operator algebras. Twenty years later Berberian had shown that an analogous ring can be constructed for some AW*-algebras [2]. It was shown that these regular rings were isomorphic to the maximal ring of quotients of those operator algebras by Loos [10] and Handelman [4]. Recently moreover Handelman constructed "$\aleph_0$-continuous regular ring" which coordinated the lattice of projections of finite Rickart C*-algebras as a subring of the maximal ring of quotients of the C*-algebras [5]. The general properties of $\aleph_0$-continuous regular ring are studied by Goodearl in [9], and Goodearl, Handelman and Lawrence are investigating the properties which are applied to the Rickart C*-algebras in [10]. In this note, we try to introduce the affine function representation of $K_0(R)$ for a $\aleph_0$-continuous regular ring R which is studied by Goodearl, Handelman and Lawrence in [10].

On the other hand, among general regular rings, there is a reasonably manageable classes, namely the "ultramatricial algebras" over a fixed field defined by Goodearl [9, Ch.15]. As an application of the affine function representation, we show that any simple ultramatricial algebras over a field F can be embedded into a $\aleph_0$-continuous regular ring which has no simple artinian homomorphic images and which is an also F-algebra.

1. $\aleph_0$-continuous regular rings

Definition [9, Ch.14]. A regular ring is $\aleph_0$-continuous if both the lattice of all principal right ideals and the lattice of all principal left ideals are upper $\aleph_0$-continuous. A regular ring is $\aleph_0$-continuous if and only if every countably generated right (resp. left) ideals of R is essential in a principal right (resp. left) ideal of R.

Definition [2, p.13]. A Rickart C*-algebra is a C*-algebra with unit (e.g.,[13]) T such that for all t in T, there exists a projection (i.e., a self-adjoint idempotent) p in T such that the right annihilator of t is pT. This is a weak form of the defining axiom for AW*-algebras, in which it is required that right annihilators of arbitrary subsets of T are of the form pT. A C*-algebra T is said to be finite if $xx^* = 1$ always implies $x^*x = 1$ for any x in T.

Definition [2, p.229]. A *-regular ring is a ring with involution in which every principal right ideal is generated by a projection. Equivalently, *-regular rings may be characterized as regular rings with involution in which $xx^* = 0$ implies $x = 0$.

Handelman proved the following fundamental theorem concerning finite Rickart C*-algebra.

Proposition 1 [5, Th.2.1]. Let T be a finite Rickart C*-algebra. Then there exists a ring of quotients R of T with the following property:

(1) R is a subring of the maximal ring of quotients of T.

(2) The involution on T extends to an involution on R.

(3) R is *-regular and $\aleph_0$-continuous.

(4) All projections of R lie in T.

2. Pseudo-rank functions.

Definition [9, Ch.16]. A pseudo-rank function on a regular ring R is a map $N: R \longrightarrow [0,1]$ such that

(a) $N(1) = 1$.

(b) $N(xy) \leq N(x)$ and $N(xy) \leq N(y)$ for all x, y in R.

(c) $N(e + f) = N(e) + N(f)$ for all orthogonal idempotents e, f in R.

A rank function on R is a pseudo-rank function N with the additional property

(d) $N(x) > 0$ for all non-zero x in R.

For any regular rings R, we use P(R) to denote the set of all pseudo-rank functions on R. We view P(R) as a subset of the real vector space $\mathbb{R}^R$, which we equip with the product topology.

Proposition 2 [9, Prop.16.17]. For any regular rings, the set P(R) is a compact convex subset of $\mathbb{R}^R$.

Definition [9, Appendix]. Let S be a convex subset of a real vector space. An extreme point of S is any point x in S which cannot be expressed as a positive convex combination of two distinct points. We use $\partial_e S$ to denote a set of all extreme points in S.

There are connections between geometric properties of P(R) and algebraic properties of R. Put kerN = { $x \in R$; $N(x) = 0$ } for $N \in P(R)$, then kerN is a two-sided ideal of R.

Proposition 3 [10, Prop.II.14.5]. Let R be $\aleph_0$-continuous regular ring. Then there is a bijection from $\partial_e P(R)$ to the set of all maximal two-sided ideals of R by the rule: $N \longrightarrow \ker N$.

Proposition 4 [9, Th.14.33 and 10, Th.II.12.6]. Let R be a $\aleph_0$-continuous regular ring and let $N \in \partial_e P(R)$. Then we have the following results;

(1) R / kerN is a simple regular self-injective ring.

(2) Given a positive integer n, we have $N(R) = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ if and only if $R / \ker N \cong M_n(D)$ for some division ring D.

(3) $N(R) = [0, 1]$ if and only if R / kerN is not artinian.

3. Affine function representation of K_0.

Definition [9, Ch.15]. Recall that the Grothendieck group K_0 of a ring R is an abelian group with generators [A] corresponding to the finitely generated projective right R-modules A and with relations [A] + [B] = [C] whenever $A \oplus B \cong C$. All elements of $K_0(R)$ are of the form [A] - [B], for suitable A and B. We set

$K_0(R)^+ = \{[A];$ A is a finitely generated projective R-module$\}$, and we define a relation $\leq$ on $K_0(R)$ so that $x \leq y$ if and only if $y - x$ lies in $K_0(R)^+$. This relation is a translation invariant pre-order on $K_0(R)$, so that $K_0(R)$ becomes a pre-ordered abelian group. The element $[R]$ is an order-unit in $K_0(R)$, meaning that for any x in $K_0(R)$, there exists $n \in \mathbb{N}$ such that $x \leq n[R]$. For a unit-regualr ring R, the relations between $K_0(R)$ and the finitely generated projective right R-modules are much cleaner than in general. Namely for finitely generated projective right R-modules A, B, C, D, we have

$$[A] - [B] = [C] - [D] \text{ if and only if } A \oplus D \cong B \oplus C$$

$$[A] - [B] = [C] - [D] \text{ if and only if } A \oplus D \lesssim B \oplus C$$

by [7, Prop.15.2].

For a compact convex set S, we use $\mathrm{Aff}(S)$ to denote the partially ordered Banach space with pointwise ordering and supremum norm of all affine continuous real valued functions on S.

Let R be a $\aleph_0$-continuous regular ring. Then R is unit-regular by [5, Th.3.2], threfore $K_0(R)$ is a compact convex subset of the real vector space $\mathbb{R}^R$ Then we have the natural map $\alpha : K\,(R) \longrightarrow \mathrm{Aff}(P(R))$ given by the rule $\alpha([xR])(N) = N(x)$ for all x in R and for all N in $P(R)$. This map α can be extended to a homomorphism as a partially ordered abelian group, because $K_0(R)$ is generated as a group by the elements $[xR]$ for x in R. Now we can describe the affine function representation for $K_0(R)$ proved by Goodearl, Handelmen and Lawrence [10].

Proposition 5 [10, Th.II.15.1]. Let R be a $\aleph_0$-continuous regular ring. Whenever $N \in \partial_e P(R)$ and $R / \ker N$ is not artinian, set $A_n = \mathbb{R}$. Whenever $N \in \partial_e P(R)$ and $R / \ker N \cong M_n(D)$ for some positive integer n and some division ring D, set $A_n = \frac{1}{n}\mathbb{Z}$. Set

$A = \{p \in \mathrm{Aff}(P(R));\ p(N) \in A_n \text{ for all } N \in \partial_e P(R)\}$

Then the natural map $\alpha: K_0(R) \longrightarrow \mathrm{Aff}(P(R))$ provides an isomorphism of $K_0(R)$ onto A as partially ordered abelian group wtih order-unit.

Corollary [10, Cor.II.15.2]. Let R be a $\aleph_0$-continuous regular ring which has no simple artinian homomorphic images. Then the natural map $\alpha: K\ (R) \longrightarrow \mathrm{Aff}(P(R))$ is an isomorphism of partially ordered abelian groups.

Definition [9, p.219]. Given a field F, an ultramatricial F-algebra is any F-algebra which is a union of an increasing sequence of finite-dimensional subalgebras each of which is a finite direct product of full matrix algebra over F.

In [8], we studied the relation between a simple regular self-injective ring R which is not artinian and an ultramatricial algebra over its center F which is a direct limit of $M_2(F) \longrightarrow M_4(F) \longrightarrow \ldots.$ Now we can prove the more general theorem by a corollary of Proposition 5.

Theorem 6. Let R be a $\aleph_0$-continuous regular ring which has no simple artinian homomorphic images and moreover it is an algebra over a field F. For any ultamatricial

F-algebra S, there exists a non-zero F-algebra homomorphism from S to R. Consequently for each simple ultramatricial F-algebra S, there exists a subalgebra of R which is isomorphic to S.

Proof. Since S is unit-regular, then we have $n[S] > 0$ in $K_0(S)$ for all positive integers n by [7,Prop. 2.1]. By [9, Cor.18.2], there exists a order-preserving homomorphism $\beta: K_0(S) \longrightarrow R$ such that $\beta([S]) = 1$. R is naturally identified with the subgroup consisting of all constant functions in Aff(P(R)). On the other hand, there exists a isomorphism γ from Aff(P(R)) to $K_0(R)$ such that $\gamma(1) = [R]$. Put $\delta = \gamma\beta$. Since δ is a order-preserving homomorphism by Corollary of Proposition 5 from $K_0(S)$ to $K_0(R)$ such that $\delta([S]) = [R]$, we have a non-zero F-algebra homomorphism $f: S \longrightarrow R$ such that $K_0(f) = \delta$ by [8,Lemma 3]. If S is moreover simple evidently we have $\ker f = 0$.

4. AF-C*-algebras

Among general C*-algebras, there is a reasonably manageable class which still exhibits many of the general phenomena, namely the approximately finite-dimensional (or AF-) C*-algebra introduced by Bratteli [1], these are the C*-algebras possessing a norm dense subalgebra which is the union of an increasing sequence of finite-dimensional sub-C*-algebra. Since finite-dimensional C*-algebra is isomorphic to finite direct product of full matrix over $\mathbb{C}$ [13, p.50], we say on other word that AF-C*-algebra possess an ultramatricial algebra over $\mathbb{C}$ as norm-dense subalgebra.

Bratteli showed that such a C*-algebra and such a dense subalgebra determine each other, in the following manner: If A and A' are AF-C*-algebras which posses ultramatricial algebras S and S' over $\mathbb{C}$ as a norm-dense subalgebra, then A and A' are isomorphic as C*-algebra if and only if S and S' are isomorphic just as algebra [3, Appendix].

Theorem 7. Let T be a finite Rickart C*-algebra with identity whose all simple homomorphic images are infinite dimensional. Then for each simple AF-C*-algebra A, there exists a sub-C*-algebra A' of T which is isomorphic to A.

Proof. Let R be the regular ring of T defined in Proposition 1. By [10, Props.III.16.5 and III.16.7], R has no simple artinian homomorphic images. Let S be an ultramatricial algebra over $\mathbb{C}$ which is dense subalgebra of A. Then S is algebraically simple , because there is a bijection between the set of norm-closed ideals of A and the set of two-sided ideals of S by [1, Th.3.3]. Therefore there is a monomorphism $f: S \longrightarrow R$ as algebra by Theorem 6. Put $S' = f(S)$. On the other hand, T is *-isomorphic to an $n \times n$ matrix ring over a finite Rickart C*-algebra for every integer $n > 1$ by [10, Th.III.16.8], because T has no finite-dimensional homomorphic images. Then Handelman showed that every partially isometries of R lie in T in the proof of [6, Th.7.8]. Then the ultramatricial algebra S' is contained in T, because any element of S' is a linear combination over $\mathbb{C}$ of partially isometries of R. Let A' be the closure of S' in T, then A' is AF-C*-algebra which is isomorphic to A.

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Department of Mathematics

Osaka City University

NOTE ON COMMUTATIVITY OF RINGS

Isao MOGAMI

In the last few years, several authors has been investigating commutativity of rings. Recently, when ring R (with identity 1) satisfying the polynomial identities $(xy)^{\alpha} = x^{\alpha}y^{\alpha}$, $\alpha = n_1,\ldots,n_r$ where n_i are positive integer (with some conditions), then R is commutative, as was discussed by Y. Kobayashi in this symposium [7] and in his paper [6]. This result is very interest and elegant. In this note we present a survey of the results of commutativity of s-unital ring R satisfying the identities $(xy)^{\alpha} = x^{\alpha}y^{\alpha}$, $\alpha = n_1(x,y),\ldots,n_r(x,y)$ where n_i are positive integers (with some conditions) depending on x and y in R. Also, we consider the identity $(xy)^{\beta} = y^{\beta}x^{\beta}$ and we pose some conjectures.

Throught, R will represent an associative ring with center, J the (Jacobson) radical of R. A ring R is called s-unital if for each $x \in R$, $x \in Rx \cap xR$ [14]. As stated in [10, Lemma 1 (a)], if R is s-unital ring, then for any finite subset F of R, there exists an element e in R such that $ex = xe = x$ for all x in R. Such an element e will be called a pseudo-identity of F.

We consider the following properties of rings:

(1) For each pair of elements x, y in R, there exists positive integer $m = m(x,y)$ such that

$$(xy)^{\alpha} = x^{\alpha}y^{\alpha}, \quad \alpha = m,\ m+1,\ m+2.$$

(2) For each pair of elements x, y in R, there exist positive integers $m = m(x,y)$, $m' = m'(x,y)$ such that $(m,m') = 1$ and

$$(xy)^{\alpha} = x^{\alpha}y^{\alpha}, \quad \alpha = m,\ m+1,\ m',\ m'+1.$$

(3) For each pair of elements x, y in R, there exist positive integers $m = m(x,y)$, $m' = m'(x,y)$ such that $(m,m') = 2$ and

$$(xy)^{\alpha} = x^{\alpha}y^{\alpha}, \quad \alpha = m,\ m+1,\ m',\ m'+1.$$

(4) For each pair of elements x, y in R, there exists an even positive integer $m = m(x,y)$ such that

$$(xy)^{\alpha} = x^{\alpha}y^{\alpha}, \quad \alpha = m,\ m+2,\ m+4.$$

(5) For each pair of elements x, y in R, there exists an odd positive integer $m = m(x,y)$ such that

$$(xy)^{\alpha} = x^{\alpha}y^{\alpha}, \quad \alpha = m,\ m+2,\ m+4.$$

(6) For each pair of elements x, y in R, there exists positive integer $m = m(x,y)$ such that

$$(xy)^{\alpha} = x^{\alpha}y^{\alpha}, \quad \alpha = m,\ m+2,\ m+4.$$

(1') For each pair of elements x, y in R, there exists positive integer $m = m(x,y)$ such that

$$(xy)^{\beta} = y^{\beta}x^{\beta}, \quad \beta = m,\ m+1,\ m+2.$$

(2') For each pair of elements x, y in R, there exist positive integers $m = m(x,y)$, $m' = m'(x,y)$ such that $(m+1, m'+1) = 1$ and

$$(xy)^{\beta} = y^{\beta}x^{\beta}, \quad \beta = m,\ m+1,\ m',\ m'+1.$$

(3') For each pair of elements x, y in R, there exist positive integers $m = m(x,y)$, $m' = m'(x,y)$ such that $(m + 1, m' + 1) = 2$ and

$$(xy)^{\beta} = y^{\beta}x^{\beta}, \quad \beta = m,\ m + 1,\ m',\ m' + 1.$$

(4′) For each pair of elements x, y in R, there exists an odd positive integer $m = m(x,y)$ such that

$$(xy)^{\beta} = y^{\beta}x^{\beta}, \quad \beta = m,\ m + 2,\ m + 4.$$

(5′) For each pair of elements x, y in R, there exists an even positive integer $m = m(x,y)$ such that

$$(xy)^{\beta} = y^{\beta}x^{\beta}, \quad \beta = m,\ m + 2,\ m + 4.$$

(6′) For each pair of elements x, y in R, there exists positive integer $m = m(x,y)$ such that

$$(xy)^{\beta} = y^{\beta}x^{\beta}, \quad \beta = m,\ m + 2,\ m + 4.$$

Although several authors has been concerning above properties of rings, at the present day we summarize as follows:

Theorem. If R is an s-unital ring having one of the properties (1), (2), (3), (4), (1′), (2′), (3′), (4′) and (5′), then R is commutative.

Of course, if R (with 1) having one of the properties (1), (2), (3) and (4), as polynomial identity, then R is commutative by Kobayashi's Theorem stated below:

Kobayashi's Theorem ([6],[7]). Let R be a ring with 1. If $E(R)$ contains integers $n_1, \ldots, n_r \geqq 2$ such that $(n_1(n_1-1), \ldots, n_r(n_r-1)) = 2$ and some of n_i is even, then

R is commutative. where $E(R) = \{ n \in P \mid (xy)^n = x^n y^n$ for all $x, y \in R \}$ and P the set of positive integers.

As is easily seen, the following implication are valid:

$$\begin{array}{ccccccccccc} (1') & \Longrightarrow & (2') & & (3') & & (4') & \Longrightarrow & (6') & \Longleftarrow & (5') \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ (1) & \Longrightarrow & (2) & & (3) & & (4) & \Longrightarrow & (6) & \Longleftarrow & (5) \end{array}$$

Therefore,properties (2), (3), and (6) are essential but, unfotunately we do not know if one of the properties (5), (6) and (6') implies R is commutative.

Proof of the Theorem

(A) (1) $\Longrightarrow$ R is commutative. Historically, the first result in this type was obtained by J. Luh [9] in 1971 when he was able to prove in the special case of R primary ring with 1 (i.e. R/J is simple) and having the property (1) as polynomial identity. This result has subsequently been generalized by A. Kaya, A. Kaya & C. Koç, S. Ligh & A.Richoux, M. Hongan & author and by B. Felzenswalb as follows: In 1976 A. Kaya [4] proved that a ring having the property (1) is commutative if it is a primary ring with 1 or if it is a semiprime ring with 1. A. Kaya & C. Koç [5] in 1976 proved that if all zero divisor of R with 1 are contained in a proper left (or right) ideal, then R is commutative. S. Ligh & A. Richoux [8] in 1977 showed that any ring with 1 which possesses the property (1) as polynomial identity is commutative. In 1978 M. Hongan & author [3],[10] proved it in general for s-unital ring R. A. Richoux [13] in 1979 showed that a ring which possesses the property (1) is commutative if it has 1 or if it has no nilpotent elements. Also, B. Felzenswalb [2] in 1979 proved that if an element a in R having the property (1), together with for every x in R and if R has no nonzero nil right ideals, then a is central.

(B) (2) $\Longrightarrow$ R is commutative. In 1978 H. Bell [1] showed that any ring with 1 which possesses the property (2) as polynomial identity is commutative. In 1980 author [11] proved it in general for s-unital ring R.

(C) (3) $\Longrightarrow$ R is commutative. In 1980 C. T. Ten [15] showed that a ring which possesses the property (3) as polynomial identity is commutative if it is primary ring with 1. In 1981 author [12] proved it in general for s-unital ring R.

(D) (4) $\Longrightarrow$ R is commutative. (5') $\Longrightarrow$ R is commutative. In 1981 author showed both of them in general for s-unital ring R.

We conclude this note by listing several conjectures as follows: Let R be an s-unital ring or a ring with 1.

Conjecture (I). For each pair of elements x, y in R there exist positive integers $n_1(x,y),\dots,n_r(x,y) \geqq 2$ such that $(n_1(n_1 - 1),\dots,n_r(n_r - 1)) = 2$ and some of n_i is even and such that $(xy)^\alpha = x^\alpha y^\alpha$, $\alpha = n_1,\dots,n_r$, then R is commutative.

Conjecture (II). For each pair of elements x, y in R there exist positive integers $n_1(x,y),\dots,n_r(x,y) \geqq 2$ such that $(n_1(n_1 - 1),\dots,n_r(n_r - 1)) = 2$ and such that

$(xy)^{\alpha} = x^{\alpha}y^{\alpha}$, $\alpha = n_1,\dots,n_r$, then R is commutative.

Conjecture (I'). For each pair of elements x, y in R there exist positive integers $n_1(x,y),\dots,n_r(x,y)$ such that $(n_1(n_1 + 1),\dots,n_r(n_r + 1)) = 2$ and some of n_i is odd and such that $(xy)^{\beta} = y^{\beta}x^{\beta}$, $\beta = n_1,\dots,n_r$, then R is commutative.

Conjecture (II'). For each pair of elements x, y in R there exist positive integers $n_1(x,y),\dots,n_r(x,y)$ such that $(n_1(n_1 + 1),\dots,n_r(n_r + 1)) = 2$ and such that $(xy)^{\beta} = y^{\beta}x^{\beta}$, $\beta = n_1,\dots,n_r$, then R is commutative.

Evidently, Conjecture (I) includes Kobayashi's Theorem and among the conditions of Conjectures, there holds the following:

$$\begin{array}{ccc} \text{Conditions of (I')} & \Longrightarrow & \text{Conditions of (II')} \\ \Downarrow & & \Downarrow \\ \text{Conditions of (I)} & \Longrightarrow & \text{Conditions of (II)} \end{array}$$

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Tsuyama College of Technology